

Towards field theory amplitudes from the cohomology of pure spinor superspace

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ABSTRACT: A simple BRST-closed expression for the color-ordered super-Yang-Mills 5-point amplitude at tree-level is proposed in pure spinor superspace and shown to be BRST-equivalent to the field theory limit of the open superstring 5-pt amplitude. It is manifestly cyclic invariant and each one of its five terms can be associated to the five Feynman diagrams which use only cubic vertices. Its form also suggests an empirical method to find superspace expressions in the cohomology of the pure spinor BRST operator for higher-point amplitudes based on their kinematic pole structure. Using this method, Ansätze for the 6- and 7-point 10D super-Yang-Mills amplitudes which map to their 14 and 42 color-ordered diagrams are conjectured and their 6- and 7-gluon expansions are explicitly computed.

KEYWORDS: Superstrings and Heterotic Strings, Superspaces, BRST Symmetry

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1 Introduction

As Parke and Taylor have shown for MHV amplitudes [1], it is sometimes possible to obtain simple expressions for seemingly complicated Yang-Mills amplitudes in four space-time dimensions. Using the pure spinor formalism [2] and its pure spinor superspace [3] (see also [4]) it will be proved that the tree-level color-ordered five-point super-Yang-Mills amplitude in ten dimensions can be written simply as

$$\mathcal{A}_5(1, 2, 3, 4, 5) = \frac{\langle L_{45} L_{12} V^3 \rangle}{s_{45} s_{12}} + \text{cyclic}(12345), \tag{1.1}$$

where V^j is the unintegrated massless vertex operator and L_{ij} is related to the OPE of a unintegrated and an integrated vertex operator in a way to be defined below.

It will also be suggested that higher-point amplitudes might have simple forms like the above, as there seems to be a direct correspondence between superspace expressions and Feynman diagrams which use only cubic vertices as in the arguments of [5]. Using the empirical method described in subsection 3.1, it will be argued that the super-Yang-Mills 6- and 7-point color-ordered amplitudes are proportional to

$$\begin{aligned} \mathcal{A}_6(1, 2, 3, 4, 5, 6) = & \frac{\langle L_{12} L_{34} L_{56} \rangle}{3s_1 s_3 s_5} \tag{1.2} \\ & + \frac{1}{2} \frac{\langle T_{123} (V^4 L_{56} + L_{45} V^6) \rangle}{s_1 t_1 s_5} + \frac{L_{45} V^6}{s_4} - \frac{1}{2} \frac{\langle T_{126} (V^3 L_{45} + L_{34} V^5) \rangle}{s_1 t_3 s_4} + \frac{L_{34} V^5}{s_3} + \text{cyclic}(1 \dots 6) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_7(1, 2, 3, 4, 5, 6, 7) = & \frac{\langle T_{231} L_{45} L_{67} \rangle}{s_2 t_1 s_4 s_6} + \frac{\langle T_{123} T_{564} V_7 \rangle}{s_1 t_1 s_5 t_4} + \frac{\langle T_{127} T_{345} V_6 \rangle}{s_1 t_7 s_3 t_3} \\ & - \frac{\langle T_{123} T_{456} V_7 \rangle}{s_1 t_1 s_4 t_4} - \frac{\langle T_{127} T_{453} V_6 \rangle}{s_1 t_7 s_4 t_3} - \frac{\langle T_{123} L_{45} L_{67} \rangle}{s_1 t_1 s_4 s_6} + \text{cyclic}(1 \dots 7) \end{aligned} \quad (1.3)$$

where T_{ijk} is related to the OPE of one unintegrated and two integrated vertices in a way to be defined below and s_1, \dots, s_6 and t_1, \dots, t_3 (s_1, \dots, s_7 and t_1, \dots, t_7) are the 6-point (7-point) generalized Mandelstam variables of [6, 7]. Using a computer program [8], the 6- and 7-gluon expansions of (1.2) and (1.3) are computed in appendix B.¹

Furthermore, given that the tree-level SYM 4-point amplitude can be written as [12]

$$\mathcal{A}_4(1, 2, 3, 4) = \frac{1}{s_{12}} \langle L_{12} V^3 V^4 \rangle + \frac{1}{s_{41}} \langle L_{41} V^2 V^3 \rangle, \quad (1.4)$$

it is pointed out that the four-point Jacobi-like Bern-Carrasco-Johansson kinematic identity [5] becomes

$$\langle L_{\{12} V_3 \} V_4 \rangle = 0, \quad (1.5)$$

where $\{ijk\}$ means a sum over cyclic permutations of (ijk) . Its vanishing is explained by noting that it is BRST trivial. For the five-point amplitude (1.1), the generalized BCJ identities of [13, 14] hold in the form of

$$-\frac{L_{45}}{s_{45}} L_{\{12} V_3 \} } + \frac{L_{42}}{s_{24}} L_{\{13} V_5 \} } - \frac{L_{12}}{s_{12}} L_{\{34} V_5 \} } + \frac{L_{51}}{s_{51}} L_{\{23} V_4 \} } = 0, \quad (1.6)$$

etc. It is well-known that there are powerful four-dimensional methods to compute scattering amplitudes recursively (see [15] and references therein). The hints of a simplified ten-dimensional parametrization of field theory tree-level amplitudes using pure spinors² seem to suggest that there might be similar methods in a ten-dimensional pure spinor superspace setup — which is desirable since there is no need to differentiate between MHV and NMHV contributions as in the four-dimensional methods.

