

Modular realizations of hyperbolic Weyl groups

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Abstract

We study the recently discovered isomorphisms between hyperbolic Weyl groups and unfamiliar modular groups. These modular groups are defined over integer domains in normed division algebras, and we focus on the cases involving quaternions and octonions. We outline how to construct and analyse automorphic forms for these groups; their structure depends on the underlying arithmetic properties of the integer domains. We also give a new realization of the Weyl group $W(E_8)$ in terms of unit octavians and their automorphism group.

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1 Introduction

Following the work of Feingold and Frenkel [1] it was realized in [2] that very generally there are isomorphisms between the Weyl groups of hyperbolic Kac-Moody algebras and (extensions or quotients of) modular groups over integer domains in the division algebras \mathbb{A} of real, complex, quaternionic or octonionic numbers; $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The simplest case is associated with the rational integers $\mathbb{Z} \subset \mathbb{R}$: the modular group $\mathrm{PSL}(2, \mathbb{Z})$ is isomorphic to the even part of the Weyl group of the canonical hyperbolic extension of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ [1]. This is but the first example of a rich and interesting family of novel isomorphisms that rely on the existence of integers within the division algebras of higher dimensions [2], culminating in the relation $W^+(E_{10}) \cong \mathrm{PSL}(2, \mathbb{O})$

between the even Weyl group of E_{10} (the hyperbolic extension of the exceptional Lie algebra E_8) and the integer octonions.

These isomorphisms are interesting for several reasons. The underlying arithmetic structure of the various integer domains could help in understanding the structure of the still elusive hyperbolic Kac-Moody algebras, especially for the ‘maximally extended’ hyperbolic algebra E_{10} which is expected to possess very special properties.¹ This is certainly a tantalizing possibility but we will not pursue it further in this work. Instead, we focus on the modular theory associated with the novel modular groups that appear in the isomorphisms, and the associated ‘generalized upper half planes’. More specifically, we will discuss the quaternionic and the octonionic cases, corresponding to the hyperbolic extensions of the split real $D_4 \equiv \mathfrak{so}(4, 4)$ and $E_{8(8)}$ Lie algebras. In the quaternionic case in particular, we give a detailed description of the ‘generalized upper half plane’ of dimension five on which the modular group acts naturally.

We note that generalized modular groups and upper half planes have already appeared in the mathematical literature. In [3] they are defined as cosets $GL(n, \mathbb{R})/O(n) \times GL(1, \mathbb{R})$, on which the discrete groups $GL(n, \mathbb{Z})$ act as generalized modular groups. This definition fits with our definition only for $n = 2$ (corresponding to $\mathbb{A} = \mathbb{R}$). More generally, we can take any non-compact group G with maximal compact subgroup $K(G)$ and take the ‘upper half plane’ to be $G/K(G)$. On this space one considers the action of some arithmetic group $\Gamma \equiv G(\mathbb{Z}) \subset G$ and studies invariant (or equivariant) functions that are also eigenfunctions of the Laplacian [4].

Our main motivation derives from potential applications of this modular theory in quantum gravity and M-theory. As shown recently, the generalized upper half planes are the configuration spaces for certain models of (anisotropic) mini-superspace quantum gravity, and a subclass of the automorphic forms associated with the modular groups studied here appear as solutions to the Wheeler-DeWitt equation in the cosmological billiards limit [5–7], a version of mini-superspace quantum gravity which we refer to as ‘arithmetic quantum gravity’. Automorphic functions (particularly Eisenstein series) also feature prominently in recent studies of non-perturbative effects in string and M-theory, see [8, 9] for early work and [10–12] for recent progress concerning the exceptional groups $G = E_{7(7)}$ and $E_{8(8)}$, with corresponding arithmetic groups $E_7(\mathbb{Z})$ and $E_8(\mathbb{Z})$.² There exist far reaching conjectures concerning the infinite-dimensional extensions E_9 , E_{10} and E_{11} , but it is less clear how to make sense of (or even *define*) their discrete subgroups (see, however, [13]). Nevertheless, our results can be viewed as a first step towards extending these ideas to the infinite-dimensional duality group E_{10} , since $PSL(2, \mathbb{O})$, being the even Weyl group, is expected to

¹For instance, the (so far unknown) arithmetic structure of the modular group $PSL(2, \mathbb{O})$ may impose more stringent constraints on the characters and root multiplicities of E_{10} than the corresponding modular groups do for the lower rank hyperbolic algebras. E_8 and its hyperbolic extension E_{10} are also special because they are the only eligible algebras with (Euclidean or Lorentzian) even self-dual root lattices.

²In some cases, the relevant ‘automorphic forms’ are not eigenfunctions of the Laplacian but have source terms [14].

be contained in any hypothetical arithmetic subgroup of E_{10} . A possible link with the work of [5–7] is provided by a conjecture of [15] according to which the solution of the Wheeler-DeWitt equation in M-theory is a vastly generalized automorphic form with respect to $E_{10}(\mathbb{Z})$. Our results on $D_4 \equiv \mathfrak{so}(4,4)$ can also be viewed from a string theory perspective, since $SO(4,4)$ can be realized as the symmetry of type I supergravity in ten dimensions without any vector multiplets on a four-torus. It is also the continuous version of T-duality of type II strings.

Within the general theory the main feature that distinguishes our construction is therefore the link with the division algebras \mathbb{A} and the integer domains $\mathcal{O} \subset \mathbb{A}$ (which always contain \mathbb{Z} as a ‘real’ subset). In particular, the arithmetic modular groups considered here are all identified with the even subgroups of certain hyperbolic Weyl groups. That is, each of the modular groups considered here is isomorphic to the *even* Weyl group of a canonical hyperbolic extension \mathfrak{g}^{++} of an associated simple finite-dimensional split Lie algebra \mathfrak{g} ,

$$\Gamma \equiv W_{\text{hyp}}^+ \equiv W^+(\mathfrak{g}^{++}). \quad (1.1)$$

More specifically, the groups Γ are discrete subgroups of the groups $SO(1, n+1)$ acting on the (Lorentzian) root space of \mathfrak{g}^{++} , but with additional restrictions implying special (and as yet mostly unexplored) arithmetic properties. The standard Poincaré series construction then implies a simple general expression for an automorphic function on the generalized upper half plane as a sum over images of the group action as

$$f(u, v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\gamma \cdot v)^s, \quad (1.2)$$

where $u \in \mathbb{A}$ and $v \in \mathbb{R}_{>0}$ parametrize the upper half plane. (This expression is convergent for $\text{Re}(s)$ sufficiently large and can be analytically continued to most complex values of s by means of a suitable functional relation.) The subgroup Γ_∞ appearing in this sum is conventionally defined as the subgroup of the modular group Γ leaving invariant the cusp at infinity; here this group becomes identified with the even Weyl group of the affine subalgebra $\mathfrak{g}^+ \subset \mathfrak{g}^{++}$,

$$\Gamma_\infty \equiv W_{\text{aff}}^+ \equiv W^+(\mathfrak{g}^+) \subset \Gamma. \quad (1.3)$$

As we will explain, the cusp at infinity in the generalized upper half plane here corresponds to the affine null root of the affine algebra \mathfrak{g}^+ , in accord with the fact that W_{aff} can be defined as the subgroup of W_{hyp} stabilizing the affine null root [16]. At the same time W_{aff}^+ is the *maximal parabolic subgroup* of $\Gamma = W_{\text{hyp}}^+$. Because $W_{\text{aff}} = W_{\text{fin}} \times \mathcal{T}$, where $W_{\text{fin}} \equiv W(\mathfrak{g})$ and \mathcal{T} is the abelian group of affine translations, the minimal parabolic subgroup is $\mathcal{T} \subset \Gamma_\infty$.

This article is structured as follows. In section 2, we review the construction of normed division algebras and introduce the generalized upper half planes that we will utilize in the remainder of the text. Section 3 is devoted to the isomorphism between modular groups and hyperbolic Weyl groups in general;

section 4 then deals in detail with the case of quaternions and $D_4 \equiv \mathfrak{so}(4, 4)$ and the associated automorphic forms in section 5. In section 6, we present some progress in the understanding of the modular group $\mathrm{PSL}(2, \mathcal{O})$ over the non-associative integers octonions \mathcal{O} that appear for E_{10} , the hyperbolic extension of E_8 . This section also gives a novel realization of the even (and finite) Weyl group $W^+(E_8)$ in terms of the 240 octavian units and the finite automorphism group $G_2(2)$ of the octavians. Two appendices contain some technical details that we use in the main text.

We emphasize that it is not our intention here to give a complete account of the theory of automorphic forms for the new modular groups (after all, even for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ this is a subject which easily fills a whole book [17]). Rather, we regard the present work merely as a first step towards a theory that remains to be fully developed in future work.

2 Division algebras and upper half planes

2.1 Cayley-Dickson doubling

The division algebras of complex numbers, quaternions and octonions can be defined recursively from the real numbers by a procedure called *Cayley-Dickson doubling* [18, 19]. Let \mathbb{A} be a real finite-dimensional algebra with a conjugation, i.e. a vector space involution $x \mapsto \bar{x}$ such that $\overline{ab} = \bar{b}\bar{a}$ for any $a, b \in \mathbb{A}$. In the Cayley-Dickson doubling one introduces a vector space $i\mathbb{A}$, isomorphic to \mathbb{A} , where i is a new imaginary unit, and considers the direct sum $\mathbb{A} \oplus i\mathbb{A}$ of these two vector spaces. The product of two elements in $\mathbb{A} \oplus i\mathbb{A}$ is defined as [19]

$$(a + ib)(c + id) := (ac - d\bar{b}) + i(cb + \bar{a}d), \quad (2.1)$$

and conjugation as

$$\overline{a + ib} := \bar{a} - ib \quad (\text{with } ai = i\bar{a}). \quad (2.2)$$

Starting from $\mathbb{A} = \mathbb{R}$ (with the identity map as conjugation) and successively applying the Cayley-Dickson doubling one thus obtains an infinite sequence of so-called *Cayley-Dickson algebras*. The first doubled algebra is of course the familiar algebra \mathbb{C} of complex numbers. The following two algebras in the sequence are \mathbb{H} and \mathbb{O} , consisting of quaternions and octonions, respectively. Studying these algebras one finds that \mathbb{H} is associative but not commutative, whereas \mathbb{O} is neither associative nor commutative. Nevertheless, \mathbb{O} is an *alternative* algebra, which means that any subalgebra generated by two elements is associative.

Any Cayley-Dickson algebra admits a positive-definite inner product

$$(a, b) := \frac{1}{2}(a\bar{b} + b\bar{a}) \quad (2.3)$$

with the associated norm $|a|^2 \equiv (a, a) = a\bar{a}$. As a consequence of (2.1) and (2.2) the norm of the ‘doubled’ expression $a + ib \in \mathbb{A} \oplus i\mathbb{A}$ is

$$|a + ib|^2 = |a|^2 + |b|^2. \quad (2.4)$$

From the doubling formulas (2.1) and (2.2) it follows

$$\begin{aligned} |(a + ib)(c + id)|^2 &= |ac - d\bar{b}|^2 + |cb + \bar{a}d|^2 \\ &= (|a|^2 + |b|^2)(|c|^2 + |d|^2) + 2 \operatorname{Re}\left((\bar{d}a)(cb) - (b\bar{d})(ac)\right), \end{aligned} \quad (2.5)$$

where $\operatorname{Re} a := \frac{1}{2}(a + \bar{a})$. Because for $a, b, c \in \mathbb{A}$ (for all \mathbb{A})

$$\operatorname{Re} a(bc) = \operatorname{Re} (ab)c, \quad \operatorname{Re} ab = \operatorname{Re} ba, \quad (2.6)$$

we get

$$|(a + ib)(c + id)|^2 = |a + ib|^2|c + id|^2 + 2 \operatorname{Re}\left(\bar{d}\{a, c, b\}\right), \quad (2.7)$$

where the associator $\{a, b, c\} := a(bc) - (ab)c$ vanishes for associative algebras. The fact that \mathbb{R} , \mathbb{C} and \mathbb{H} are associative thus implies that the composition property

$$|ab| = |a||b| \quad (2.8)$$

holds for their doubled algebras \mathbb{C} , \mathbb{H} and \mathbb{O} . Algebras with this property are called *normed division algebras*. It follows from (2.8) that if $ab = 0$ then a or b must be zero, so these algebras have no zero divisors. Hurwitz' theorem (see [18,19]) states that \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} are the only (real and finite-dimensional) normed division algebras with unity. For instance, since the octonions are non-associative, their Cayley-Dickson double does not satisfy the norm composition property (2.8). This is a real 16-dimensional algebra, often called the *sedonions*. It is of course non-associative (since it contains the octonions) but also non-alternative, and does possess zero divisors.

