

GENERAL VARIATIONAL PRINCIPLE FOR NORMAL MODES OF ROTATING STARS

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ABSTRACT

A variational principle is constructed for the eigenfrequencies of the normal modes of perfect fluid stars which are axisymmetric, differentially rotating, but without meridional circulation. It is the first such principle to be capable of yielding the complex as well as the real eigenfrequencies. The key to the principle is the discovery of a simple way of turning a right eigenvector for a normal mode into a left one. This is based on a symmetry of the unperturbed star, that it is invariant under reflection in time and azimuth. The variational expression also gives a new approach to approximating the linear perturbation problem by one restricted to a finite-dimensional set of trial functions. In cases where it differs from the more conventional approach, the new method should be more accurate.

Subject headings: stars: pulsation — stars: rotation

Variational expressions which help one to calculate eigenvalues of physically important operators are well known and often used in physics. Problems involving a compact self-adjoint operator (e.g., Sturm-Liouville) can be solved by the usual method of constrained minima (Courant and Hilbert 1953, Chap. 6). Chandrasekhar (1961) constructed a number of variational expressions in hydrodynamics and used them to calculate either eigenfrequencies of normal modes or wavenumbers of marginally stable modes. Chandrasekhar (1964) also was the first to give a variational principle for the eigenfrequencies of a nonradially pulsating spherical star. This has since been extended to general relativity by Chandrasekhar (1965) for spherical modes of spherical stars, and by Detweiler (1975) for dipole modes; in both cases the variational principles provided a convenient approach to a stability criterion. Against this background, it was perhaps surprising that Lynden-Bell and Ostriker (1967) were able to show that the usual method of construction failed to produce a variational principle that would work for the complex eigenfrequencies of pulsation of rotating stars, although it did work when the frequencies were real. The purpose of this paper is to point out that a small modification of Lynden-Bell and Ostriker's method does give a full variational principle in the restricted but important case of an axisymmetric (perfect-fluid) stellar model without meridional circulation.¹ The key step in the construction is Theorem 1 below, which gives a relation between left and right eigenfunctions for a given eigenvalue. Further implications of Theorem 1 for the stellar perturbation problem will be explored elsewhere (Schutz 1979*a, b*).

To avoid confusion, let me begin by stressing that the variational principle given here is *not* the usual dynamical one, based on a Lagrangian whose first variation gives the equation of motion. Rather, it is a formula for calculating a trial eigenfrequency from a trial eigenfunction, and it has the property that a true eigenfrequency is unchanged to first order by changes away from the corresponding true eigenfunction. The dynamical variational principle (cf. Lynden-Bell and Ostriker 1967) leads to the dynamical equations

$$A\xi_{tt} + B'\xi_t + C'\xi = 0, \quad (1)$$

where ξ is the Lagrangian displacement vector of a fluid element from its equilibrium position; A , B' , and C' are linear operators; and the subscript t denotes a partial derivative with respect to time. The eigenvalue problem is a search for solutions to (1) of the form

$$\xi = \xi(r, \theta)e^{im\phi}e^{i\sigma t}, \quad (2)$$

where ϕ is the angle about the axis of symmetry, and the vector $\xi(r, \theta)$ is an element of some complex Hilbert space H . The usual inner product in H will be denoted by parentheses:

$$(\xi, \eta) = \int \bar{\xi} \cdot \eta d^3x \quad (3)$$

¹ In fact, the variational principle holds for modes of any stationary system with the reflection symmetry $(t, \phi) \rightarrow (-t, -\phi)$. This is proved in Schutz (1979*a*). This excludes meridional circulation but does not force axisymmetry. Since the axisymmetric case is of most interest and is easier to demonstrate, I shall confine myself to it here.

where a bar denotes complex conjugation.² We shall adopt the spherical coordinates (r, θ) to describe vectors in H , but of course none of the results depend on this. For a solution of the form (2), equation (1) becomes

$$-\sigma^2 A\xi + \sigma B\xi + C\xi = 0, \quad (4)$$

where here and from now on ξ is just $\xi(r, \theta) \in H$ and the dependence on m has been taken into the operators B and C . For instance, for stars without meridional circulation, B is

$$B\xi = -2\rho\Omega(m\xi - i\mathbf{e}_z \times \xi), \quad (5)$$

where \mathbf{e}_z is the unit vector parallel to the axis of symmetry, and Ω is the (position-dependent) angular velocity about that axis. The operator C is more complicated, but its exact form is not important. What is important is that A , B , and C are all self-adjoint in H (Dyson and Schutz 1979), and A (which is just the star's density, ρ) is positive-definite.