This paper is organized as follows. In section 2 an ansatz will be given for the tree-level five-point SYM amplitude by analogy with the structure of the known four-point amplitude. In section 3 the five-point ansatz will be derived from the field theory limit of a BRST-equivalent expression of the superstring amplitude computed in [12]. In subsection 3.1 an empirical method to write down similar Ansätze for higher-point amplitudes is presented, and expressions for the 6- and 7-point super-Yang-Mills amplitudes in ten-dimensional space-time are conjectured. In appendix A the BCJ kinematic relations and

¹In the amplitude computations of [6, 7] the results were written in the 4D helicity formalism language, so a 10D comparison of results is not straightforward. However a comparison to the result [9] should be made [10]. After the first version of this paper came out, the 6-gluon amplitude has been successfully matched against the results of Zvi Bern, which he kindly provided [11]. The 7-gluon amplitude still remains to be checked.

²It was suggested a long time ago that pure spinors simplify the description of super-Yang-Mills and supergravity theories [16, 17]. The superspace results obtained with the pure spinor formalism seem to realize those expectations.

its generalization [13, 14] are written down using the pure spinor representations of the previous sections. Finally, in appendix B the first few terms of the (rather long) 5-, 6- and 7-gluon expansions from (1.1), (1.2) and (1.3) are written down (the full expansions can be easily generated with a computer using [8] or other methods).

2 Tree-level amplitudes with the pure spinor formalism

The prescription to compute n-point tree-level open string amplitudes with the pure spinor formalism is given by [2]³

$$\mathcal{A}_n = \langle V^1(0)V^{(n-1)}(1)V^n(\infty) \int dz_2 U^2(z_2) \dots \int dz_{(n-2)} U^{(n-2)}(z_{(n-2)}) \rangle, \quad (2.1)$$

where $V^i(z) = \lambda^\alpha A_\alpha^i$ and $U^i(z) = \partial\theta^\alpha A_\alpha^i + \Pi^m A_m^i + d_\alpha W_i^\alpha + \frac{1}{2}\mathcal{F}_{mn}^i N^{mn}$ are the unintegrated and integrated vertices with conformal weight zero and one, respectively, and i is the label denoting the different strings being scattered. The massless sector of the open superstring is described by the ten-dimensional super-Yang-Mills superfields $[A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}]$ which satisfy the equations of motion [4, 19, 20],

$$Q\mathcal{F}_{mn} = 2k_{[m}(\lambda\gamma_n]W), \quad QW^\alpha = \frac{1}{4}(\lambda\gamma^{mn})^\alpha \mathcal{F}_{mn}, \quad QA_m = (\lambda\gamma_m W) + k_m(\lambda A), \quad QV = 0, \quad (2.2)$$

where $\lambda^\alpha(z)$ is a pure spinor satisfying $\lambda^\alpha\gamma_{\alpha\beta}^m\lambda^\beta = 0$, $Q = \lambda^\alpha D_\alpha$ is the pure spinor BRST operator and $D_\alpha = \partial_\alpha + \frac{1}{2}k_m(\gamma^m\theta)_\alpha$ is the supersymmetric derivative.⁴ They have the following θ -expansions, [21–24]

$$\begin{aligned} A_\alpha(x, \theta) &= \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \dots \\ A_m(x, \theta) &= a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \dots \\ W^\alpha(x, \theta) &= \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta) + \frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} + \dots \\ \mathcal{F}_{mn}(x, \theta) &= F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \dots, \end{aligned} \quad (2.3)$$

where $a_m(x) = e_m e^{ik \cdot x}$, $\xi^\alpha(x) = \chi^\alpha e^{ik \cdot x}$ are the bosonic and fermionic polarizations and $F_{mn} = 2\partial_{[m}a_{n]}$ is the field-strength.

After using the OPE's to eliminate the conformal weight-one variables from (2.1), the integration of the zero-modes of λ^α and θ^α is carried out by taking only the terms which contain three λ 's and five θ 's in the correlator which are proportional to the pure spinor measure

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 1, \quad (2.4)$$

³For background material in the pure spinor formalism, see [18, 19]. The conventions for the OPE's however follow the appendix A of [12].

⁴In what follows spinor index contractions are denoted by parenthesis, e.g. $\lambda^\alpha D_\alpha = (\lambda D)$ and the worldsheet positions are mostly omitted.

where the normalization can be chosen arbitrarily.⁵ The normalization condition (2.4) defines the action of the pure spinor angle-brackets $\langle \cdot \rangle$. Arbitrary *pure spinor superspace* expressions are written down as

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \rangle, \tag{2.5}$$

where $f_{\alpha\beta\gamma}(\theta)$ is given in terms of super-Yang-Mills superfields, e.g. $f_{\alpha\beta\gamma}(\theta) = A_\alpha^i(\theta)A_\beta^j(\theta)A_\gamma^k(\theta)$. The measure (2.4) is in the cohomology of the pure spinor BRST operator and can not be written as the supersymmetry variation of a BRST-closed object, so amplitudes computed from (2.1) are supersymmetric [2].