2.2 Generalized upper half planes

For every normed division algebra \mathbb{A} we define the *generalized upper half plane*

$$\mathcal{H} \equiv \mathcal{H}(\mathbb{A}) := \{z = u + iv \quad \text{with } u \in \mathbb{A} \text{ and real } v > 0\}. \quad (2.9)$$

By i we will always denote the new imaginary unit not contained in \mathbb{A} , while $u \equiv u^0 + \sum u^i e_i$ with the imaginary units e_i in \mathbb{A} . Hence $\mathcal{H}(\mathbb{A})$ is contained in a hyperplane in the Cayley-Dickson double $\mathbb{A} \oplus i\mathbb{A}$, and of (real) dimension $\dim_{\mathbb{R}} \mathcal{H}(\mathbb{A}) = (\dim_{\mathbb{R}} \mathbb{A}) + 1$. By complex conjugation (2.2) we get $\bar{z} = \bar{u} - iv$, and thus \bar{z} parametrizes the corresponding 'lower half plane' $\overline{\mathcal{H}}$. From (2.7) we see that the composition property

$$|zz'| = |z||z'| \quad z, z' \in \mathcal{H}(\mathbb{A}) \quad (2.10)$$

continues to hold for *all* normed division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. In particular, it still holds for $\mathbb{A} = \mathbb{O}$, even though in this case z is a sedenion, because

the associator in (2.7) still vanishes as long as v remains real. Likewise, the alternative laws

$$(aa)z = a(az), \quad (az)a = a(za), \quad (za)a = z(aa) \quad (2.11)$$

remain valid for $a \in \mathbb{O}$ and $z \in \mathcal{H}(\mathbb{O})$ by Cayley-Dickson doubling (2.1). This will be important below when we define modular transformations.

The line element in $\mathcal{H}(\mathbb{A})$ is

$$ds^2 = \frac{|du|^2 + dv^2}{v^2} \quad (2.12)$$

for $u \in \mathbb{A} (= \mathbb{R}, \mathbb{C}, \mathbb{H} \text{ or } \mathbb{O})$ and $v > 0$, with $|du|^2 \equiv d\bar{u} du$. Using v to parametrize the geodesic, the geodesic equation is (with $u' \equiv du/dv$)

$$\frac{d}{dv} \left(\frac{1}{v\sqrt{1+|u'|^2}} \frac{du}{dv} \right) = 0. \quad (2.13)$$

It is then straightforward to see that geodesics are given by straight lines parallel to the ‘imaginary’ ($= v$) axis, or by half circles starting at $u_1 \in \mathbb{A}$ and ending at $u_2 \in \mathbb{A}$ on the boundary $v = 0$ of $\mathcal{H}(\mathbb{A})$. The equation of this geodesic half circle for $z = u + iv \in \mathcal{H}(\mathbb{A})$ reads

$$(u - u_1)(\bar{u} - \bar{u}_2) + v^2 = 0. \quad (2.14)$$

The length of any geodesic segment connecting two points $z_1, z_2 \in \mathcal{H}(\mathbb{A})$ is

$$d(z_1, z_2) = \log \frac{|z_1 - z_2^*| + |z_1 - z_2|}{|z_1 - z_2^*| - |z_1 - z_2|} \quad (2.15)$$

or, equivalently,

$$d(z_1, z_2) = 2 \operatorname{artanh} \frac{|z_1 - z_2|}{|z_1 - z_2^*|} = \operatorname{arcosh} \left(1 + \frac{|z_1 - z_2|^2}{v_1 v_2} \right), \quad (2.16)$$

where we defined

$$z = u + iv \quad \Rightarrow \quad z^* := u - iv \in \overline{\mathcal{H}(\mathbb{A})}. \quad (2.17)$$

The formula (2.16) generalizes the familiar formula from complex analysis to all division algebras. Likewise, the volume element on $\mathcal{H}(\mathbb{A})$ is given by

$$d\operatorname{vol}(z) := \frac{d^n u dv}{v^{n+1}} \quad (2.18)$$

and the Laplace-Beltrami operator reads³

$$\Delta_{\text{LB}} = v^{n+1} \frac{\partial}{\partial v} \left(v^{1-n} \frac{\partial}{\partial v} \right) + v^2 \frac{\partial}{\partial u} \frac{\partial}{\partial \bar{u}}, \quad (2.19)$$

³Note that $\frac{\partial}{\partial u} \frac{\partial}{\partial \bar{u}} |u|^2 = n$ in our conventions.

with $n = 1, 2, 4$ and 8 for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} , respectively.

As far as the geometry is concerned, the ‘generalized upper half planes’ $\mathcal{H}(\mathbb{A})$ are special examples of the general coset spaces

$$\mathcal{H}_n = \frac{\mathrm{SO}(1, n+1)}{\mathrm{SO}(n+1)} \quad (2.20)$$

for $n = 1, 2, 4, 8$. These are all hyperbolic spaces of constant negative curvature, which can be embedded as unit hyperboloids

$$\mathcal{H}_n = \{x^\mu \in \mathbb{R}^{1, n+1} \mid x^\mu x_\mu = -x^+ x^- + x \cdot x = -1; x^\pm > 0\} \quad (2.21)$$

into the forward lightcone of $(n+2)$ -dimensional Minkowski space $\mathbb{R}^{1, n+1}$, with $x \in \mathbb{R}^n$. These hyperboloids are isometric to the upper half planes introduced above by means of the mapping

$$x^- = \frac{1}{v}, \quad x^+ = v + \frac{|u|^2}{v}, \quad x = \frac{u}{v}. \quad (2.22)$$

where the last equation identifies the components $x^j = u^j/v$ in the Euclidean subspace \mathbb{R}^n . The line element (2.12) is the pull-back of the Minkowskian line element $ds^2 = -dx^+ dx^- + dx \cdot dx$. Similarly, it is straightforward to check that

$$\int d\mathrm{vol}(z)(\dots) = \int d^n x dx^+ dx^- \delta(x^+ x^- - \bar{x}x - 1)(\dots) \quad (2.23)$$

and to derive the Laplace-Beltrami operator (2.19) from the Klein-Gordon operator in $\mathbb{R}^{1, n+1}$. Consequently, the geodesic length (2.16), the volume element (2.18) and the Laplace-Beltrami operator (2.19) are left invariant under the isometry group $\mathrm{SO}(1, n+1)$ of the embedding space (and the unit hyperboloid). Because the even Weyl groups (or modular groups) to be considered below are all discrete subgroups of $\mathrm{SO}(1, n+1)$, these geometric objects are *a fortiori* invariant under these discrete groups as well. The main new feature here distinguishing the cases $n = 1, 2, 4, 8$ from the general case (2.21) is the link with the division algebras and their algebraic structure, which is evident in particular in the form of the modular transformations (3.17) below.

3 Weyl groups as modular groups

As vector spaces with a positive-definite inner product, the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ can be identified with the root spaces of finite-dimensional Kac-Moody algebras of rank $1, 2, 4, 8$, respectively. The root lattices are then identified with lattices in these algebras. In several cases of interest, the lattices close under multiplication and thus endow the root lattice with the structure of a (possibly non-associative) ring [2].

The fundamental Weyl reflection with respect to a simple root a is defined as a reflection in the hyperplane orthogonal to a ,

$$x \mapsto x - 2 \frac{(a, x)}{(a, a)} a. \quad (3.1)$$

When we identify the roots with elements in the division algebra \mathbb{A} this can be written

$$x \mapsto x - |a|^{-2}(a\bar{x} + x\bar{a})a = x - |a|^{-2}(a\bar{x}a + x|a|^2) = -\frac{a\bar{x}a}{|a|^2}, \quad (3.2)$$

where we have used the definition of the inner product, and the alternativity of \mathbb{A} . For simply-laced algebras (the only case we will consider in this paper), we normalize the simple roots to unit norm. In particular, if ε_i denote the so normalized simple roots of the finite algebra, then the associated finite Weyl group W_{fin} is generated by the fundamental Weyl reflections

$$x \mapsto w_i(x) = -\varepsilon_i \bar{x} \varepsilon_i, \quad (3.3)$$

for $x \in \mathbb{A}$ with i ranging over the rank of the algebra. We will always choose the orientation of the simple roots such that the highest root θ of the algebra is equal to the real unit, i.e., $\theta = 1$.

3.1 Hyperbolic overextensions

To any simple finite-dimensional Lie algebra (finite Kac-Moody algebra) of rank r one can associate an infinite-dimensional (indefinite) Kac-Moody algebra of rank $r + 2$ as follows. One first constructs the non-twisted affine extension, thereby increasing the rank by one. Then one adds an additional node with a single line to the affine node in the Dynkin diagram. The resulting algebras are often called ‘over-extended’ [20,21] and in many cases turn out to be hyperbolic Kac-Moody algebras.⁴ The Cartan-Killing metric on the Cartan subalgebra is always Lorentzian. We will number the affine and over-extended nodes by 0 and -1 , respectively. Denoting the finite-dimensional split real algebra by \mathfrak{g} , the associated over-extension will be denoted by \mathfrak{g}^{++} and we use the index $I = -1, 0, 1, \dots, r$ to denote the simple roots of \mathfrak{g}^{++} . The affine extension will be denoted by \mathfrak{g}^+ .

The infinite Weyl groups associated with these over-extended algebras were studied in [2]. The Weyl group acts on the Lorentzian vector space $\mathbb{A} \oplus \mathbb{R}^{1,1}$, *alias* the Minkowski space $\mathbb{R}^{1,n+1}$, consisting of real linear combinations of the simple roots. We can identify this vector space with the Jordan algebra⁵ $H_2(\mathbb{A})$ of Hermitean matrices

$$X := \begin{pmatrix} x^+ & x \\ \bar{x} & x^- \end{pmatrix} = X^\dagger \quad (3.4)$$

where $x^\pm \in \mathbb{R}$ and $x \in \mathbb{A}$. This vector space is Lorentzian with respect to the norm

$$\|X\|^2 := -\det X = -x^+ x^- + \bar{x}x \quad (3.5)$$

⁴But not always: For instance, the finite algebra A_8 extends to AE_{10} which is indefinite, but not hyperbolic.

⁵A Jordan algebra is a commutative (but possibly non-associative) algebra where any two elements X and Y satisfy $X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X$. This identity holds for $H_2(\mathbb{A})$ with \circ being the symmetrized matrix product. However, we will here only consider the Jordan algebras $H_2(\mathbb{A})$ as vector spaces, without using the Jordan algebra property.

and the associated bilinear form

$$(X, Y) = \frac{1}{2} (\|X + Y\|^2 - \|X\|^2 - \|Y\|^2). \quad (3.6)$$

This definition of the norm differs from the one in [2] by a factor of 1/2. We also define the simple roots by

$$\alpha_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \varepsilon_i \\ \bar{\varepsilon}_i & 0 \end{pmatrix}, \quad (3.7)$$

where $|\varepsilon_i| = 1$, so that they have unit length, $\|\alpha_I\|^2 = 1$, instead of length 2, which is the standard normalization for a simply-laced algebra. The reason for this is that we want the norm to coincide with the standard norm in the division algebra \mathbb{A} , as soon as we restrict the root space to the finite subalgebra.