Equation (4) is a quadratic eigenvalue problem with self-adjoint operators. The commonest variational principles for eigenfrequencies usually involve only linear eigenvalue problems. For instance, modes of a nonrotating star ($B = 0$) solve the equation

$$-\lambda A\xi + C\xi = 0, \quad (6)$$

where I have renamed σ^2 as λ to emphasize the essential linearity of the problem in the eigenvalue. The variational expression is obtained simply by taking the inner product of equation (6) with ξ :

$$-\lambda(\xi, A\xi) + (\xi, C\xi) = 0,$$

which is to be regarded as an equation for λ as a function of any trial function $\xi \in H$. If ξ is changed to $\xi + \delta\xi$, the change $\delta\lambda$ in the trial eigenvalue is found from the first variation:

$$-\delta\lambda(\xi, A\xi) + (\xi, (-\lambda A + C)\delta\xi) + (\delta\xi, (-\lambda A + C)\xi) = 0. \quad (7)$$

For this to provide a variational principle, one requires $\delta\lambda$ to vanish for all $\delta\xi$ if λ and ξ are a solution of (6). This clearly requires not only that $(-\lambda A + C)\xi$ vanish, but also (in order to get rid of the middle term of [7]) that $(\xi, (-\lambda A + C)\delta\xi)$ vanish for all $\delta\xi$. In more familiar language, ξ must be a right eigenvector and its adjoint must be a left eigenvector for the same eigenvalue λ . This relation between the left and right eigenvectors does in fact hold for linear self-adjoint problems, but, as Lynden-Bell and Ostriker pointed out, it is not true for complex eigenvalues of the quadratic problem (4). This causes the analogous variation principle derived from (4) to fail.

As Chandrasekhar (1961) has stressed, a successful principle must clearly be built by taking the inner product of (4) with a left eigenvector. Lynden-Bell and Ostriker gave a generalized variational principle by calling this left eigenvector η , and treating it as independent of ξ . This does indeed work, but it has twice as many unknowns, since ξ and η are unrelated and must both be determined by the variations. A more conventional variational principle can be constructed if one knows a relation between left and right eigenvectors. Knowing a right eigenvector, can one derive from it the left eigenvector for the same eigenvalue? In general the answer is no: for a general matrix one must find *all* the right eigenvectors in order to determine any left eigenvectors. (In tensor language, the left eigenvectors are the basis of H^* dual to the right eigenvector basis of H .) But for stars without meridional circulation we are fortunate, for there is a simple way of obtaining a left eigenvector from its corresponding right eigenvector.

From now on I shall suppose that the unperturbed star is invariant under the reflection symmetry $(t, \phi) \rightarrow (-t, -\phi)$. Moreover, the original operators A , B , C in (1) are real. It therefore follows that if

$$\xi = [\xi^r(t, r, \theta, \phi), \xi^\theta(t, r, \theta, \phi), \xi^\phi(t, r, \theta, \phi)]$$

solves (1), then so does

$$S\xi = [\bar{\xi}^r(-t, r, \theta, -\phi), \bar{\xi}^\theta(-t, r, \theta, -\phi), -\bar{\xi}^\phi(-t, r, \theta, -\phi)], \quad (8)$$

where, as before, a bar denotes complex conjugation. This defines the fundamental symmetry operator. Applied to a vector with azimuthal eigenvalue m , it produces another vector of the same m . Therefore S has a natural restriction to the space H in which the eigenfunctions $\xi(r, \theta)$ reside:

$$S(\xi^r, \xi^\theta, \xi^\phi) = (\bar{\xi}^r, \bar{\xi}^\theta, -\bar{\xi}^\phi). \quad (9)$$

Notice that S is antilinear and is its own inverse:

$$S(\alpha\xi) = \bar{\alpha}S\xi, \quad S^2 = 1. \quad (10)$$

² Strictly speaking, one should use a slightly larger Hilbert space with the inner product $\langle \xi, \eta \rangle = (\xi, A\eta)$, as discussed by Dyson and Schutz (1979). But this is a detail which need not be worried about here.