As an illustration of the above steps, the supersymmetric tree-level 3-point amplitude following from (2.1) is given by⁶

$$\mathcal{A}_3 = \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) \rangle. \tag{2.6}$$

Evaluating the explicit component expansion for e.g. the 3-gluon amplitude, is a matter of plugging in the expansions (2.3) and selecting the components with five θ 's which contain the gluon fields. Doing that one obtains,

$$\mathcal{A}_3 = -\frac{1}{64} (k_m^3 e_r^1 e_s^2 e_n^3 - k_m^2 e_r^1 e_n^2 e_s^3 + k_m^1 e_n^1 e_r^2 e_s^3) \langle (\lambda \gamma^r \theta)(\lambda \gamma^s \theta)(\lambda \gamma_p \theta)(\theta \gamma^{pmn} \theta) \rangle. \tag{2.7}$$

As mentioned in the appendix of [26], symmetry arguments and the normalization condition (2.4) fix all pure spinor correlators. Among the list of [26] one finds

$$\langle (\lambda \gamma^r \theta)(\lambda \gamma^s \theta)(\lambda \gamma_p \theta)(\theta \gamma^{pmn} \theta) \rangle = \frac{1}{120} \delta_{pmn}^{rsp} = \frac{1}{45} \delta_{mn}^{rs},$$

so the 3-gluon amplitude (2.7) is given by

$$\mathcal{A}_3 = -\frac{1}{2880} ((e^1 \cdot e^2)(k^2 \cdot e^3) + (e^1 \cdot e^3)(k^1 \cdot e^2) + (e^2 \cdot e^3)(k^3 \cdot e^1)). \tag{2.8}$$

Given the systematic nature of the above procedure, an implementation using FORM [27, 28] has been written which performs these expansions automatically [8]. So although component expansions can have many thousand terms as in the 7-gluon amplitude discussed in appendix B, they come from much simpler superspace expressions which can be analysed by hand.

3 The 5-pt field theory amplitude ansatz

When the amplitude involves more than three strings, the prescription (2.1) requires the computation of the OPE's with integrated vertices. In this section we will be concerned with the field theory limit (FT) of the string scattering. The 5-point FT amplitude will

⁵See however the tree-level, one-loop and two-loop calculations of [25] to check how the choice has to be taken into account at higher-loops.

⁶One also has to evaluate the functional integration of the exponentials $\prod : e^{ik^i \cdot X(z_i)} :$, but they will not appear explicitly in this paper.

be given an Ansatz motivated by the superspace form of the FT 4-point amplitude, which will later be obtained from a BRST equivalent expression of the first principles superstring 5-point amplitude evaluated in [12].

In superspace, the OPE between the unintegrated and integrated vertex operators is given by $V^i(z)U^j(w) \rightarrow \frac{\tilde{L}_{ij}}{z-w}$, with [29]

$$\tilde{L}_{ij}(\theta) = A_m^i(\lambda\gamma^m W^j) + (\lambda A^i)(k^i \cdot A^j). \quad (3.1)$$

Using the equations of motion (2.2) it follows that

$$Q\tilde{L}_{ij} = -s_{ij}(\lambda A^i)(\lambda A^j), \quad Q(A^i \cdot A^j) = \tilde{L}_{ij} + \tilde{L}_{ji} \equiv 2\tilde{L}_{(ij)} \quad (3.2)$$

where⁷ $s_{ij} = (k^i \cdot k^j)$. Using (3.2) and defining $L_{ij} = 1/2(\tilde{L}_{ij} - \tilde{L}_{ji})$ the superfield \tilde{L}_{ij} can be written as⁸

$$\tilde{L}_{ij} = L_{ij} + \frac{1}{2}Q(A^i \cdot A^j). \quad (3.3)$$

The massless 4-point super-Yang-Mills amplitude obtained from the field theory limit of the open string amplitude is given by [12]

$$\mathcal{A}(1, 2, 3, 4) = \frac{1}{s_{12}}\langle\tilde{L}_{12}V^3V^4\rangle + \frac{1}{s_{41}}\langle\tilde{L}_{41}V^2V^3\rangle = \frac{1}{s_{12}}\langle L_{12}V^3V^4\rangle + \frac{1}{s_{41}}\langle L_{41}V^2V^3\rangle \quad (3.4)$$

where we used that $\langle Q(A^i \cdot A^j)V^kV^l\rangle = 0$, which follows from integrating the BRST charge by parts. The other sub-amplitudes are obtained from (3.4) by relabeling,

$$\begin{aligned} \mathcal{A}(1, 3, 4, 2) &= -\frac{1}{s_{13}}\langle L_{13}V^2V^4\rangle - \frac{1}{s_{12}}\langle L_{12}V^3V^4\rangle \\ \mathcal{A}(1, 4, 2, 3) &= -\frac{1}{s_{14}}\langle L_{41}V^2V^3\rangle + \frac{1}{s_{13}}\langle L_{13}V^2V^4\rangle. \end{aligned} \quad (3.5)$$

It is easy to check that the amplitudes in (3.5) are BRST-closed.

As emphasized in [5], a color-ordered 5-point tree-level amplitude consists of five diagrams with purely cubic vertices specifying the poles,

$$\mathcal{A}(1, 2, 3, 4, 5) = \frac{n_1}{s_{45}s_{12}} + \frac{n_2}{s_{51}s_{23}} + \frac{n_3}{s_{12}s_{34}} + \frac{n_4}{s_{23}s_{45}} + \frac{n_5}{s_{34}s_{51}}. \quad (3.6)$$

As the BRST variation of L_{ij} is proportional to s_{ij} , the idea now is to construct a pure spinor superspace expression using L_{ij} and L_{kl} in the numerators of the terms containing poles in s_{ij} and s_{kl} , in such a way as to obtain a BRST-closed expression. It is straight-

⁷Note that the usual definition for massless particles is $s_{ij} = 2(k^i \cdot k^j)$.