The fundamental Weyl reflections with respect to the simple roots (3.7) are given by [2]

$$w_I : X \mapsto M_I \bar{X} M_I^\dagger \quad (3.8)$$

where

$$M_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & -\bar{\varepsilon}_i \end{pmatrix}, \quad (3.9)$$

and we denote the thus generated hyperbolic Weyl group by $W \equiv W_{\text{hyp}}$.

Since Weyl transformations preserve the norm, we can consider their action on elements of fixed norm, and in particular on those elements that lie on the unit hyperboloid $\|X\|^2 = -1$ inside the forward light-cone of the Lorentzian space. As we showed in the previous section, this unit hyperboloid is isometric to the generalized upper half plane $\mathcal{H}(\mathbb{A})$ via the isometric embedding (2.22). The lightcone $\|X\|^2 = 0$ in $H_2(\mathbb{A})$ then corresponds to (a double cover of) the boundary $\partial\mathcal{H}(\mathbb{A})$, consisting of the subspace \mathbb{A} (that is, $v = 0$) and the single point $z = i\infty$. The latter point is called ‘cusp at infinity’, and plays a special role because it is associated with the affine null root

$$\delta = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.10)$$

To see this more explicitly we note that the lines with constant u parallel to the imaginary axis in $\mathcal{H}(\mathbb{A})$ are obtained via (2.22) by projection of the null line

$$x^- = 1, \quad x^+ = t + |u|^2, \quad x = u \quad (3.11)$$

onto the unit hyperboloid inside the forward lightcone of $\mathbb{R}^{1,n+1}$. The tangent vector of this null line is then identified with (3.10) again via (2.22), and the boundary point on the unit hyperboloid corresponding to the cusp $z = i\infty$ is reached for $t \rightarrow \infty$. Choosing any other null direction inside the forward lightcone one can reach all other points in the boundary, that is \mathbb{A} with $v = 0$.

Because the Weyl transformations generated by the repeated action (3.8) preserve the norm $\|X\|^2$, they leave invariant the unit hyperboloid. As shown in [5,6] the projection of the fundamental reflections of the infinite Weyl group W onto the unit hyperboloid, and hence the generalized upper half plane, induce the following modular action on $z \in \mathcal{H}(\mathbb{A})$

$$w_{-1}(z) = \frac{1}{\bar{z}}, \quad w_0(z) = -\bar{z} + 1, \quad w_i(z) = -\varepsilon_i \bar{z} \varepsilon_i, \quad (3.12)$$

where ε_i are the (unit) simple roots of the underlying finite algebra as above. Furthermore, $1/\bar{z} = z/|z|^2$.

3.2 Even Weyl group

We are here mainly interested in the *even* part of the Weyl group, which is realized by ‘holomorphic’ transformations (see (3.17) below) and which we wish to interpret as a generalized modular group. The even Weyl group $W^+ \subset W$ consists of the words of even length in W . As the generators of the even Weyl group we take the transformations

$$s_I := w_I w_\theta \quad (3.13)$$

with $I = -1, 0$ or $I = i$, where w_θ is the reflection on the highest root $\theta = 1$. These generators act as

$$s_I : X \mapsto S_I X S_I^\dagger, \quad (3.14)$$

where

$$S_{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & \bar{\varepsilon}_i \end{pmatrix}. \quad (3.15)$$

On the upper half plane it follows from (3.3) (with $\varepsilon = \theta = 1$) that

$$w_\theta(z) = -\bar{z}, \quad (3.16)$$

and the generators (3.13) act as

$$\boxed{s_{-1}(z) = -\frac{1}{z}, \quad s_0(z) = z + 1, \quad s_i(z) = \varepsilon_i z \varepsilon_i.} \quad (3.17)$$

These transformations are ‘holomorphic’ on the upper half plane and generalize the well-known expression for modular transformations of the two-dimensional upper half plane $\mathcal{H}(\mathbb{R})$ under $\text{PSL}(2, \mathbb{Z})$.

In [2] $\tilde{s}_0 = w_{-1} w_0$ and $\tilde{s}_i = w_{-1} w_i$ were used as generating elements. Our choice here is more convenient since it refers to the universal element $\theta = 1$, but unlike the one in [2], the generating set (3.13) is not minimal. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ this redundancy is expressed in the extra relations

$$s_1 = 1 \text{ for } \mathbb{A} = \mathbb{R}, \quad s_1 s_2 = 1 \text{ for } \mathbb{A} = \mathbb{C}, \quad s_1 s_3 s_4 = 1 \text{ for } \mathbb{A} = \mathbb{H} \quad (3.18)$$

with appropriate numbering of the roots, and always in the basis where $\theta = 1$. For the octonions the relevant relation is more tricky because of non-associativity.

Quite generally, the even Weyl groups of the over-extended algebras considered in [2] are of the form

$$W^+ \equiv W_{\text{hyp}}^+ \cong \text{PSL}(2, \mathcal{O}), \quad (3.19)$$

where $\mathcal{O} \subset \mathbb{A}$ is a suitable set of algebraic integers in \mathbb{A} . The roots of the associated finite Kac-Moody algebra then correspond to the units of \mathcal{O} (a unit being an invertible element of the ring). In some cases one has to consider finite extensions of this group or finite quotients (depending on whether the algebra is not simply-laced or has diagram automorphisms); a detailed analysis of all cases can be found in [2]. The even Weyl group W^+ is a normal subgroup of index two in the full Weyl group and for many purposes it is sufficient to study it. In the following sections, we will focus on two distinguished cases, one related to D_4 and the quaternions (where $\mathcal{O} = \mathbb{H}$, the ring of ‘Hurwitz numbers’) and one related to E_8 and the octonions (where $\mathcal{O} = \mathbb{O}$, the ‘octavians’).

The Weyl transformations (3.12) define a fundamental domain $\mathcal{F}_0 \subset \mathcal{H}(\mathbb{A})$, which is the image of the fundamental Weyl chamber in $H_2(\mathbb{A})$ under the projection (2.22). Likewise, the *even* Weyl transformations (3.17) define a fundamental domain \mathcal{F} , which contains two copies of \mathcal{F}_0 , and a ‘projection’ K of \mathcal{F} onto \mathbb{A} ; see Figure 1 below for an illustration. As we already explained, the cusp $z = i\infty$ is the image of the boundary point of the unit hyperboloid which is reached by following and projecting any null ray inside the forward lightcone along the affine null root δ .

From (2.18) and the compactness of K it is immediately obvious that

$$\int_{\mathcal{F}} d\text{vol}(z) < \infty. \quad (3.20)$$

The finite volume of the fundamental domain is in accord with the hyperbolicity of the underlying Kac-Moody algebra [21].

For all $\mathcal{H}(\mathbb{A})$, the geodesic length (2.16) is invariant under the continuous isometry group $\text{SO}(1, n+1)$ and therefore also under the generating elements of the Weyl group (3.12), that is under the full Weyl group, as can also be verified explicitly for the generating Weyl reflections (3.12). For instance, for the inversion we get

$$w_{-1} : |z_1 - z_2| \rightarrow |\bar{z}_2^{-1} - \bar{z}_1^{-1}| = (|z_1||z_2|)^{-1}|z_1 - z_2| \quad (3.21)$$

using alternativity. By iteration it then follows that

$$d(z_1, z_2) = d(w(z_1), w(z_2)) \quad (3.22)$$

for all elements w of the Weyl group $w \in W_{\text{hyp}}$. Similarly, for all $w \in W_{\text{hyp}}$ the hyperbolic volume element is left invariant:

$$d\text{vol}(z) = d\text{vol}(w(z)) \quad (3.23)$$

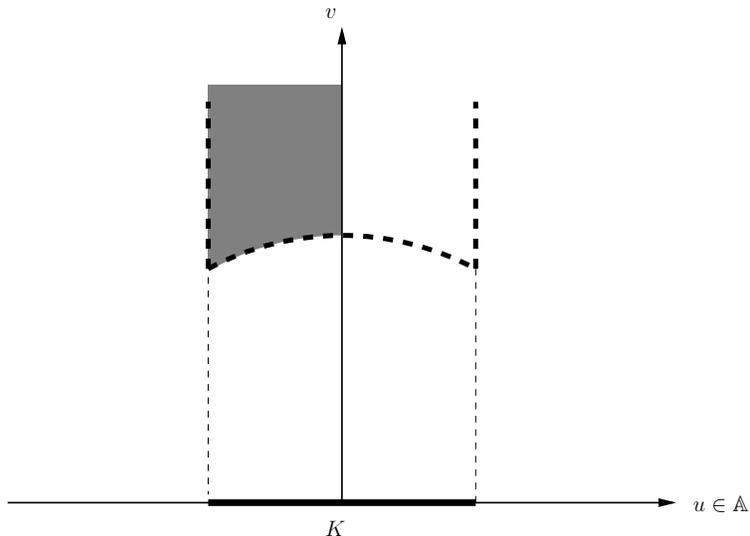


Figure 1: Sketch of the fundamental domains \mathcal{F}_0 (in grey) and \mathcal{F} (within heavy dashed lines). The shape of \mathcal{F} is always that of a ‘skyscraper’ over a compact domain $K \subset \mathbb{A}$ whose ‘bottom’ has been cut off, and that extends to infinite height in v .

Finally, the Laplace-Beltrami operator is also invariant in the following sense. Setting $z' := w(z)$ and denoting by Δ'_{LB} the Laplace-Beltrami operator with respect to the primed coordinates z' we have for any function $f(z)$ and for all $w \in W_{\text{hyp}}$

$$\Delta'_{\text{LB}} f(z') = \Delta_{\text{LB}} f(w(z)) . \quad (3.24)$$

3.3 Affine Weyl group

The hyperbolic Weyl group has a natural subgroup associated with the affine subalgebra, corresponding to the embedding $\mathfrak{g}^+ \subset \mathfrak{g}^{++}$. We denote this subgroup by $W_{\text{aff}} \equiv W(\mathfrak{g}^+)$. It is known [22] that it has the structure

$$W_{\text{aff}} = W_{\text{fin}} \ltimes \mathcal{O}, \quad (3.25)$$

where \mathcal{O} is the integer domain corresponding to the root lattice of the finite-dimensional algebra \mathfrak{g} . In (3.25), \mathcal{O} is to be thought of as a free abelian group of translations on the finite root lattice on which the finite Weyl group W_{fin} acts. Written in matrix form the translations take the form

$$T_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad (3.26)$$

and obey $T_x T_y = T_{x+y}$ for all $x, y \in \mathcal{O}$. In the associative cases one can write the finite Weyl transformations as matrices as

$$M_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (3.27)$$

for units $a, b \in \mathcal{O}$, possibly with restrictions depending on \mathfrak{g} .⁶ It is easy to see that the transformations leaving invariant the null root (3.10) are exactly the T_x and the $M_{a,b}$. This is to be expected as the affine Weyl group can be defined as the stabilizer of the affine null root [16].

4 D_4 and its hyperbolic extension

Now we specialize to $\mathbb{A} = \mathbb{H}$ in which case the relevant ring of algebraic integers is $\mathcal{O} = \mathbb{H}$, the ‘Hurwitz numbers’, to be defined below. The results of this section generalize many well known results for the usual upper half plane ($\mathbb{A} = \mathbb{R}$). Since they contain similar results for $\mathbb{A} = \mathbb{C}$ we will not specially consider this case (which has been dealt with in the literature [23]). Our main task will be to find closed formulas for arbitrary elements of the group $\mathrm{PSL}(2, \mathbb{H})$ and its modular realization on $\mathcal{H}(\mathbb{H})$. In comparison to the commutative cases $\mathbb{A} = \mathbb{R}, \mathbb{C}$ this case presents new features because of the non-commutativity of the matrix entries. However, a direct generalization to $\mathbb{A} = \mathbb{O}$ is not possible because of the non-associativity of the matrix entries; this case will be treated separately in section 6.