The existence of the symmetry operator S means that if $\xi(t)$ is a solution of (1) for initial data $\xi(0) = \eta$, $\xi_t(0) = \zeta$, then $S\xi(t)$ is a solution for initial data $(S\eta, -S\zeta)$. Since $S(\xi_{tt}) = (S\xi)_{tt}$ and $S^2 = 1$, equation (1) for arbitrary η and ζ implies the operator symmetries $SAS = A$, $SB'S = -B'$, $SC'S = C'$. Then the invariance of m under S implies the following symmetries on A , B , and C :

$$SAS = A, \quad SBS = B, \quad SCS = C. \quad (11)$$

(For B this can be proved directly from [5] by noting that the ϕ -reflection in S changes the handedness of the coordinates, causing the cross-product in [5] to change sign.) The importance of S is that it turns a right eigenvector into the adjoint of a left one.

THEOREM 1. *If ξ is a right eigenvector of (4) for eigenfrequency σ and azimuthal eigenvalue m , then the adjoint of $S\xi$ is a left eigenvector for the same σ and m .³*

$$(S\xi, (-\sigma^2 A + \sigma B + C)\eta) = 0, \quad \forall \eta \in H. \quad (12)$$

Proof. By hypothesis,

$$-\sigma^2 A\xi + \sigma B\xi + C\xi = 0.$$

Since $S^2 = 1$, we can multiply by S and insert S^2 before ξ :

$$-(S\sigma^2 A S)S\xi + (S\sigma B S)S\xi + (S C S)S\xi = 0.$$

Using (10) and (11), we find

$$-\bar{\sigma}^2 A S\xi + \bar{\sigma} B S\xi + C S\xi = 0,$$

i.e., that $S\xi$ is a right eigenvector for eigenfrequency $\bar{\sigma}$.⁴ (This is obvious from the original definition of S in eq. [8].) Then for any vector η it follows that

$$(\eta, (-\bar{\sigma}^2 A + \bar{\sigma} B + C)S\xi) = 0,$$

whose complex conjugate is (by the self-adjointness of A , B , and C)

$$(S\xi, (-\sigma^2 A + \sigma B + C)\eta) = 0.$$

Q.E.D.

This leads directly to the variational principle.

THEOREM 2. *Let σ be a root of the equation*

$$(S\xi, (-\sigma^2 A + \sigma B + C)\xi) = 0. \quad (13)$$

If σ is unchanged to first order for all changes in ξ , then ξ is a right eigenvector whose eigenvalue is σ .

Proof. The first variation of (13) is

$$\delta\sigma(S\xi, (-2\sigma A + B)\xi) + (S\delta\xi, (-\sigma^2 A + \sigma B + C)\xi) + (S\xi, (-\sigma^2 A + \sigma B + C)\delta\xi) = 0. \quad (14)$$

The last two terms are in fact equal. To see this, we first note that, for any two vectors ξ and η ,

$$(\bar{\xi}, \eta) = (S\xi, S\eta), \quad (15)$$

which follows directly from (9). Next, if D is any self-adjoint operator invariant under S (as in [11]), then

$$(S\eta, D\xi) = (S\xi, D\eta). \quad (16)$$

This follows complex-conjugating $(S\eta, D\xi)$ to get, by (15),

$$(\overline{S\eta}, \overline{D\xi}) = (\eta, SD\xi) = (\eta, DS\xi) = (D\eta, S\xi),$$

which is the complex conjugate of $(S\xi, D\eta)$. In turn, equation (16) implies that the last two expressions in (14) are equal term-by-term in σ , for any complex σ . Since the hypothesis of the theorem asserts that the first term of (14) vanishes, we are left with

$$(S\delta\xi, (-\sigma^2 A + \sigma B + C)\xi) = 0 \quad \forall \delta\xi.$$

Since $\delta\xi$ is arbitrary, $S\delta\xi$ is equally arbitrary, and the theorem is proved. Q.E.D.

³ In expressions like (12) in which the adjoint of $S\xi$ appears, the reader should bear in mind that *two* complex conjugations are being performed on ξ : one by S and the other by using ξ as the left member of an inner product.