⁸I thank Dimitrios Tsimpis for suggesting the separation of the BRST-trivial part of \tilde{L}_{ij} .

forward to see that the amplitudes

$$\begin{aligned}
 \mathcal{A}(1, 2, 3, 4, 5) &= \frac{\langle L_{45}L_{12}V^3 \rangle}{s_{45}s_{12}} + \frac{\langle L_{51}L_{23}V^4 \rangle}{s_{51}s_{23}} + \frac{\langle L_{12}L_{34}V^5 \rangle}{s_{12}s_{34}} + \frac{\langle L_{23}L_{45}V^1 \rangle}{s_{23}s_{45}} + \frac{\langle L_{34}L_{51}V^2 \rangle}{s_{34}s_{51}} \\
 \mathcal{A}(1, 3, 2, 4, 5) &= \frac{\langle L_{45}L_{13}V^2 \rangle}{s_{45}s_{13}} - \frac{\langle L_{51}L_{23}V^4 \rangle}{s_{51}s_{23}} - \frac{\langle L_{13}L_{42}V^5 \rangle}{s_{13}s_{24}} - \frac{\langle L_{23}L_{45}V^1 \rangle}{s_{23}s_{45}} - \frac{\langle L_{42}L_{51}V^3 \rangle}{s_{24}s_{51}} \\
 \mathcal{A}(1, 4, 3, 2, 5) &= \frac{\langle L_{25}L_{14}V^3 \rangle}{s_{25}s_{14}} + \frac{\langle L_{34}L_{51}V^2 \rangle}{s_{51}s_{43}} + \frac{\langle L_{23}L_{14}V^5 \rangle}{s_{14}s_{32}} + \frac{\langle L_{25}L_{34}V^1 \rangle}{s_{43}s_{25}} + \frac{\langle L_{51}L_{23}V^4 \rangle}{s_{32}s_{51}} \\
 \mathcal{A}(1, 3, 4, 2, 5) &= \frac{\langle L_{25}L_{13}V^4 \rangle}{s_{25}s_{13}} - \frac{\langle L_{34}L_{51}V^2 \rangle}{s_{51}s_{34}} + \frac{\langle L_{13}L_{42}V^5 \rangle}{s_{13}s_{42}} - \frac{\langle L_{25}L_{34}V^1 \rangle}{s_{34}s_{25}} + \frac{\langle L_{42}L_{51}V^3 \rangle}{s_{42}s_{51}} \\
 \mathcal{A}(1, 2, 4, 3, 5) &= \frac{\langle L_{35}L_{12}V^4 \rangle}{s_{35}s_{12}} + \frac{\langle L_{42}L_{51}V^3 \rangle}{s_{51}s_{43}} - \frac{\langle L_{12}L_{34}V^5 \rangle}{s_{12}s_{43}} + \frac{\langle L_{35}L_{42}V^1 \rangle}{s_{42}s_{35}} - \frac{\langle L_{34}L_{51}V^2 \rangle}{s_{43}s_{51}} \\
 \mathcal{A}(1, 4, 2, 3, 5) &= \frac{\langle L_{35}L_{14}V^2 \rangle}{s_{35}s_{14}} - \frac{\langle L_{42}L_{51}V^3 \rangle}{s_{51}s_{24}} - \frac{\langle L_{23}L_{14}V^5 \rangle}{s_{14}s_{23}} - \frac{\langle L_{35}L_{42}V^1 \rangle}{s_{24}s_{35}} - \frac{\langle L_{51}L_{23}V^4 \rangle}{s_{23}s_{51}}
 \end{aligned} \tag{3.7}$$

are BRST-closed. One can also check that all sub-amplitudes in (3.7) are related to $\mathcal{A}(1, 2, 3, 4, 5)$ by index relabeling, taking into account the antisymmetry of L_{ij} and its fermionic nature. The signs in (3.7) precisely match the ones presented in equation (4.5) of [5], so one can identify

$$\begin{aligned}
 n_1 &= \langle L_{45}L_{12}V^3 \rangle, & n_2 &= \langle L_{51}L_{23}V^4 \rangle, & n_3 &= \langle L_{12}L_{34}V^5 \rangle, & n_4 &= \langle L_{23}L_{45}V^1 \rangle \\
 n_5 &= \langle L_{34}L_{51}V^2 \rangle, & n_6 &= \langle L_{25}L_{14}V^3 \rangle, & n_7 &= \langle L_{23}L_{14}V^5 \rangle, & n_8 &= \langle L_{25}L_{34}V^1 \rangle \\
 n_9 &= \langle L_{25}L_{13}V^4 \rangle, & n_{10} &= \langle L_{13}L_{42}V^5 \rangle, & n_{11} &= \langle L_{42}L_{51}V^3 \rangle, & n_{12} &= \langle L_{35}L_{12}V^4 \rangle \\
 n_{13} &= \langle L_{35}L_{42}V^1 \rangle, & n_{14} &= \langle L_{35}L_{14}V^2 \rangle, & n_{15} &= \langle L_{45}L_{13}V^2 \rangle.
 \end{aligned} \tag{3.8}$$

As will be mentioned in the appendix, the above “solution” for the n_i ’s of [5] do not satisfy the strict Bern-Carrasco-Johansson (BCJ) kinematic identities, but they do satisfy the generalized BCJ’s of [13, 14]. As explained in [13, 14], a general parametrization of the sub-amplitudes in terms of poles does not necessarily satisfy the BCJ Jacobi-like identities of [5]. They must however satisfy “generalized BCJ identities”, for which the original BCJ relations are just one out of many possible solutions.

The amplitudes in (3.7) will now be obtained from the field theory limit of a BRST-equivalent expression of the pure spinor superstring amplitude computed in [12].