4.1 D_4 and the integer quaternions

The quaternions are obtained by applying the Cayley-Dickson doubling procedure described in section 2.1 to $\mathbb{A} = \mathbb{C}$. An arbitrary quaternion $x \in \mathbb{H}$ is written as

$$x = x_0 + x_1 e_1 + x_2 e_5 + x_3 e_6 \quad (4.1)$$

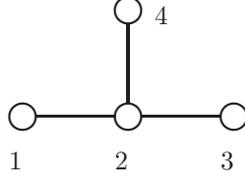
where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and we have the usual rules of quaternionic multiplication $e_1 e_5 = -e_5 e_1 = e_6$, etc. for the imaginary quaternionic units e_1, e_5 and e_6 .⁷ The norm is $|x|^2 \equiv x \bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

We are here interested in quaternionic integers. The *Hurwitz integers* (or just *Hurwitz numbers*) \mathbb{H} are those quaternions (4.1) for which the coefficients x_0, x_1, x_2, x_3 are either all integers or all half-integers. They constitute a lattice $\mathbb{H} \subset \mathbb{H}$ and form a non-commutative ring of integers (actually, a ‘maximal order’)

⁶We will encounter an explicit example of such restrictions in the case $\mathfrak{g} = D_4$ below.

⁷The imaginary units are conventionally designated as i, j, k but here we prefer to use $e_1 \equiv i, e_5 \equiv j$ and $e_6 \equiv k$ because we want to reserve the letter i for the new imaginary unit used in the Cayley-Dickson doubling. For the doubling $\mathbb{H} = \mathbb{C} \oplus i\mathbb{C}$ we thus identify $i \equiv e_5$ and $x = (x_0 + x_1 e_1) + i(x_2 - x_3 e_1)$. Our notation conforms with the one used in section 6 for the octonions, such that the obvious D_4 roots inside E_8 are simply obtained from (4.2) by multiplication with E_7 .

inside \mathbb{H} . The key feature is that \mathbb{H} at the same time can be identified with the root lattice of the algebra $D_4 \equiv \mathfrak{so}(4, 4)$. Indeed, the simple roots of D_4 , labeled according to the Dynkin diagram



can be identified with the following set of *Hurwitz units*:

$$\varepsilon_1 = e_1, \quad \varepsilon_2 = \frac{1}{2}(1 - e_1 - e_5 - e_6), \quad \varepsilon_3 = e_5, \quad \varepsilon_4 = e_6. \quad (4.2)$$

The D_4 root lattice is spanned by integer linear combinations of these simple roots, and with the above choice of basis, the highest root $\theta = \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4$ is indeed equal to 1. The new feature is that such combinations also close under multiplication, thereby endowing the D_4 root lattice with the structure of a non-commutative ring. The 24 roots of D_4 are then identified with the 24 *units* in \mathbb{H} . Comparison with (3.17) shows that we now have the extra relation $s_1 s_3 s_4 = 1$ for the generating set (3.13), see (3.18).

The elements in $W^+(D_4)$ are the words of even length where the letters are the fundamental Weyl reflections. For example, those of length two act as

$$x \mapsto \varepsilon_i(\overline{\varepsilon_j x \varepsilon_j})\varepsilon_i = \varepsilon_i(\bar{\varepsilon}_j x \bar{\varepsilon}_j)\varepsilon_i = (\varepsilon_i \bar{\varepsilon}_j)x(\bar{\varepsilon}_j \varepsilon_i). \quad (4.3)$$

It follows [2, 19] that any element in $W^+(D_4)$ has the form of a combined left and right multiplication,

$$x \mapsto axb, \quad (4.4)$$

where a and b are (in general different) Hurwitz units, subject to the constraint that the product ab is an element of the quaternionic group

$$ab \in \mathcal{Q} \equiv \{\pm 1, \pm e_1, \pm e_5, \pm e_6\}. \quad (4.5)$$

The restriction to the quaternionic group arises from triality of the D_4 root system as the outer triality automorphisms are not elements of the Weyl group. In [2] this result was generalized to the hyperbolic extension $W(D_4^{++})$ in a way that we will describe next.

4.2 The even hyperbolic Weyl group $W^+(D_4^{++})$

Consider transformations acting on $X \in H_2(\mathbb{H})$ by

$$X \rightarrow s(X) = SXS^\dagger \quad (4.6)$$

with

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{H}. \quad (4.7)$$

(Of course, these formulas trivially specialize to $\mathbb{A} = \mathbb{R}, \mathbb{C}$.) Such transformations preserve the norm of X if and only if

$$\det(SS^\dagger) = 1. \quad (4.8)$$

This is true for all the normed division algebras up to and including the quaternions.⁸ The associated continuous group of matrices will be denoted by

$$SL(2, \mathbb{H}) = \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{H}, \quad \det(SS^\dagger) = 1 \right\}. \quad (4.9)$$

Its projective version

$$\mathrm{PSL}(2, \mathbb{H}) := SL(2, \mathbb{H}) / \{\mathbf{1}, -\mathbf{1}\} \quad (4.10)$$

is isomorphic to $\mathrm{SO}_0(1, 5)$, see [18, 24]. Let us note the explicit expressions

$$\begin{aligned} \det(SS^\dagger) &= |a|^2|d|^2 + |b|^2|c|^2 - 2 \operatorname{Re}(a\bar{c}d\bar{b}) \\ &= |ad - bc|^2 - 2 \operatorname{Re}(a[\bar{c}, d]\bar{b}) \end{aligned} \quad (4.11)$$

and the explicit form of the (left and right) inverse matrix for $S \in SL(2, \mathbb{H})$

$$S^{-1} = \begin{pmatrix} |d|^2\bar{a} - \bar{c}d\bar{b} & |b|^2\bar{c} - \bar{a}b\bar{d} \\ |c|^2\bar{b} - \bar{d}c\bar{a} & |a|^2\bar{d} - \bar{b}a\bar{c} \end{pmatrix}. \quad (4.12)$$

The discrete groups $SL(2, \mathbb{H}) \subset SL(2, \mathbb{H})$ and $\mathrm{PSL}(2, \mathbb{H}) \subset \mathrm{PSL}(2, \mathbb{H})$ are obtained from the above groups by restricting the entries a, b, c, d to be Hurwitz integers. From (4.11) it then follows that in order for S to be an element of $SL(2, \mathbb{H})$ or $\mathrm{PSL}(2, \mathbb{H})$, the pairs (a, b) and (c, d) must each be *left coprime*, and the pairs (a, c) and (b, d) must each be *right coprime* (that is, share no common left or right factor $g \in \mathbb{H}$ with $|g| > 1$). In fact, these conditions are not only necessary, but also sufficient (for some choice of the remaining two entries), as we show in appendix A.

Straightforward computation using (3.4) and (4.6) gives the transformations of the various components of X

$$\begin{aligned} s(x^+) &= \bar{a}a x^+ + \bar{b}b x^- + ax\bar{b} + b\bar{x}a, \\ s(x) &= a\bar{c}x^+ + b\bar{d}x^- + ax\bar{d} + b\bar{x}c, \\ s(x^-) &= \bar{c}c x^+ + \bar{d}d x^- + cx\bar{d} + d\bar{x}c. \end{aligned} \quad (4.13)$$

This formula is, of course, also valid for $\mathbb{A} = \mathbb{R}, \mathbb{C}$. Likewise, it holds for general quaternionic matrices S as long as these are invertible.

⁸For real and complex numbers the determinant can be factorized to give $|\det(S)| = 1$. Making use of commutativity then allows the reduction to $\det(S) = 1$ in the projective version.

The Weyl groups of the hyperbolic over-extensions of rank four algebras are discrete subgroups of $\mathrm{PSL}(2, \mathbb{H})$ (see [2] for details). For the hyperbolic over-extension D_4^{++} of D_4 one finds

$$W^+(D_4^{++}) \cong \mathrm{PSL}^{(0)}(2, \mathbb{H}) := \{S \in \mathrm{PSL}(2, \mathbb{H}) \mid ad - bc \equiv 1 \pmod{\mathbb{C}}\} \quad (4.14)$$

where \mathbb{C} is the (two-sided) commutator ideal $\mathbb{C} = \mathbb{H}[\mathbb{H}, \mathbb{H}]\mathbb{H}$ introduced in [2]. In other words, the Weyl group of D_4^{++} is not simply $\mathrm{PSL}(2, \mathbb{H})$: the elements S in $\mathrm{PSL}^{(0)}(2, \mathbb{H})$ must satisfy the additional constraint $ad - bc \equiv 1$ modulo \mathbb{C} , which is an index 4 sublattice in \mathbb{H} . As shown in [2], $\mathrm{PSL}^{(0)}(2, \mathbb{H})$ is an index 3 subgroup of $\mathrm{PSL}(2, \mathbb{H})$: if one extends the Weyl group by outer automorphisms related to the diagram automorphisms (‘trianality’) one obtains all of $\mathrm{PSL}(2, \mathbb{H})$, corresponding to all even symmetries of the D_4^{++} root lattice written in quaternionic coordinates.

Below, we will encounter two important subgroups of $\mathrm{PSL}^{(0)}(2, \mathbb{H})$. One is the (even) affine subgroup $W_{\mathrm{aff}}^+ \subset W^+(D_4^{++})$, which is the semi-direct product $W_{\mathrm{fin}} \times \mathcal{T}$ of the finite Weyl group $W_{\mathrm{fin}} = W^+(D_4)$ and the abelian group \mathcal{T} of affine translations. This is a maximal parabolic subgroup of $W^+(D_4^{++})$ whose elements in the quaternionic matrix representation are given by

$$S = \begin{pmatrix} a & u \\ 0 & b \end{pmatrix}, \quad (4.15)$$

where a, b are unit quaternions with $ab \in \mathcal{Q}$ and $u \in \mathbb{H}$ is an arbitrary Hurwitz integer. The other important subgroup is the translation subgroup \mathcal{T} itself, consisting of the matrices

$$T_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad (4.16)$$

where again $u \in \mathbb{H}$ is an arbitrary Hurwitz number. The following little lemma will be useful in section 5.4.

Lemma 4.1 *The left coprime pairs $(c, d) \in \mathbb{H}^2$ parametrize the coset spaces $W^+(D_4^+) \backslash W^+(D_4^{++}) \cong W_{\mathrm{aff}}^+ \backslash W_{\mathrm{hyp}}^+$. The equivalence classes $(c, d) \sim (ec, ed)$ for a unit $e \in \mathbb{H}$ uniquely parametrize the cosets.*

Proof. Suppose that the left coprime lower entries (c, d) are given and that a, b and \tilde{a}, \tilde{b} are two different pairs of Hurwitz numbers completing c, d to two different matrices S and \tilde{S} in $\mathrm{PSL}^{(0)}(2, \mathbb{H})$. Then an easy computation using (4.12) shows that

$$\tilde{S}S^{-1} = \begin{pmatrix} q & * \\ 0 & 1 \end{pmatrix}, \quad (4.17)$$

where $q \in \mathcal{Q}$ is in the quaternionic group (4.5). Hence all such matrices are related by an upper triangular element of $\mathrm{PSL}^{(0)}(2, \mathbb{H})$ of type (4.15), that is,

an element of W_{aff}^+ . Since (4.15) also allows for left multiplication of the pair (c, d) by a Hurwitz unit we arrive at the claim. \square

In the following sections we will be concerned with the generalization of the standard theory of automorphic functions to $\mathcal{H}(\mathbb{H})$ and $\mathcal{H}(\mathbb{O})$. For this reason we will also make use of the established identification (1.1) for the relevant modular group and its subgroup stabilizing the cusp at infinity.

5 Automorphic functions on $\mathcal{H}(\mathbb{H})$

In this section, we take first steps towards developing a theory of automorphic functions and Maass wave forms on $\mathcal{H}(\mathbb{H})$. This should be viewed as a preparation for the development of a similar theory on $\mathcal{H}(\mathbb{O})$ which is our main goal.

