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GRAVITATIONAL RADIATION INSTABILITY IN ROTATING STARS

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ABSTRACT

We have shown elsewhere that gravitational radiation leads to an instability to nonaxisymmetric perturbations in stars that are rotating rapidly enough to be secularly unstable in Newtonian theory, and we explore here some astrophysical consequences. Neutron stars are unstable when $\langle\Omega\rangle$ exceeds 10^3–10^4 s^{-1} and lose their excess angular momentum in seconds. Unstable white dwarfs spin down to stability or to collapse in times that vary from weeks to 10^3 years. All stars with ergoregions are unstable; this fact imposes stringent upper limits on the rotation of relativistic configurations.
Subject headings: neutron stars — relativity — white dwarf stars

I. INTRODUCTION

Gravitational radiation induces a generic instability in rapidly rotating stars: a configuration with too much rotational energy will radiate away its excess angular momentum (or perhaps fission) until a stable configuration is attained. In the special case of uniform-density, uniformly rotating Maclaurin spheroids, the behavior was first described by Chandrasekhar (1970); we recently generalized his result to arbitrary relativistic stars and obtained a minimum principle for the onset of instability along a sequence of equilibrium configurations (Friedman and Schutz 1975). Like the general-relativistic pulsational instability, the rotational instability can play a decisive role *even in nearly Newtonian stars* (e.g., in ruling out rapidly rotating white dwarfs); here the minimum principle locates the point of Newtonian secular stability. For highly relativistic configurations, an apparently more stringent limit on rotation is set by a related fact that stars with ergoregions are unstable. We will discuss below circumstances in which the gravitational radiation instability will be important, the associated growth rates of unstable modes, and—in the more relativistic regime—the ergoregion instability.

II. NEWTONIAN CONFIGURATIONS

In the nearly Newtonian domain our work implies that Newtonian stars which are secularly unstable by the Lynden-Bell–Ostriker (1967) minimum principle (see also Chandrasekhar and Lebovitz 1973) are in fact unstable in the presence of gravitational radiation. The point at which secular instability sets in along a sequence of rotating stars characterized (say) by increasing angular momentum is the first point at which an energy integral can be made to vanish for some trial function representing the perturbation. Stable stars minimize their total energy, whereas the secularly unstable stars adjoin lower-energy nonaxisymmetric configurations. In the absence of dissipation, conservation of angular momentum and of vorticity make the lower energy states dynamically inaccessible; but because gravitational radiation can carry off angular momentum, it excites the nonaxisymmetric instability.

The question of when a Newtonian star is secularly stable has been elucidated in a series of papers by Ostriker, Tassoul, and Bodenheimer; we will briefly sketch their work and clear up a misconception concerning the validity of their "tensor virial" method. An ingenious paper by Ostriker and Tassoul (1968) applied the tensor virial formalism developed in earlier work by Chandrasekhar and Lebovitz (see Chandrasekhar 1969) to differentially rotating, compressible stars. In the subsequent papers, Ostriker, Tassoul, and Bodenheimer found that the ratio $t = T/|W|$ of the kinetic to potential energy at the point of secular instability was always about 0.14 and was remarkably insensitive to the rotation law or the (polytropic) equation of state. It was further claimed that in locating the secular point (the point along a sequence of rotating stars at which a lower-energy nonaxisymmetric sequence first bifurcates) the tensor virial theorem was exact; this, however, is not the case. Instead, the tensor virial method is equivalent to choosing a trial function linear in the coordinates in evaluating the Newtonian energy integral. But the physical perturbation is not in general linear at the bifurcation point,[1] and the tensor virial criterion is correspondingly only a *sufficient* condition for instability. Of course, the rule of thumb that axisymmetric configurations are unstable when $t \gtrsim 0.14$ is approximate in any case and remains valid.

The gravitational radiation instability can be important for a rapidly rotating star only when its growth

[1] In fact, stationary nonaxisymmetric deformations of differentially rotating polytropes cannot arise from displacements linear in the Cartesian coordinates: the fluid velocity $\boldsymbol{v}$ of a linearly deformed barotropic configuration will fail to satisfy the consistency requirement for equilibrium, $\nabla \times (\boldsymbol{v}\cdot\nabla\boldsymbol{v}) = 0$, unless the unperturbed configuration rotates uniformly.