4 First principles derivation of the 5-pt ansatz (3.7)

The massless 5-point open superstring amplitude is given by [12]⁹

$$\begin{aligned}
 \mathcal{A}_5(1, 2, 3, 4, 5) &= \langle L_{2131}V^4V^5 \rangle K_1 - \langle L_{2134}V^5 \rangle K_2 - \langle L_{2434}V^1V^5 \rangle K'_1 + \langle L_{2431}V^5 \rangle K_3 \\
 &\quad - \langle L_{2331}V^4V^5 \rangle K_5 - \langle L_{2334}V^1V^5 \rangle K'_4 + \langle D_{23}V^1V^4V^5 \rangle (1 + s_{23}) K_6,
 \end{aligned} \tag{4.1}$$

⁹The notation here slightly differs from [12], but should not lead to confusion.

where K_j and K'_j denote integrals which satisfy [30, 31]

$$\begin{aligned} s_{34}K_2 &= s_{13}K_1 + s_{23}K_4, & s_{24}K_3 &= s_{12}K_1 - s_{23}K_5, & K_1 &= K_4 - K_5 \\ s_{12}K_2 &= s_{24}K'_1 + s_{23}K'_4, & s_{13}K_3 &= s_{34}K'_1 - s_{23}K'_5, & K'_1 &= K'_4 - K'_5 \\ (1 + s_{23})K_6 &= s_{34}K'_4 - s_{13}K_5 = s_{12}K_4 - s_{24}K'_5. \end{aligned} \quad (4.2)$$

The various L_{ijkl} kinematic building blocks have the following expressions¹⁰

$$L_{2131} = +\tilde{L}_{12}((k^1 + k^2) \cdot A^3) + (\lambda\gamma^m W^3)[A_m^1(k^1 \cdot A^2) + A^{1n}\mathcal{F}_{mn} - (W^1\gamma_m W^2)] \quad (4.3)$$

$$L_{2134} = \tilde{L}_{12}\tilde{L}_{43}, \quad D_{23} = -(A^2 \cdot A^3). \quad (4.4)$$

Relabeling $1 \leftrightarrow 4$ determines L_{2434} from (4.3) and L_{2431} from (4.4). Finally, the OPE identities of [12] (which are related to the BCJ dualities of [5]) imply that

$$L_{2331} = L_{3121} - L_{2131}, \quad L_{2334} = L_{3424} - L_{2434}, \quad (4.5)$$

which are used to obtain the remaining kinematic factors appearing in (4.1) from the expression for (4.3) and relabelings thereof.

Using the integral relation for K_6 and the expression for D_{23} ,

$$\langle D_{23}V^1V^4V^5 \rangle K_6 = -(1 + s_{23})K_6 \langle (A^2 \cdot A^3)V^1V^4V^5 \rangle = (s_{13}K_5 - s_{34}K'_4) \langle (A^2 \cdot A^3)V^1V^4V^5 \rangle$$

the amplitude (4.1) becomes

$$\begin{aligned} \mathcal{A}_5(1, 2, 3, 4, 5) &= \langle L_{2131}V^4V^5 \rangle K_1 - \langle L_{2134}V^5 \rangle K_2 - \langle L_{2434}V^1V^5 \rangle K'_1 + \langle L_{2431}V^5 \rangle K_3 \\ &\quad - \langle (L_{2331} - s_{13}(A^2 \cdot A^3)V^1)V^4V^5 \rangle K_5 - \langle (L_{2334} - s_{34}(A^2 \cdot A^3)V^4)V^1V^5 \rangle K'_4. \end{aligned} \quad (4.6)$$

A key point is to note from (4.3) is that it obeys the identity

$$QL_{2131} = s_{12}(\tilde{L}_{23}V_1 - \tilde{L}_{13}V_1 + \tilde{L}_{12}V_3) - (s_{12} + s_{13} + s_{23})\tilde{L}_{12}V_3, \quad (4.7)$$

and by defining¹¹

$$T_{ijk} \equiv L_{jiki} - S_{jiki}, \quad S_{jiki} = \frac{1}{2}s_{ij}((A^j \cdot A^k)V^i - (A^i \cdot A^k)V^j) - \frac{1}{2}(s_{ik} + s_{jk})(A^i \cdot A^j)V^k, \quad (4.8)$$

the BRST-trivial parts from L_{jiki} are removed and one obtains a BRST variation written in terms of L_{ij} instead of \tilde{L}_{ij} ,

$$QT_{ijk} = s_{ij}L_{\{ij}V_k\} - (s_{jk} + s_{ki} + s_{ij})L_{ij}V_k. \quad (4.9)$$

¹⁰In the computations of [12] there were terms with factors of $(A^i W^j)V^k$ in the expressions for L_{jiki} . But it was shown that using the relations (4.2) those terms drop out from the amplitude, so they are not written in this paper for brevity.

¹¹I thank Dimitrios Tsipiras for suggesting the relevance of using this definition in the context of an ansatz for the 6-pt amplitude. It turns out to clean up the 5-pt formulæ too.