5.1 Action of $\text{PSL}^{(0)}(2, \mathbb{H})$ on $\mathcal{H}(\mathbb{H})$

We now wish to interpret the even Weyl group $W^+(\mathbb{D}_4^{++}) \equiv \text{PSL}^{(0)}(2, \mathbb{H})$ as a modular group Γ acting on the quaternionic upper half plane $\mathcal{H}(\mathbb{H})$. To exhibit the nonlinear ‘modular’ action of Γ we map the forward unit hyperboloid $\|X\|^2 = -1$ in $\mathbb{R}^{1,5}$ to the upper half plane $\mathcal{H}(\mathbb{H})$ by means of the projection (2.22). Accordingly, we consider (2.9), but now specialize to $u \in \mathbb{H}$. Using the formulas of the previous section, in particular (2.22) and (4.13), we obtain

$$v' = \frac{v}{D} \quad , \quad u' = \frac{1}{D} \left[(au + b)(\bar{u}\bar{c} + \bar{d}) + a\bar{c}v^2 \right] \quad (5.1)$$

with $z' = u' + iv' \equiv s(z) \equiv s(u + iv)$, where

$$D \equiv D(S, u, v) := |cu + d|^2 + |c|^2v^2 = |cz + d|^2 \quad (5.2)$$

and (2.4) has been used. Observe that now $cz + d \in \mathbb{O}$. The transformations (5.1) can be combined into a single formula

$$z' = \frac{(au + b)(\bar{u}\bar{c} + \bar{d}) + a\bar{c}v^2 + iv}{|cz + d|^2} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}^{(0)}(2, \mathbb{H}), \quad (5.3)$$

which is our expression for the most general quaternionic modular transformation. Here z' is manifestly in the upper half plane $\mathcal{H}(\mathbb{H})$, and also reduces to the standard formula $z' = (az + b)(cz + d)^{-1}$ for the commutative cases $u \in \mathbb{R}$ (or \mathbb{C}) and $a, b, c, d \in \mathbb{R}$ (or \mathbb{C}). However, (5.3) fails for $\mathbb{A} = \mathbb{O}$ because of non-associativity. One can check explicitly that the transformations (5.1) and the identifications (2.22) are consistent with (4.13).

As a special case we have the modular transformations (4.15) corresponding to the even affine Weyl group. The induced modular action of the most general element $\gamma \in \Gamma_\infty \equiv W_{\text{aff}}^+ \equiv W_{\text{fin}}^+ \times \mathbb{H}$ is

$$\gamma : z \rightarrow \gamma(z) = aub + y\bar{b} + iv \quad (5.4)$$

with units a, b such that $ab \in \mathcal{Q}$ (cf. (4.5)) and $y \in \mathbb{H}$ an arbitrary Hurwitz integer. This transformation corresponds to a finite (even) Weyl transformation followed by a constant shift along the root lattice \mathbb{H} . In particular, it follows that such transformations leave invariant the ‘cusp’ $z = i\infty$ in $\mathcal{H}(\mathbb{H})$ — in analogy with the action of the usual shift matrix T for the real modular group $\mathrm{PSL}(2, \mathbb{Z})$ (in that case, W_{fin}^+ is trivial). As we explained, this is in accord with the fact that the affine Weyl group W_{aff} is the subgroup of W_{hyp} leaving invariant the affine null root δ [16].

5.2 Geodesics in $\mathcal{H}(\mathbb{H})$ and periodic orbits

Geodesics in $\mathcal{H}(\mathbb{H})$ are given by half-circles and straight lines, as already explained. For the usual complex upper half plane it is known (see e.g. [25]) that we can associate to each hyperbolic element⁹ $M \in \mathrm{PSL}(2, \mathbb{R})$ a geodesic that is mapped onto itself by the action of M (the same is true for all matrices conjugate to M). Let us consider the ‘imaginary axis’ in $\mathcal{H}(\mathbb{H})$, parametrized by $z(t) = it$ with $0 < t < \infty$. For this geodesic, the hyperbolic motion leaving it invariant is obviously given by $\gamma_t = \mathrm{diag}(t^{1/2}, t^{-1/2})$, whereby i is mapped to $\gamma_t(i) = \gamma_t(z(1)) = it$. It is straightforward to see that all geodesics in $\mathcal{H}(\mathbb{H})$ are $\mathrm{PSL}(2, \mathbb{H})$ images of one another. First, straight line geodesics centered at $u \neq 0$ are trivially obtained by acting on $z(t) = it$ with the shift matrix

$$T_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}. \quad (5.5)$$

For arbitrary $u_1 \neq u_2 \in \mathbb{H}$ the circular geodesic with endpoints u_1 and u_2 can (for example) be obtained by acting on $z(t)$ with the matrix

$$S_{u_1, u_2} = \frac{1}{\sqrt{|u_1 - u_2|}} \begin{pmatrix} u_2 & u_1 \\ 1 & 1 \end{pmatrix}. \quad (5.6)$$

Use of (5.3) results in the explicit parametrization of the geodesic half-circle

$$z'(t) = \frac{u_1 + u_2 t^2 + it|u_1 - u_2|}{1 + t^2} \quad (0 < t < \infty). \quad (5.7)$$

The hyperbolic motions leaving invariant this geodesic half-circle are now simply given by (for all t)

$$M_t = S \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} S^{-1}, \quad (5.8)$$

where S is either T_u or S_{u_1, u_2} . The formula expressing M_t explicitly in terms of t, u_1, u_2 follows directly from (5.6) and (4.12) but is not very illuminating. However, it implies the inequality

$$\mathrm{Re}(\mathrm{Tr} M_t) = t^{1/2} + t^{-1/2} \geq 2. \quad (5.9)$$

⁹Recall that a hyperbolic element $M \in \mathrm{SL}(2, \mathbb{R})$ has reciprocal real eigenvalues or, equivalently, satisfies $|\mathrm{Tr} M| > 2$.

Note that the real part of the trace is cyclic even over quaternionic matrices and hence furnishes an invariant for matrices up to conjugation. In other words, we may adopt (5.9) as the quaternionic generalization of the usual condition of hyperbolicity for an $SL(2, \mathbb{R})$ matrix.

Consider now the case when $M \equiv M_{t_0}$ happens to be integral for some value $t_0 > 1$, i.e., $M_{t_0} \in \mathrm{PSL}^{(0)}(2, \mathbb{H})$. Then the associated geodesic gives rise to a *periodic orbit in the fundamental domain* \mathcal{F} of the modular group $\mathrm{PSL}^{(0)}(2, \mathbb{H})$. This is most easily seen by following the geodesic circle successively through the images of the fundamental domain \mathcal{F} until we reach the image of \mathcal{F} under the special transformation M , where we connect up again to the original geodesic curve intersecting \mathcal{F} . (The integrality condition $M \in \mathrm{PSL}^{(0)}(2, \mathbb{H})$ may be viewed as the analog of the rationality condition for periodic orbits on tori.) The length ℓ_p of this periodic orbit is easily calculated by mapping it back to the imaginary axis, whence

$$\ell_p = \int_1^{t_0} \frac{dv}{v} = \log t_0 \quad (5.10)$$

by the invariance of the geodesic length under the modular group. With (5.9) we recover the quaternionic analog of the well known formula for $\mathrm{PSL}(2, \mathbb{Z})$ [25]

$$2 \cosh \frac{\ell_p}{2} = |\mathrm{Re}(\mathrm{Tr} M)|. \quad (5.11)$$

The limiting case $\ell_p \rightarrow \infty$ corresponds to the infinite geodesic along the imaginary axis. Consequently, the number of periodic orbits in the fundamental domain \mathcal{F} increases exponentially with the geodesic length of the orbit, as it does for $\mathrm{PSL}(2, \mathbb{Z})$ (see e.g. [25]).

5.3 Maass wave forms

The definitions of this subsection apply to all the normed division algebras (including $\mathbb{A} = \mathbb{O}$). Following [3, 17] we define the scalar product between two functions $f, g : \mathcal{H}(\mathbb{A}) \rightarrow \mathbb{C}$ by means of the invariant measure (2.18)

$$(f, g) := \int_{\mathcal{F}} \overline{f(z)} g(z) d\mathrm{vol}(z). \quad (5.12)$$

A *Maass wave form of type s* with respect to the modular group Γ acting on $\mathcal{H}(\mathbb{A})$ is then defined to be a non-zero complex function $f \in L^2(\mathcal{F}, \mathbb{C})$ on $\mathcal{F} \subset \mathcal{H}(\mathbb{A})$ obeying

- $f(z) = f(\gamma(z))$ for $\gamma \in \Gamma$,
- $(-\Delta_{\mathrm{LB}})f(z) = s(n - s)f(z)$,

where $n = \dim_{\mathbb{R}} \mathbb{A}$ as always, and $s \in \mathbb{C}$. If we also demand

- $\int_K f(u + iv) d^n u = 0$ for all $v > 0$,

where K is the ‘projection’ of \mathcal{F} onto \mathbb{A} , cf. Figure 1, we would obtain so-called ‘cusp forms’, i.e. automorphic functions that vanish at the cusp at infinity. However, it will be sufficient for our purposes here to require the functions to be Fourier expandable around $v = \infty$.

Among the Maass wave forms one can introduce a further distinction according to whether they are even or odd. Because $\Gamma \equiv W_{\text{hyp}}^+$ is an index 2 subgroup in the full hyperbolic Weyl group W_{hyp} we have two possibilities for extending the group action, namely

$$f(w(z)) = \begin{cases} f(z) \\ \det(w) f(z) \end{cases}, \quad w \in W_{\text{hyp}}. \quad (5.13)$$

Taking the special reflection $w_\theta \in W_{\text{hyp}}$ as a reference (see (3.16)), we see that these definitions respectively are equivalent to

$$f(-\bar{z}) = \begin{cases} +f(z) \\ -f(z) \end{cases} \quad (5.14)$$

Maass wave forms obeying the first (second) condition are referred to as *even* (*odd*) Maass wave forms. Employing the generating Weyl reflections (3.12) it is straightforward to see that odd Maass wave forms vanish on $\partial\mathcal{F}_0$, hence obey Dirichlet boundary conditions on \mathcal{F}_0 (that is, on all faces of the fundamental Weyl chamber), while even Maass wave forms have vanishing normal derivatives on $\partial\mathcal{F}_0$, hence obey Neumann boundary conditions. If one interprets f as a quantum mechanical wave function, both possibilities are compatible with the rules of quantum mechanics because only scalar products of wave functions (for which the sign factors cancel) are observable [5–7].

A main goal of the theory is to analyse the spectral decomposition of the Laplace-Beltrami operator on $L^2(\mathcal{F}, \mathbb{C})$ [3, 17]. As it turns out, for the discrete eigenvalues the associated eigenfunctions must be determined numerically (see e.g. [23] for results on $\mathbb{A} = \mathbb{C}$). For odd Maass wave forms the spectrum is purely discrete, and one can establish the bound [5, 6]¹⁰

$$-\Delta_{\text{LB}} \geq \frac{n^2}{4} \quad (5.15)$$

generalizing an argument of [17]. For even Maass wave forms one has in addition to the discrete spectrum a continuous spectrum that can be constructed by means of Eisenstein series and extends from $n^2/4$ over the positive real axis (see below).

5.4 Eisenstein series

We recall that the Laplacian on hyperbolic space $\mathcal{H}(\mathbb{A})$ of dimension $n + 1$ is given by (2.19) again with $n = \dim_{\mathbb{R}} \mathbb{A}$. For $I_s(z) = v^s$ or $I_{n-s}(z) = v^{n-s}$ with

¹⁰In fact, for $SL(2, \mathbb{Z})$ Maass wave forms a better bound is $E \geq 3\pi^2/2$ (see [3], p. 71), and we expect such improved lower bounds also to exist for the other normed division algebras \mathbb{A} .

$s \in \mathbb{C}$ we compute

$$-\Delta_{\text{LB}} I_s(z) = E_s I_s(z) = s(n-s) I_s(z). \quad (5.16)$$

Note that for real values of s we have $E_s > 0$ if and only if $0 < s < n$.