⁴ It follows that if σ is real, $S\xi$ differs from ξ by at most an overall phase. By adjusting the phase, one can take ξ to be invariant under S .

I have not asserted the converse of Theorem 2 because there is one exceptional case, which occurs when

$$(S\xi, (-2\sigma A + B)\xi) = 0 \quad (17)$$

for an eigenvector ξ with eigenvalue σ . I will show elsewhere (Schutz 1979a) that if the eigensubspace for σ is one-dimensional, then this is exactly the condition that the eigenvectors for σ do not span the characteristic subspace defined by σ : a Jordan chain is present. (See Bogнар 1974 for a discussion of Jordan chains in the quadratic eigenvalue problem.) It seems likely that this only occurs at points of marginal stability for the mode in question. A complete discussion of this case is beyond the scope of this paper, but will be undertaken elsewhere. But it should be remarked that (17) will rarely be found in practice. Theorem 2 also does not give a method for finding points of the continuous spectrum, because for these a true eigenvector does not exist. It may be possible, however, to overcome this problem with a more elaborate principle. For numerical calculations, of course, the spectrum is always discrete because the problem is always approximated by a finite-dimensional one. It is also worth noting that when the eigenfrequency σ is real, the remark in note 4 shows that this principle reduces to the one given by Lynden-Bell and Ostriker (1967).

In a later paper (Schutz 1979a) I will discuss a number of related points in more detail: the orthogonality relations between right and left eigenvectors (and Jordan chains); the complex structure of H when S is taken to be a conjugation; the reformulation of this variational principle in the space $H \oplus H$ ("phase space," in which ξ and ξ_t are the basic elements) in a particularly instructive version involving the total energy operator and the symplectic form; and a criterion for the existence of Jordan chains (incompleteness of eigenvectors). A further paper (Schutz 1979b) will use Theorem 1 to formulate a perturbation theory for eigenvalues. This will allow the effects on a particular mode of secular terms (viscosity and radiation reaction, for example) to be calculated. The second paper will also examine closely the onset of dynamical instability, showing that Jordan chains mark those stars which are marginally unstable.

As a final remark I would like to point out that equation (13) provides more than a basis for calculating eigenfrequencies by variational methods. It is an eigenfrequency formula which ought to be used whenever approximate calculations are made. For example, suppose one wishes to approximate H by a space spanned by a few convenient complex basis functions, $\{f_j, j = 1, \dots, N\}$. Then there are two natural matrix approximations to, say, the operator C : the matrix

$${}_{(S)}C_{jk} = (Sf_j, Cf_k),$$

which is symmetric (${}_{(S)}C_{jk} = {}_{(S)}C_{kj}$) by virtue of (16), and the matrix

$${}_{(H)}C_{jk} = (f_j, Cf_k),$$

which is Hermitian (${}_{(H)}C_{jk} = {}_{(H)}\bar{C}_{kj}$) because C is self-adjoint. Will these representations be equivalent, and if not, which is preferred? They will be equivalent if every function Sf_k is expressible as a complex linear combination of the f_j 's. Presumably in most applications the trial basis will have this property. If one is chosen which does not, then the symmetric representation is preferred to the Hermitian one because of the stationarity property of the eigenvalues. That is, the exact eigenfrequency is a root of equation (13) for the exact eigenfunction. The approximate eigenfunction can be thought of as a perturbation away from the true one. In the symmetric representation, the approximate eigenfrequency is a root of (13) for the approximate eigenvector. By the converse of Theorem 2 (whose exception will be discussed below), this root differs from the exact one by an amount of the order of the square of the error in the eigenfunction. We have no such guarantee for an inequivalent Hermitian representation. Note, however, that near the onset of instability in a mode the error in the eigenfrequency of the symmetric representation, while formally of second order, may be enhanced because the factor $(S\xi, (-2\sigma A + B)\xi)$ in equation (14) becomes very small. Thus, approximations of this type may give rather worse values for the marginal eigenfrequencies than for ones that are far from their instability point.⁵ It is probable that the Hermitian representation suffers from the same problem.

⁵ This problem also arises in the nonrotating star, equations (6) and (7). There the eigenfrequency's error is $\delta\sigma = \delta\lambda/2\sigma$; although $\delta\lambda$ is of second order, $\delta\sigma$ is amplified near the marginal point ($\sigma = 0$).

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