Furthermore, using (4.9) it is easy to show that $Q(T_{jik} - T_{jki} - T_{kij}) = 0$. In fact this combination is BRST-trivial,

$$T_{jik} - T_{jki} - T_{kij} = Q \left((A^i \cdot A^j)(k^i \cdot A^k) - (A^i \cdot A^k)(k^i \cdot A^j) - (A^j \cdot A^k)(k^k \cdot A^i) \right). \quad (4.10)$$

Using the definitions (3.3), (4.8), the relations (4.2) obeyed by the integrals and the identity (4.10) the superstring five point amplitude (4.6) becomes

$$\begin{aligned} \mathcal{A}_5(1, 2, 3, 4, 5) &= \langle L_{12}L_{34}V_5 \rangle K_2 + \langle L_{13}L_{24}V_5 \rangle K_3 \\ &+ \langle T_{123}V_4V_5 \rangle K_1 - \langle T_{432}V_1V_5 \rangle K'_1 + \langle T_{321}V_4V_5 \rangle K_5 - \langle T_{234}V_1V_5 \rangle K'_5. \end{aligned} \quad (4.11)$$

As discussed in [30, 31], under the twist $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ of the vertex operators on the disc, the integrals behave as

$$K_1 \leftrightarrow K'_1, \quad K_4 \leftrightarrow K'_4, \quad K_5 \leftrightarrow K'_5, \quad K_2 \leftrightarrow K_2, \quad K_3 \leftrightarrow K_3, \quad (4.12)$$

from which one can easily check that the 5-pt superstring amplitude (4.11) is anti-symmetric, as it should on general grounds.

Writing the five point integrals in the two dimensional basis (T, K_3) of [30, 31] where

$$T = s_{12}s_{34}K_2 + (s_{12}s_{51} - s_{12}s_{34} + s_{34}s_{45})K_3 \quad (4.13)$$

as follows [12]

$$K_1 = \frac{T}{s_{12}s_{45}} - \left(\frac{s_{34}}{s_{12}} + \frac{s_{23}}{s_{45}} \right) K_3, \quad K'_1 = \frac{T}{s_{34}s_{51}} - \left(\frac{s_{12}}{s_{34}} + \frac{s_{23}}{s_{51}} \right) K_3 \quad (4.14)$$

$$K_5 = \frac{T}{s_{23}s_{45}} - \left(\frac{s_{12}}{s_{45}} + \frac{s_{51}}{s_{23}} - 1 \right) K_3, \quad K'_5 = \frac{T}{s_{23}s_{51}} - \left(\frac{s_{34}}{s_{51}} + \frac{s_{45}}{s_{23}} - 1 \right) K_3 \quad (4.15)$$

the amplitude (4.11) becomes

$$\mathcal{A}_5(1, 2, 3, 4, 5) = T A_{\text{YM}}(\theta) + K_3 A_{F^4}(\theta), \quad (4.16)$$

where,

$$A_{\text{YM}}(\theta) = \frac{\langle T_{123}V^4V^5 \rangle}{s_{12}s_{45}} - \frac{\langle T_{234}V^1V^5 \rangle}{s_{23}s_{51}} + \frac{\langle L_{12}L_{34}V^5 \rangle}{s_{12}s_{34}} + \frac{\langle T_{321}V^4V^5 \rangle}{s_{23}s_{45}} - \frac{\langle T_{432}V^1V^5 \rangle}{s_{34}s_{51}} \quad (4.17)$$

and

$$\begin{aligned} A_{F^4}(\theta) &= \langle L_{12}L_{34}V^5 \rangle + \langle L_{13}L_{24}V^5 \rangle - \langle T_{234}V^1V^5 \rangle + \langle T_{321}V^4V^5 \rangle \\ &- \langle L_{12}L_{34}V^5 \rangle \left(\frac{s_{45}}{s_{12}} + \frac{s_{51}}{s_{34}} \right) - \langle T_{123}V^4V^5 \rangle \left(\frac{s_{34}}{s_{12}} + \frac{s_{23}}{s_{45}} \right) + \langle T_{234}V^1V^5 \rangle \left(\frac{s_{45}}{s_{23}} + \frac{s_{34}}{s_{51}} \right) \\ &- \langle T_{321}V^4V^5 \rangle \left(\frac{s_{51}}{s_{23}} + \frac{s_{12}}{s_{45}} \right) + \langle T_{432}V^1V^5 \rangle \left(\frac{s_{23}}{s_{51}} + \frac{s_{12}}{s_{34}} \right). \end{aligned} \quad (4.18)$$

One can also find a BRST-equivalent form for the amplitude by using the fact that $Q(L_{mn}/s_{mn}) = -V^mV^n$ to rewrite $\langle T_{ijk}V^mV^n \rangle$ as $-\langle T_{ijk}Q(L_{mn}/s_{mn}) \rangle$, which upon integration of the BRST charge by parts using (4.9) implies that

$$\langle T_{ijk}V_mV_n \rangle = -\left\langle \frac{L_{mn}}{s_{mn}} (s_{ij}L_{\{ij}V_k\}} - s_{ijk}L_{ij}V_k) \right\rangle. \quad (4.19)$$

A somewhat tedious but straightforward use of (4.19) in the expressions (4.17) and (4.18) allows them to be rewritten as

$$A_{\text{YM}}(\theta) = \frac{\langle L_{45}L_{12}V^3 \rangle}{s_{45}s_{12}} + \frac{\langle L_{51}L_{23}V^4 \rangle}{s_{51}s_{23}} + \frac{\langle L_{12}L_{34}V^5 \rangle}{s_{12}s_{34}} + \frac{\langle L_{23}L_{45}V^1 \rangle}{s_{23}s_{45}} + \frac{\langle L_{34}L_{51}V^2 \rangle}{s_{34}s_{51}} \quad (4.20)$$

and

$$\begin{aligned} A_{F^4}(\theta) = & -\langle L_{45}L_{12}V^3 \rangle \left(\frac{s_{23}}{s_{45}} + \frac{s_{34}}{s_{12}} \right) - \langle L_{51}L_{23}V^4 \rangle \left(\frac{s_{34}}{s_{15}} + \frac{s_{45}}{s_{23}} \right) \\ & - \langle L_{12}L_{34}V^5 \rangle \left(\frac{s_{45}}{s_{12}} + \frac{s_{51}}{s_{34}} \right) - \langle L_{23}L_{45}V^1 \rangle \left(\frac{s_{51}}{s_{23}} + \frac{s_{12}}{s_{45}} \right) - \langle L_{34}L_{51}V^2 \rangle \left(\frac{s_{12}}{s_{34}} + \frac{s_{23}}{s_{51}} \right) \\ & + \langle L_{12}L_{34}V^5 + L_{51}L_{23}V^4 - L_{13}L_{42}V^5 + L_{23}L_{45}V^1 \rangle \\ & + \frac{s_{13}}{s_{51}} \langle L_{51}L_{\{23}V_4\} \rangle - \frac{s_{24}}{s_{45}} \langle L_{45}L_{\{12}V_3\} \rangle. \end{aligned} \quad (4.21)$$