We are here interested in functions that are both eigenfunctions of the Laplacian Δ_{LB} and automorphic (but not necessarily square integrable). To turn the function I_s into an automorphic form, we have to make it symmetric under modular transformation by ‘averaging’ over the modular group $\Gamma \equiv W_{\text{hyp}}^+$. Abstractly, this is achieved by defining the *Poincaré series* (or restricted Eisenstein series)

$$\mathcal{P}_s(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} I_s(\gamma(z)). \quad (5.17)$$

Whenever this sum converges it is an eigenfunction of the Laplace-Beltrami operator by virtue of the invariance property (3.24) which implies that every term in the sum is separately an eigenfunction. We emphasize again that the above definition, though standard [3, 17], is special inasmuch all our ‘modular groups’ Γ are hyperbolic Weyl groups, with the corresponding affine Weyl subgroups Γ_∞ as the stabilizer groups of the cusp at infinity. The above definition also works for E_{10} where so far no simple realization of $W^+(E_{10})$ in terms of octavian matrices is available (but see following section). Because the (even part of the) affine Weyl group stabilizes the affine null root (*alias* the cusp), the associated sum over cosets $\gamma \in W_{\text{aff}}^+ \backslash W_{\text{hyp}}^+$ records only different images of v , and thereby avoids overcounting, as required for the sum to be well defined. Furthermore $I_s(w_\theta(z)) = I_s(-\bar{z}) = +I_s(z)$, so we conclude that $\mathcal{P}_s(z)$ satisfies Neumann boundary conditions on \mathcal{F}_0 .

We now specialize again to $\mathbb{A} = \mathbb{H}$ with the aim of rewriting the sum (5.17) in a form more familiar from complex analysis. To this aim we restrict attention to $\mathbb{A} = \mathbb{H}$. Using (5.3) we can re-express (5.17) in the form

$$\mathcal{P}_s(z) := \sum_{(c,d)_{L=1}} \frac{v^s}{|cz+d|^{2s}} \quad (5.18)$$

for $z = u + iv \in \mathcal{H}(\mathbb{H})$ and $s \in \mathbb{C}$, where the sum is restricted to run only over left coprime pairs (c, d) with greatest common left divisor a unit. Indeed, the Lemma in section 4.2. guarantees that the left coprime pairs (c, d) up to left multiplication by Hurwitz units uniquely characterize the cosets $\Gamma_\infty \backslash \Gamma$, and therefore the sum (5.18) includes precisely all independent matrices $S \in \Gamma = \text{PSL}^{(0)}(2, \mathbb{H})$ modulo matrices in Γ_∞ (an explicit construction of a representative matrix for each pair (c, d) by means of the Euclidean algorithm is given in appendix A).

Thanks to (3.24) we can alternatively define the *unrestricted* Eisenstein series

$$\mathcal{E}_s(z) := \sum_{(c,d) \in \mathbb{H}^2 \setminus \{(0,0)\}} \frac{v^s}{|cz+d|^{2s}}. \quad (5.19)$$

This Eisenstein series is related to (5.18) by

$$\mathcal{E}_s(z) = \zeta_{\mathbb{H}}(s)\mathcal{P}_s(z), \quad (5.20)$$

where

$$\zeta_{\mathbb{H}}(s) = \sum_{0 \neq a \in \mathbb{H}} |a|^{-s} = \sum_{n \in \mathbb{N}} \frac{\sigma_{\mathbb{H}}(n)}{n^s} \quad (5.21)$$

is the (Dedekind) zeta function associated with the Hurwitz numbers, and $\sigma_{\mathbb{H}}(n)$ counts the number of roots of (squared) length $2n$ on the D_4 root lattice (which is always a multiple of 24).

To determine the domain of convergence of the Eisenstein series (5.17) in the complex s -plane, we note that both (5.17) and (5.18) are majorized by

$$|\mathcal{P}_s(z)| \text{ or } |\mathcal{E}_s(z)| \leq \sum_{(c,d) \in \mathbb{H}^2 \setminus \{(0,0)\}} \frac{v^\sigma}{|cz + d|^{2\sigma}} \quad (5.22)$$

where $\sigma \equiv \operatorname{Re} s$. For any fixed $|c| < N$, the sum over d converges for $\sigma > 2$ by standard estimates. Taking N sufficiently large the remainder of the sum can be majorized by means of the approximation

$$\begin{aligned} & \left(\sum_{|c| > N} \frac{1}{|c|^{2\sigma-4}} \right) \left(\frac{1}{|c|^4} \sum_{d \in \mathbb{H}} \frac{1}{(|u + c^{-1}d|^2 + v^2)^\sigma} \right) \sim \\ & \sim \left(\sum_{|c| > N} \frac{1}{|c|^{2\sigma-4}} \right) \int_{\mathbb{R}^4} \frac{d^4 y}{(|u + y|^2 + v^2)^\sigma} \end{aligned} \quad (5.23)$$

which shows that the sum (5.17) converges for $\sigma > 4$ (a similar argument shows that $\mathbb{A} = \mathbb{C}$ requires $\operatorname{Re} s > 2$ for convergence). For other values in the complex s -plane, $\mathcal{P}_s(z)$ and $\mathcal{E}_s(z)$ are defined by analytic continuation in s .

We expect that with an appropriately defined completed zeta function $\xi_{\mathbb{H}}(s)$ the following functional relation holds¹¹

$$\xi_{\mathbb{H}}(s)\mathcal{P}_s(z) = \xi_{\mathbb{H}}(n-s)\mathcal{P}_{n-s}(z), \quad (5.24)$$

where n is the dimension of the division algebra and this functional relation should be related to the ones studied in the real rank one case in [26].

Because the sum (5.17) (or equivalently (5.18)) is periodic under shifts of $u \equiv \operatorname{Re} z$ by elements of the lattice of integers, it can be expanded into a Fourier

¹¹For example, the completed Riemann zeta function is

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

and satisfies the functional relation $\xi(s) = \xi(1-s)$ which can be used to define the Riemann zeta function by analytic continuation outside its domain of convergence $\operatorname{Re}(s) > 1$. There exist appropriate generalizations for other algebraic integers.

series like every eigenfunction of the Laplacian, *viz.*

$$\mathcal{E}_s(z) = v^s + a(s)v^{n-s} + v^{n/2} \sum_{\mu \in \mathcal{O}^*} a_\mu K_{s-n/2}(2\pi|\mu|v) e^{2\pi i\mu(u)}, \quad (5.25)$$

where $\mathcal{O}^* \subset \mathbb{A}$ is the lattice dual to the lattice of integers $\mathcal{O} \subset \mathbb{A}$ relevant to the hyperbolic Weyl group (here $\mathcal{O} = \mathbb{H}$, while for the octonions $\mathcal{O} = \mathbb{O}$), and $K_\nu(v)$ is the solution to the Bessel equation

$$v^2 \frac{d^2 K_\nu}{dv^2} + v \frac{dK_\nu}{dv} + (v^2 - \nu^2) K_\nu = 0, \quad (5.26)$$

which vanishes exponentially for large v . The reason for this expansion is the invariance of $\mathcal{P}_s(z)$ under integer shifts $u \rightarrow u + o$ for $o \in \mathcal{O}$. The coefficients a_μ are further constrained by the even part of the finite Weyl group which acts as a set of (generalized) rotations on \mathcal{O} , so that $a_\mu = a_{s(\mu)}$ for all $s \in W_{\text{fin}}^+$. Further constraints come from a Hecke algebra (if it can be suitably defined) and will render the coefficients a_μ multiplicative over $\mathbb{A} = \mathbb{R}, \mathbb{C}$. The first two terms in (5.25) correspond to the so-called constant terms, while the remaining terms fall off exponentially as $v \rightarrow \infty$. In string theory applications, v is associated with the string dilaton and hence the first two terms correspond to perturbative effects whereas the exponentially suppressed terms are non-perturbative in the string coupling.

The Eisenstein functions are never square integrable with respect to (5.12). For both $\text{Re } s > n/2$ and $\text{Re } s < n/2$, the functions $\mathcal{E}_s(z)$ are not normalizable with respect to the invariant measure (2.18) when integrated over the fundamental domain as in (5.12). This is due to the divergence of the v -integral for $v \rightarrow \infty$ because of the ‘constant terms’ v^s and v^{n-s} , both of which appear with non-vanishing coefficients in (5.25), whereas the ‘tail’ involving Bessel functions decays exponentially for large v . However, for the special values ($r \in \mathbb{R}$)

$$s = \frac{n}{2} + ir \quad (5.27)$$

we obtain ‘almost’ normalizable states with eigenvalues

$$E = \frac{n^2}{4} + r^2 \geq \frac{n^2}{4}. \quad (5.28)$$

Notwithstanding a rigorous proof, this can be roughly seen as follows. Exploiting the expansion (5.25) the leading terms behave like v^s or v^{n-s} , with the subleading terms being integrable in v . Substituting these ‘dangerous’ terms into (5.12) and neglecting finite contributions, we get (with $\text{Re } s = n/2$)

$$\int_{\mathcal{F}} d\text{vol}(z) \overline{\mathcal{E}_s(z)} \mathcal{E}_{s'}(z) \sim \int_K d^n u \int_{\sqrt{1-|u|^2}}^{\infty} \frac{dv}{v} \exp [i(\pm r' \pm r) \ln v]. \quad (5.29)$$

The integral over u is finite because K is compact. Changing variables to $\xi = \ln v$ and extending the range of integration to $v = 0$, we get

$$\int_{-\infty}^{\infty} d\xi \exp [i(\pm r' \pm r)\xi] = 2\pi\delta(r' \pm r). \quad (5.30)$$

Hence, for these values of s , the Eisenstein functions are δ -function normalizable up to finite corrections. As a result, there is a continuous part of the spectrum

$$\text{spec}_c(-\Delta_{\text{LB}}) = \left[\frac{n^2}{4}, \infty \right) \quad (5.31)$$

for *even* Maass wave forms. In addition there are discrete eigenvalues for even Maass wave forms, which are embedded in the continuum (but which must be determined numerically). The spectrum of odd Maass wave forms is purely discrete.

6 E_8 and its hyperbolic extension

In extending the results of the previous section to $\mathbb{A} = \mathbb{O}$ a main obstacle is the non-associativity of the octonions. We have the abstract isomorphism

$$W^+(\mathbb{E}_{10}) \cong \text{PSL}(2, \mathbb{O}) \quad (6.1)$$

but a simple and manageable representation of this modular group is not yet at hand. In particular, realizing part of this group in terms of 2×2 matrices with integer octonionic ('octavian') entries is certainly not sufficient, because such matrices violate associativity and thus cannot by themselves define any group. Nevertheless, the generating transformations (3.17) for $\Gamma \equiv \text{PSL}(2, \mathbb{O})$ are still valid in the form given there: by the alternativity of the octonions no parentheses need to be specified for nested products involving only two different octonions. In this section, we present first results towards a more manageable representation of the group $\text{PSL}(2, \mathbb{O})$. In particular, our formula (6.23) below gives a rather explicit expression for the action of an E_8 Weyl group element in terms of unit octavians and the $G_2(2)$ automorphisms of the octavians.

6.1 E_8 and the integer octonions

As a basis for the octonion algebra one usually takes the real number 1 and seven 'imaginary' units e_1, e_2, \dots, e_7 that anticommute and square to -1 . One then chooses three of them to be in a quaternionic subalgebra. For instance taking (e_1, e_5, e_6) as a basic quaternionic triplet as in section 4, the octonion multiplication has the following properties:

$$e_i e_j = e_k \quad \Rightarrow \quad e_{i+1} e_{j+1} = e_{k+1}, \quad e_{2i} e_{2j} = e_{2k}, \quad (6.2)$$

where the indices are counted mod 7. Using these rules, the whole multiplication table can be derived from only one equation, for example $e_1 e_5 = e_6$.