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time is shorter than the star's lifetime. To estimate the growth time, we will use the e-folding time obtained by Chandrasekhar (1970) in his study of secularly unstable Maclaurin spheroids, namely,

$$\tau \sim 10^{-8}\frac{R}{c}\left(\frac{R\Omega}{c}\right)^{-6}(t-t_c)^{-5} \quad (0 < t - t_c \ll 1)\,, \quad (1)$$

where t_c is the critical value of t at which secular instability sets in. The angular velocity can be obtained from

$$t = \frac{T}{|W|} = \frac{1}{3\alpha}\frac{R^3\langle\Omega\rangle^2}{GM}\,, \quad (2)$$

where the parameter α is of order unity, ranging from 0.6 for uniform spheres with slow rotation to 2 for centrally condensed objects in rapid differential rotation. Then, defining the gravitational radius

$$R_g = 2GM/c^2\,, \quad (3)$$

we have

$$\langle\Omega\rangle = (\tfrac{3}{2}\alpha t)^{1/2}\left(\frac{R_g}{R}\right)^{3/2}\frac{c}{R_g}\,, \quad (4)$$

and

$$\tau \sim 10^{-13}\frac{M}{M_\odot}\left(\frac{R_s}{R}\right)^{-4}(\tfrac{3}{2}\alpha t)^{-3}(t-t_c)^{-5}\text{ s}\,. \quad (5)$$

For the neutron stars modeled by Baym, Pethick, and Sutherland (1971) we find the following results: the maximum stable angular velocities (corresponding to $t = 0.14$) range from

$$\langle\Omega_{\max}\rangle = 8 \times 10^3 \text{ s}^{-1} \quad \text{when} \quad M = M_\odot$$

to

$$\langle\Omega_{\max}\rangle = 1 \times 10^3 \text{ s}^{-1} \quad \text{when} \quad M = 0.15M_\odot\,;$$

the growth rates are

$$\tau = 10\text{ s} \quad \text{when} \quad M = M_\odot\,, \quad t = 0.15\,,$$

$$\tau = 10^{-4}\text{ s} \quad \text{when} \quad M = M_\odot\,, \quad t = 0.24\,;$$

$$\tau = 10^{4}\text{ s} \quad \text{when} \quad M = 0.15M_\odot\,, \quad t = 0.15\,,$$

$$\tau = 10^{-1}\text{ s} \quad \text{when} \quad M = 0.15M_\odot\,, \quad t = 0.24\,.$$

Thus any neutron star whose angular velocity is appreciably above the critical value will spin down in a matter of seconds or less.

Growth times of white dwarf instabilities are also short; for the Ostriker-Bodenheimer (1968) models, the times given by equation (5) vary from a few weeks to 10^3 years. For example, model 8 ($M = 1.81M_\odot$, $t = 0.160$, $\alpha = 1.70$) had $\tau = 300$ years; model 12 (the most massive one, with $M = 4.07M_\odot$, $t = 0.184$, $\alpha = 1.99$) had $\tau = 2$ weeks. These numbers in fact overestimate the times because the angular velocity defined by equation (4) turned out always to approximate the surface angular velocity at the equator of the Ostriker-Bodenheimer models, whereas an angular velocity such as that at the half mass point (which could be a factor of 3 larger) would presumably be more appropriate. It therefore appears that any secularly unstable white dwarf is likely to spin down within a few thousand years or less.

As a result, the extent to which rapid differential rotation might raise the maximum mass of degenerate stars is sharply limited. Astrophysical models involving massive white dwarfs (such as the suggestion that the compact components of binary X-ray sources may be rapidly rotating dwarfs; Lamb and Van Horn 1973; Brecher and Morrison 1973) can be realized only if they are secularly stable; and when $M \gtrsim 3M_\odot$, stable configurations are probably nonexistent. In fact, if high-mass, rapidly rotating dwarfs or neutron stars can form, they are more likely to provide a pathway to black holes than an alternative state. That is, if the initial star spins rapidly enough, its mass may exceed the allowed limit for stable configurations, in which case spin-down will lead to collapse.[2]

III. RELATIVISTIC CONFIGURATIONS

For self-gravitating fluids in general relativity there is a minimum principle governing stability analogous to that in the Newtonian theory. On a spacelike hypersurface one chooses arbitrary time-independent trial functions satisfying the perturbed initial value equations and in a gauge regular at infinity. An integral representing the total energy of the perturbation can then be evaluated; and if at some point along a sequence of relativistic configurations, the minimum value of the integral for all trial functions passes through zero, the sequence is unstable beyond that point. A related result is that instability in any mode sets in when its frequency vanishes; that is, in the absence of a horizon, there can be no nonzero real frequency modes.