In the field theory limit $T \rightarrow 1$ and $K_3 \rightarrow 0$ [30, 31], so the first principles derivation of (3.7) is completed. The 5-gluon component expansion were already computed in [12], and shown to agree with earlier RNS results [30, 31].

4.1 Higher-point amplitudes

It is worth checking whether the simple mappings between the cubic Feynman diagrams and pure spinor building blocks persist at higher-points. The discussion in section 2 suggests a way to write down n-point field theory amplitudes. For each one of the $2^{n-2}(2n-5)!!/(n-1)!$ color-ordered diagrams specifying the kinematic poles [5], a ghost-number-three numerator whose BRST transformation is proportional to those poles should be written down. One then tries to find a combination with the correct dimension of a n-point amplitude such that the sum of all diagrams is BRST-closed.

To help finding candidates for superfield building blocks, the first principles tree-level superstring amplitude prescription [2, 32] can be used as guide. For example, the superfield \tilde{L}_{ij} appears in the OPE of $V^i(z)U^j(w)$ in the 4-pt string amplitude [29], and its BRST transformation $Q\tilde{L}_{ij} = -s_{ij}V^iV^j$ has precisely the Mandelstam variable to cancel poles in the 5-pt amplitude. Similarly, the superfield L_{jiki} comes from the numerator of the $1/z_{ij}z_{ik}$ pole in the OPE $V^i(z_i)U^j(z_j)U^k(z_k)$ appearing in the 5-pt computation [12], and its BRST transformation has the required Mandelstam variables to cancel poles in the 6-pt amplitude,

$$QL_{jiki} = s_{ij}(\tilde{L}_{jk}V^i - \tilde{L}_{ik}V^j + \tilde{L}_{ij}V^k) - (s_{jk} + s_{ki} + s_{ij})\tilde{L}_{ij}V^k. \quad (4.22)$$

As the expressions must be in the cohomology of the pure spinor BRST operator, one also removes the BRST-trivial parts of the building blocks \tilde{L}_{ij} and L_{jiki} , using L_{ij} and T_{ijk} instead.

Following the above procedure for the 14 color-ordered diagrams of the 6-point amplitude which are generated from the cyclic permutations of the diagrams in figures 1, 2

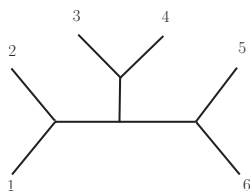


Figure 1. The diagram associated with $\langle \frac{L_{12} L_{34} L_{56}}{s_1 s_3 s_5} \rangle$.

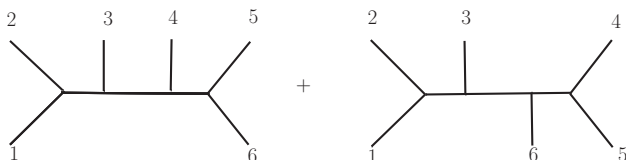


Figure 2. The diagrams associated with $\langle \frac{T_{123}}{s_1 t_1} \left(\frac{V_4 L_{56}}{s_5} + \frac{L_{45} V_6}{s_4} \right) \rangle$.

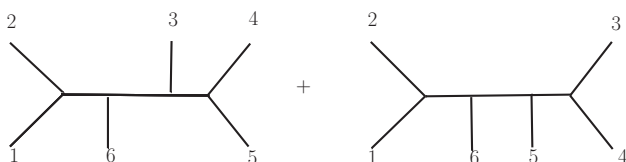


Figure 3. The diagrams associated with $\langle \frac{T_{126}}{s_1 t_3} \left(\frac{V_3 L_{45}}{s_4} + \frac{L_{34} V_5}{s_3} \right) \rangle$.

and **3**, a BRST-closed expression with the correct pole structure looks like¹²

$$\begin{aligned}
 \mathcal{A}_6(1, 2, 3, 4, 5, 6) = & \frac{\langle L_{12} L_{34} L_{56} \rangle}{3s_1 s_3 s_5} \\
 & + \frac{1}{2} \frac{\langle T_{123} (V^4 L_{56} + L_{45} V^6) \rangle}{s_1 t_1 s_5} + \frac{1}{s_4} - \frac{1}{2} \frac{\langle T_{126} (V^3 L_{45} + L_{34} V^5) \rangle}{s_1 t_3 s_4} + \frac{1}{s_3} + \text{cyclic}(1 \dots 6)
 \end{aligned} \tag{4.23}$$

where $s_1 = s_{12}, s_2 = s_{23}, \dots, s_6 = s_{61}, t_1 = (s_{12} + s_{23} + s_{13}), t_2 = (s_{23} + s_{34} + s_{24})$ and $t_3 = (s_{34} + s_{45} + s_{35})$ are the 6-point Mandelstam variables of [6]. The full component expansion for the 6-gluon amplitude obtained from (4.23) contains 6706 terms [8] and it was checked to be gauge invariant.¹³ The first few terms of this expansion are given in appendix B.