The root lattice of E_8 can be identified with the non-commutative and non-associative ring of *octavian integers*, or simply *octavians* \mathbb{O} (for their construction, see [19]). Within this ring, the 240 roots of E_8 are then identified with

the set of *unit octavians*, which are the 240 invertible elements in \mathcal{O} . More specifically, the octavian units can be divided into [27]

$$\begin{aligned}
2 \text{ real numbers:} & & \pm 1, \\
112 \text{ Brandt numbers:} & & \frac{1}{2}(\pm 1 \pm e_i \pm e_j \pm e_k), \\
126 \text{ imaginary numbers:} & & \frac{1}{2}(\pm e_m \pm e_n \pm e_p \pm e_q), \quad \pm e_r,
\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
ijk &= 124, & 137, & 156, & 236, & 257, & 345, & 467, \\
mnpq &= 3567, & 2456, & 2347, & 1457, & 1346, & 1267, & 1235,
\end{aligned} \tag{6.4}$$

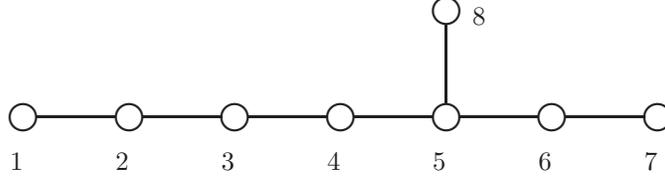
and $r = 1, 2, \dots, 7$. The lattice \mathcal{O} of octavians is the set of all integral linear combinations of these unit octavians.

Recall that the automorphism group $\text{Aut } \mathbb{O}$ of the octonionic multiplication table (6.2) is the exceptional group G_2 [18]. For the ring \mathcal{O} we have the analogous discrete result [19]

$$\text{Aut } \mathcal{O} = G_2(2), \quad |G_2(2)| = 12\,096, \tag{6.5}$$

where $G_2(2)$ is the finite (but not simple) group G_2 over the field \mathbb{F}_2 (a Chevalley group).

We choose the simple roots of E_8 , according to the Dynkin diagram



as the following unit octavians,

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2}(1 - e_1 - e_5 - e_6), & \varepsilon_2 &= e_1, \\
\varepsilon_3 &= \frac{1}{2}(-e_1 - e_2 + e_6 + e_7), & \varepsilon_4 &= e_2, \\
\varepsilon_5 &= \frac{1}{2}(-e_2 - e_3 - e_4 - e_7), & \varepsilon_6 &= e_3, \\
\varepsilon_7 &= \frac{1}{2}(-e_3 + e_5 - e_6 + e_7), & \varepsilon_8 &= e_4.
\end{aligned} \tag{6.6}$$

The identification (6.6) of the simple roots of E_8 as unit octavians is not unique. The main advantage of the choice of basis here (different from the one in [2]) is that the 126 roots of the E_7 subalgebra are given by *imaginary* octavians (defined to obey $e^2 = -1$), while the 112 roots corresponding to the two **56** representations of E_7 in the decomposition of the adjoint **248** of E_8 are given by *Brandt numbers*, and the highest root is $\theta = 1$ in accord with our general conventions. By definition, a Brandt number a obeys $a^3 = \pm 1$. As explained in [19, 29], for such numbers (and only for them) the map $x \rightarrow axa^{-1}$ is an automorphism of the octonions, see also appendix B.

Note that only the leftmost node of the Dynkin diagram corresponds to a Brandt number, while all others correspond to imaginary unit octavians. Also, the root basis for the obvious D_4 subalgebra inside E_8 is simply obtained from the one of section 4 by multiplying $\varepsilon_4, \varepsilon_5, \varepsilon_6$ and ε_8 with e_7 .

Using the scalar product (2.3) it is straightforward to reproduce from (6.6) the E_8 Cartan matrix via

$$A_{ij} = 2(\varepsilon_i, \varepsilon_j). \quad (6.7)$$

We saw in section 4 that any element in the finite Weyl group $W^+(D_4)$ can be written as a bi-multiplication

$$x \mapsto axb, \quad (6.8)$$

where a and b are Hurwitz units. But the number of different such transformations ($= 12 \times 24$) exceeds the order of $W^+(D_4)$ (which is $= 96$). This is why, in order to obtain only elements in $W^+(D_4)$, we needed to impose the extra condition $ab \in \mathcal{Q}$. For $W^+(E_8)$ we would like to generalize the expression (6.8) to unit *octavians* a and b . Because of the non-associativity, we have to place parentheses, for instance by taking

$$x \mapsto (ax)b. \quad (6.9)$$

Since the Dynkin diagram of E_8 does not have any symmetries (unlike the one of D_4) all such transformations do belong to $W^+(E_8)$. But counting these expressions, we find that their number (modulo sign changes $(a, b) \leftrightarrow (-a, -b)$) now is *smaller* than the order of the even Weyl group, which is

$$|W^+(E_8)| = 120 \times 240 \times 12\,096. \quad (6.10)$$

The extra factor 12 096 is exactly the order of the automorphism group of the octavians. This observation, made in [2] (see also [31]), suggests that the expression (6.9) should be generalized to

$$x \mapsto (a\varphi(x))b, \quad (6.11)$$

where φ is an arbitrary automorphism of the octavians. However, it was noted in [2] that the number of *independent* such expressions is in fact smaller than $120 \times 240 \times 12\,096$, since the transformation (6.9) is already an automorphism if a is a Brandt number (that is, $a^3 = \pm 1$) and $b = \pm \bar{a}$ (see above and appendix B). In section 6.3 we will explain how the formula (6.11) can be modified in these cases, so that it indeed expresses all even Weyl transformations of E_8 . As a first step we consider the E_7 subalgebra obtained by deleting the leftmost node in the Dynkin diagram and thus corresponding to imaginary octavians.

6.2 The even Weyl group of E_7

In this subsection we use a_B to denote bimultiplication from left and right with the same octavian a :

$$a_B : \mathcal{O} \rightarrow \mathcal{O}, \quad x \mapsto axa. \quad (6.12)$$

We will show that $W^+(E_7)$ is the set of all transformations

$$g_B h_B \varphi : \mathbf{O} \rightarrow \mathbf{O}, \quad x \mapsto g(h \varphi(x) h) g, \quad (6.13)$$

where g and h are imaginary unit octavians and φ is an automorphism of the octavians. It follows from (3.3), and the identification (6.6) of the simple E_7 roots as imaginary unit octavians, that the generators of $W^+(E_7)$ have this form. To show that this holds for all elements in $W^+(E_7)$ we thus have to show that the composition of two such transformations $g_{1B} h_{1B} \varphi_1$ and $g_{2B} h_{2B} \varphi_2$ again can be written in the form (6.13). From the definition of an automorphism we have

$$\begin{aligned} (g_{1B} h_{1B} \varphi_1)(g_{2B} h_{2B} \varphi_2) &= g_{1B} h_{1B} \varphi_1 (g_2)_B \varphi_1 (h_2)_B \varphi_1 \varphi_2 \\ &= (e_B f_B f_B e_B) g_{1B} h_{1B} \varphi_1 (g_2)_B \varphi_1 (h_2)_B \varphi_1 \varphi_2, \end{aligned} \quad (6.14)$$

where we in the last step have inserted the identity map in form of $e_B f_B f_B e_B$ for two arbitrary imaginary unit octavians e and f . Any unit octavian can be written as a product of two imaginary unit octavians, so we can choose e and f in (6.14) such that

$$f e = \pm((\varphi_1(h_2) \varphi_1(g_2)) h_1) g_1, \quad (6.15)$$

or equivalently,

$$((((f e) g_1) h_1) \varphi_1(g_2)) \varphi_1(h_2) = \pm 1. \quad (6.16)$$

Then, according to Corollary B.4, the map

$$\varphi_3 \equiv f_B e_B g_{1B} h_{1B} \varphi_1 (g_2)_B \varphi_1 (h_2)_B \quad (6.17)$$

is an automorphism. Inserting this into (6.14) we obtain

$$\begin{aligned} (g_{1B} h_{1B} \varphi_1)(g_{2B} h_{2B} \varphi_2) &= (e_B f_B f_B e_B) g_{1B} h_{1B} \varphi_1 (g_2)_B \varphi_1 (h_2)_B \varphi_1 \varphi_2 \\ &= e_B f_B (f_B e_B g_{1B} h_{1B} \varphi_1 (g_2)_B \varphi_1 (h_2)_B) \varphi_1 \varphi_2 \\ &= e_B f_B (\varphi_3 \varphi_2 \varphi_1). \end{aligned} \quad (6.18)$$

Since φ_3 , φ_1 and φ_2 all are automorphisms, their product $\varphi_3 \varphi_2 \varphi_1$ is an automorphism as well.

Conversely, any transformation (6.13) is an isometry of the root system. Since the group of isometries of the root system of a finite algebra is the semidirect product of the Weyl group and the symmetry group of the Dynkin diagram (which is trivial for E_7), it follows that any transformation (6.13) belongs to the Weyl group. Furthermore, it belongs to the *even* Weyl group since it does not involve the conjugate \bar{x} .

We have thus proven that $W^+(E_7)$ is the set of all expressions (6.13). But there is some redundancy in this set. If $g_1 h_1 = g_2 h_2$, then the transformation

$$\varphi_3 = h_2 g_2 g_1 h_1 \quad (6.19)$$

is an automorphism according to Corollary B.4, and we have

$$g_1 h_1 \varphi_1 = g_2 h_2 \varphi_2 \quad (6.20)$$

where $\varphi_2 = \varphi_3 \varphi_1$. Thus only the product $\pm gh$ matters modulo automorphisms φ , and this product can be any unit octavian. Accordingly, the order of $W^+(\mathbb{E}_7)$ is $120 \times 12\,096$.

6.3 From $W(\mathbb{E}_7)$ to $W(\mathbb{E}_8)$

Having understood the Weyl group of \mathbb{E}_7 the step to \mathbb{E}_8 is more conceptual than technical. The Weyl group of \mathbb{E}_7 is the subgroup of $W(\mathbb{E}_8)$ that stabilizes the highest root of \mathbb{E}_8 . Then it is enough to find one transformation w in $W(\mathbb{E}_8)$ for each element x in the orbit $W(\mathbb{E}_8)\theta$, such that $w(\theta) = x$. Then the set of *all* such transformations is the coset $wW(\mathbb{E}_7)$. The root system of \mathbb{E}_8 is one single orbit under $W(\mathbb{E}_8)$ and the highest root θ is 1 in our identification with the octavians. Thus we need a transformation in $W(\mathbb{E}_8)$ for each unit octavian b , such that $\theta = 1$ is mapped to b . But this is easy to find, we just take the transformation to be right (or left) multiplication by b . It follows if we linearize the composition property (2.8), or alternatively by (2.3) together with the first of the Moufang identities (B.1), that

$$(xb, yb) = |b|^2(x, y), \quad (6.21)$$

so right multiplication by b is an isometry if b has unit length. Like for \mathbb{E}_7 any isometry of the root system is an element of the Weyl group, and right multiplications of unit octavians belong to the even Weyl group $W^+(\mathbb{E}_8)$ since they do not involve conjugation.

Thus any element of $W^+(\mathbb{E}_8)$ can be written as

$$x \mapsto (f(e\varphi(x)e)f)b, \quad (6.22)$$

where e and f are imaginary unit octavians, b an arbitrary unit octavian, and φ an automorphism of the octavians. Using the third of the Moufang identities (B.1) we can write this as

$$\begin{aligned} x \mapsto & (f(e\varphi(x)e)f)b \\ & = f((e\varphi(x)e)(fb)) \\ & = f(e(\varphi(x)(e(fb)))) \\ & = f(e(\varphi(x)d)) \end{aligned} \quad (6.23)$$

where $d = e(fb)$. This differs from the formula (6.11) suggested in [2], only by the factorization of a into two imaginary unit octavians e and f . In fact, the factors e and f do not need to be imaginary but can also be *real*, that is ± 1 . Then if a in (6.11) is imaginary, the factorization is trivial; we can just take $e = a$ and $f = 1$, and (6.23) reduces to (6.11). Only when a is a Brandt number

we need to replace (6.11) by (6.23), where we can choose any factorization of a into two imaginary unit octavians e and f .