As we stated above, the Newtonian limit of the stability criterion is just the condition for secular instability, an instability excited by gravitational radiation. In general relativity the criterion locates *the* dynamical instability. Relativistic stellar models have not yet been tested for stability and it remains unclear how general relativity will modify the $t \gtrsim 0.14$ rule. In fact, in general relativity there is no unique partition of energy into kinetic energy of rotation and gravitational potential energy.

In the ultrarelativistic regime, however, the theory is simplified by the fact that every star with an ergoregion[3] is unstable (Friedman 1975); there are always trial functions which make the energy integral negative. Heuristically, the instability reflects the fact that within the ergoregion one can always choose an initial perturba-

[2] We do not mean to suggest here that dwarfs or neutron stars rotating so rapidly as to be unstable or nearly unstable arise astrophysically; there is no evidence that they do. Moreover, differential rotation may be precluded in neutron stars, in which case the equilibrium sequence would terminate (by rotational shedding) prior to the point of instability. It is nevertheless useful to elucidate constraints on equilibrium configurations that are independent of the detailed stellar structure or evolution.

[3] The ergoregion is that part of space in which no physical object can remain at rest with respect to an observer at infinity: the dragging of inertial frames is so extreme that all timelike world lines rotate with the star. Technically, it is the region in which the asymptotically timelike Killing vector becomes spacelike.

tion having negative energy with respect to an observer at infinity. Because only positive energy can be radiated at infinity, a time-dependent nonaxisymmetric perturbation with initial negative energy finds its energy growing increasingly negative; unless it can settle down to a stationary nonaxisymmetric state (which would then represent a point of marginal instability), the energy will decrease without bound and the amplitude of the linear perturbation will therefore grow without bound within the ergoregion. Unlike the Newtonian instability which corresponds to a perturbation with angular dependence $e^{2i\varphi}$, the ergoregion instability appears to set in as a limit $m \rightarrow \infty$ of modes with angular dependence $e^{im\varphi}$; in other words, modes with the highest values of m become unstable first. Butterworth and Ipser (1975) have recently calculated numerical models of differentially rotating highly relativistic stars in which ergoregions occur for eccentricities as small as $e = 0.25$ and angular velocities $\Omega^2/\pi G\rho = 0.26$. Because these values are well below those corresponding to Newtonian secular instability, the ergoregion instability is likely to be the exact limit on rotation for ultrarelativistic configurations.

The ergoregion instability also implies that in any sequence of models approaching as a limit a rotating black hole (e.g., Bardeen and Wagoner 1971), the last part of the sequence must be unstable. This is complementary to Buchdahl's (1959) theorem that no spherical star (perfect fluid) can have a radius less than 9/8 of its Schwarzschild radius. Black holes are thus isolated from other stable equilibria, and cannot form quasi-statically. Moreover, because the ergoregion appears at low values of a/M (where $a = cJ/GM$ is the angular momentum per unit mass in gravitational units), black holes may generally form with angular momenta well below the limit $a = M$.

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ERRATUM

The *Letter,* "Gravitational Radiation Instability in Rotating Stars," (*Ap. J.* [*Letters*], **199,** L157–L159 [1975]) by John L. Friedman and Bernard F. Schutz underestimated the growth time of the instability by a factor of 10^4. Equation (1) should read

$$\tau \sim 10^{-4} \frac{R}{c} \left(\frac{R\Omega}{c} \right)^{-6} (t - t_c)^{-5} ,$$

and all time scales should likewise be changed. Thus, white dwarf spin-down times range from 10^3 years to 10^7 years, while for neutron stars the times are between 1 second and 3 years. These are, as pointed out in the *Letter,* very uncertain and are likely to be overestimates. These changes do not affect our conclusions; in particular, a massive white dwarf at the center of the Cyg X-1 accretion disk should still collapse within 10^3 years.