When we consider E_8 as a subalgebra of $E_{10} = E_8^{++}$, the Weyl group $W(E_8)$ acts in a non-trivial way on the simple root α_0 (and trivially on α_{-1}). We can thus extend the action of $W(E_8)$ to the whole root space of E_{10} . The Weyl transformation (6.23) can then be written as¹²

$$\begin{pmatrix} x^+ & x \\ \bar{x} & x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ & f(e(\varphi(x)d)) \\ ((d\bar{\varphi}(\bar{x}))\bar{e})\bar{f} & x^- \end{pmatrix} \quad (6.24)$$

or in the more compact form

$$X \mapsto F(E(D^\dagger\varphi(X)D)E^\dagger)F^\dagger, \quad (6.25)$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad E = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}. \quad (6.26)$$

6.4 From $W(E_8)$ to $W(E_9)$

As mentioned in section 3.3 (see also [22]), to any point x on the root lattice of the finite Kac-Moody algebra E_8 we can associate a *translation* t_x that acts on the root space of E_{10} as

$$t_y : X \mapsto X + 2(X, \delta)y + 2(X, y)\delta - 2|y|^2(X, \delta)\delta. \quad (6.27)$$

where δ is the null root (3.10). Then $t_x t_y = t_{x+y}$ for any x and y , and all translations form a free group of translations $\mathcal{T} \equiv \mathcal{O}$ of rank eight. This group is a normal subgroup of the Weyl group of E_9 , which can then be written as a semidirect product

$$W(E_9) = W(E_8) \ltimes \mathcal{T} \equiv W(E_8) \ltimes \mathcal{O}. \quad (6.28)$$

Furthermore, the translations belong to the even Weyl group, so we have

$$W^+(E_9) = W^+(E_8) \ltimes \mathcal{T}. \quad (6.29)$$

When we identify the root space of E_{10} with the Jordan algebra $H_2(\mathbb{O})$, we can write the action of t_y as

$$t_y : X \mapsto T_y X T_y^\dagger \quad (6.30)$$

where

$$T_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}. \quad (6.31)$$

¹²The fact that the formula ' $X \mapsto SXS^\dagger$ ' requires refinement by placing parentheses inside the matrix elements was anticipated in [2].

For such matrices we indeed need not worry about non-associativity because

$$\begin{aligned} (T_y X) T_y^\dagger &= T_y (X T_y^\dagger) = \begin{pmatrix} x^+ + y\bar{x} + x\bar{y} + yx^-\bar{y} & x + yx^- \\ \bar{x} + x^-\bar{y} & x^- \end{pmatrix} \\ &= \begin{pmatrix} x^+ + 2(x, y) + x^-|y|^2 & x + x^-y \\ \bar{x} + x^-\bar{y} & x^- \end{pmatrix} = t_y(X). \end{aligned} \quad (6.32)$$

where δ is as in (3.10). It follows that any element in $W^+(\mathbb{E}_9)$ can be written in the form (6.25), but now with

$$D = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \quad (6.33)$$

where c is an arbitrary octavian, so that D^\dagger is an upper triangular matrix. Since at least one of the diagonal entries in D is equal to 1 the expression (6.25) is still well defined.

If we furthermore replace the zeros in the upper right corner of F and the lower left corner of E with arbitrary octavians we obtain some of the additional elements in $W^+(\mathbb{E}_{10})$, for example the generator s_1 with the corresponding matrix

$$S_{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \quad (6.34)$$

However, to obtain *all* elements of $W^+(\mathbb{E}_{10})$, we need to continue this alternating sequence of upper and lower triangular matrices in the same spirit as in the proof of Theorem A.2 below. It remains for future work to make this more precise.

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A Coprime condition for Hurwitz numbers

Our first theorem states the validity of the Euclidean algorithm for Hurwitz numbers \mathbb{H} (see e.g. [19] for a proof).

Theorem A.1 *Let a and c be two right coprime Hurwitz numbers with $|a| \geq |c|$. Then there are Hurwitz numbers $q_1, q_2, \dots, q_n, q_{n+1}$ and r_0, r_1, \dots, r_n , for*

some integer $n \geq 0$, such that

$$\begin{aligned}
a &= q_1c + r_1, \\
c &= q_2r_1 + r_2 \\
r_1 &= q_3r_2 + r_3, \\
&\dots \\
r_{n-2} &= q_n r_{n-1} + r_n, \\
r_{n-1} &= q_{n+1} r_n
\end{aligned} \tag{A.1}$$

and $|r_0| > |r_1| > \dots > |r_n| = 1$, where $r_0 = c$. Furthermore, r_{k-1} and r_k are right coprime for any $k = 1, 2, \dots, n$.

In section 4.2 we saw that if

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.2}$$

is an element in $SL(2, \mathbb{H})$ or $PSL(2, \mathbb{H})$, then the pairs (a, b) and (c, d) must each be *left coprime*, and the pairs (a, c) and (b, d) must each be *right coprime*. Using the Euclidean algorithm we can now show the converse for two right coprime Hurwitz numbers a and c , in the sense that we can find b and d such that $S \in SL(2, \mathbb{H})$. The other cases are analogous.

Theorem A.2 *Let a and c be two right coprime Hurwitz numbers. Then there are Hurwitz numbers b and d such that $S \in SL(2, \mathbb{H})$.*

Proof. Let n and the Hurwitz numbers q_1, q_2, \dots, q_{n+1} and r_0, r_1, \dots, r_n be as in the preceding theorem. Set

$$Q_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_k = \begin{pmatrix} 1 & q_k \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} r_n & 0 \\ 0 & 1 \end{pmatrix} \tag{A.3}$$

for $k = 1, 2, \dots, n + 1$. By induction over n it is then straightforward to show that, for some b and d ,

$$Q_1 Q_2^t Q_3 \cdots Q_{n+1}^t R = S \tag{A.4}$$

if n is odd, and

$$Q_0 Q_1^t Q_2 \cdots Q_{n+1}^t R = S \tag{A.5}$$

if n is even. Then the theorem is proven, since each of the matrices (A.3) belongs to $SL(2, \mathbb{H})$, and this set is closed under matrix multiplication. \square

B Automorphisms of octonions

In this appendix, we will prove a theorem about the automorphisms of the octonions that we use in section 6.2. First we list the Moufang identities that we also use

$$(ax)(ya) = a(xy)a, \quad ((xa)y)a = x(aya), \quad a(x(ay)) = (axa)y. \quad (\text{B.1})$$

These identities follow from the alternative laws (2.11). (For a proof, see [28].) Another useful identity that follows from the alternative laws and the Moufang identities is

$$(a^2xa)(a^{-1}ya) = a^2(xy)a. \quad (\text{B.2})$$

Indeed, we have

$$\begin{aligned} (a^2xa)(a^{-1}ya) &= (a(ax)a)(a^{-1}ya) \\ &= a((axa)(a^{-1}y))a \\ &= a(a(xy))a = a^2(xy)a. \end{aligned} \quad (\text{B.3})$$

Now we can prove the following lemma.

Lemma B.1 *For any integer $n \leq 1$ and any $z, a_1, a_2, \dots, a_n \in \mathbb{O}$ we have*

$$\begin{aligned} a_1(a_2(\cdots(a_nza_n^{-1})\cdots)a_2^{-1})a_1^{-1} \\ = a_1^{-2}(a_2^{-2}\cdots(a_n^{-2}(b_nz)a_n^{-1})\cdots a_2^{-1})a_1^{-1} \end{aligned} \quad (\text{B.4})$$

where

$$b_n = a_n^2(a_{n-1}^2\cdots(a_2^2a_1^3a_2)\cdots a_{n-1})a_n. \quad (\text{B.5})$$

Proof. We prove this by induction over n . The case $n = 1$ follows easily by alternativity. Suppose now that the identity (B.4) holds for some integer $n \geq 1$. Then we have

$$\begin{aligned} a_1(a_2\cdots(a_n(a_{n+1}za_{n+1}^{-1})a_n^{-1})\cdots a_2^{-1})a_1^{-1} \\ = a_1^{-2}(a_2^{-2}\cdots(a_n^{-2}(b_n(a_{n+1}za_{n+1}^{-1}))a_n^{-1})\cdots a_2^{-1})a_1^{-1} \end{aligned} \quad (\text{B.6})$$

so we only have to show that

$$b_n(a_{n+1}za_{n+1}^{-1}) = a_{n+1}^{-2}(b_{n+1}z)a_{n+1}^{-1}. \quad (\text{B.7})$$

To do this we use that

$$b_n = a_{n+1}^{-2}b_{n+1}a_{n+1}^{-1} \quad (\text{B.8})$$

so that the left hand side of (B.7) becomes

$$b_n(a_{n+1}za_{n+1}^{-1}) = (a_{n+1}^{-2}b_{n+1}a_{n+1}^{-1})(a_{n+1}za_{n+1}^{-1}). \quad (\text{B.9})$$

and then (B.7) follows from (B.3). \square

Lemma B.2 For any integer $n \leq 1$ and any $x, y, a_1, a_2, \dots, a_n \in \mathbb{O}$ we have

$$\begin{aligned} & (a_1(\cdots(a_n x a_n^{-1})\cdots)a_1^{-1})(a_1(\cdots(a_n y a_n^{-1})\cdots)a_1^{-1}) \\ &= a_1^{-2}(a_2^{-2} \cdots (a_n^{-2}((b_n x)y)a_n^{-1}) \cdots a_2^{-1})a_1^{-1} \end{aligned} \quad (\text{B.10})$$

where

$$b_n = a_n^2(a_{n-1}^2 \cdots (a_2^2 a_1^3 a_2) \cdots a_{n-1})a_n. \quad (\text{B.11})$$

Proof. By Lemma B.1 we have

$$\begin{aligned} & (a_1(\cdots(a_n x a_n^{-1})\cdots)a_1^{-1})(a_1(\cdots(a_n y a_n^{-1})\cdots)a_1^{-1}) \\ &= (a_1^{-2} \cdots (a_n^{-2}(b_n x)a_n^{-1}) \cdots a_1^{-1})(a_1(\cdots(a_n y a_n^{-1})\cdots)a_1^{-1}) \end{aligned} \quad (\text{B.12})$$

and then (B.10) follows by using (B.3) successively. \square

No we can prove the main result of this appendix.

Theorem B.3 A transformation

$$\varphi : x \mapsto a_1(a_2(\cdots(a_n x a_n^{-1})\cdots)a_2^{-1})a_1^{-1} \quad (\text{B.13})$$

of the octonions, where $a_1, \dots, a_n \neq 0$, is an automorphism if and only if

$$b_n = a_n^2(a_{n-1}^2(\cdots(a_2^2 a_1^3 a_2) \cdots a_{n-1})a_n \in \mathbb{R}. \quad (\text{B.14})$$

Proof. Setting $xy = z$, we have in Lemma B.1 and Lemma B.2 obtained expressions for $\varphi(xy)$ and $\varphi(x)\varphi(y)$, respectively. The difference between these two expressions is

$$\varphi(xy) - \varphi(x)\varphi(y) = a_1^{-2}(a_2^{-2} \cdots (a_n^{-2}\{b_n, x, y\}a_n^{-1}) \cdots a_2^{-1})a_1^{-1}. \quad (\text{B.15})$$

Since there are no zero divisors, this difference is zero if and only if the associator $\{b_n, x, y\} = b_n(xy) - (b_n x)y$ vanishes. Since this must happen for all x and y , the necessary and sufficient condition is $b_n \in \mathbb{R}$. \square

The theorem simplifies considerably if we take a_1, \dots, a_n to be unit octonions, and furthermore imaginary.

Corollary B.4 A transformation

$$\varphi : x \mapsto a_1(a_2(\cdots(a_n x a_n)\cdots)a_2)a_1 \quad (\text{B.16})$$

of the octonions, where a_1, \dots, a_n are imaginary unit octonions or real unit octonions, is an automorphism if and only if

$$(((a_1 a_2) a_3) \cdots a_n) = \pm 1. \quad (\text{B.17})$$

Proof. This follows directly from Theorem B.3 if we use that $a^{-1} = \pm a$ and $a^2 = a^{-2} = \pm 1$ for any imaginary unit octonion or real unit octonion a . \square

We use this corollary in section 6.2 when we identify the imaginary unit octonions with the simple roots of E_7 , and the corresponding bimultiplications with generators of $W^+(E_7)$.

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