For the 7-point amplitude there are 6 diagrams which generate the 42 color-ordered cubic diagrams upon cyclic symmetrization. The corresponding BRST-closed expression with the correct pole structure is given by

$$\begin{aligned}
 \mathcal{A}_7(1, 2, 3, 4, 5, 6, 7) = & + \frac{\langle T_{231} L_{45} L_{67} \rangle}{s_2 t_1 s_4 s_6} + \frac{\langle T_{123} T_{564} V_7 \rangle}{s_1 t_1 s_5 t_4} + \frac{\langle T_{127} T_{345} V_6 \rangle}{s_1 t_7 s_3 t_3} \\
 & - \frac{\langle T_{123} T_{456} V_7 \rangle}{s_1 t_1 s_4 t_4} - \frac{\langle T_{127} T_{453} V_6 \rangle}{s_1 t_7 s_4 t_3} - \frac{\langle T_{123} L_{45} L_{67} \rangle}{s_1 t_1 s_4 s_6} + \text{cyclic}(1 \dots 7)
 \end{aligned} \tag{4.24}$$

¹²I thank Oliver Schlotterer and Dimitrios Tsimpis for many valuable discussions.

¹³After the first version of this paper appeared, Zvi Bern kindly provided his Mathematica file with the field theory 6-gluon amplitude written in terms of polarization and momenta. A perfect match was obtained.

where s_1, \dots, s_7 and t_1, \dots, t_7 are the 7-point Mandelstam variables of [7]. The ten-dimensional 7-gluon expansion of (4.24) contains more than 130 thousand terms [8] and a few are written in appendix B. As the results of [7] are written in the four-dimensional helicity formalism, a direct comparison with the results quoted there is not possible.

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A The Bern-Carrasco-Johansson kinematic identities

The 4-pt BCJ kinematic relation $n_u = n_s - n_t$ is mapped to the superspace expression $\langle L_{13} V^2 V^4 \rangle = \langle L_{12} V^3 V^4 \rangle - \langle L_{41} V^2 V^3 \rangle$. Using $\langle L_{41} V^2 V^3 \rangle = -\langle L_{23} V^1 V^4 \rangle$ it can be rewritten as

$$\langle L_{\{12} V_3 \} V^4 \rangle = 0, \quad (\text{A.1})$$

where $\{ijk\}$ means to sum over the cyclic permutation of the labels. Note that (A.1) can be explained from the fact that BRST-trivial quantities vanish. Explicitly,

$$0 = \langle Q(T_{123} V_4) \rangle = s \langle L_{\{12} V_3 \} V^4 \rangle - (s + t + u) \langle L_{12} V_3 V_4 \rangle, \quad (\text{A.2})$$

which implies (A.1) because $s + t + u = 0$.

The 5-pt *extended BCJ relations* of [13, 14] are given by

$$\frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_3 - n_1 + n_{12}}{s_{12}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} = 0 \quad (\text{A.3})$$

$$\frac{n_7 - n_6 + n_{14}}{s_{14}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_8 - n_6 + n_9}{s_{25}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} = 0 \quad (\text{A.4})$$

$$\frac{n_{10} - n_9 + n_{15}}{s_{13}} + \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_4 - n_2 + n_7}{s_{23}} + \frac{n_8 - n_6 + n_9}{s_{25}} = 0 \quad (\text{A.5})$$

$$\frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_9 + n_{15}}{s_{13}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_3 - n_5 + n_8}{s_{34}} = 0. \quad (\text{A.6})$$

Using the mappings of (3.8) they become

$$-\frac{L_{45}}{s_{45}} L_{\{12} V_3 \} + \frac{L_{42}}{s_{24}} L_{\{13} V_5 \} - \frac{L_{12}}{s_{12}} L_{\{34} V_5 \} + \frac{L_{51}}{s_{51}} L_{\{23} V_4 \} = 0, \quad (\text{A.7})$$

$$-\frac{L_{14}}{s_{14}} L_{\{23} V_5 \} + \frac{L_{42}}{s_{24}} L_{\{13} V_5 \} - \frac{L_{25}}{s_{25}} L_{\{13} V_4 \} + \frac{L_{51}}{s_{51}} L_{\{23} V_4 \} = 0, \quad (\text{A.8})$$

$$+\frac{L_{13}}{s_{13}} L_{\{25} V_4 \} - \frac{L_{51}}{s_{51}} L_{\{23} V_4 \} - \frac{L_{23}}{s_{23}} L_{\{14} V_5 \} + \frac{L_{25}}{s_{25}} L_{\{13} V_4 \} = 0, \quad (\text{A.9})$$

$$-\frac{L_{45}}{s_{45}} L_{\{12} V_3 \} - \frac{L_{13}}{s_{13}} L_{\{25} V_4 \} + \frac{L_{51}}{s_{51}} L_{\{23} V_4 \} + \frac{L_{34}}{s_{34}} L_{\{12} V_5 \} = 0, \quad (\text{A.10})$$

which one can check to hold true when expanding in components. Using the momentum conservation relations

$$\begin{aligned} s_{13} &= s_{45} - s_{12} - s_{23}, & s_{14} &= s_{23} - s_{51} - s_{45}, & s_{24} &= s_{51} - s_{23} - s_{34} \\ s_{25} &= s_{34} - s_{12} - s_{51}, & s_{35} &= s_{12} - s_{45} - s_{34}, \end{aligned} \tag{A.11}$$

one finds that the l.h.s. of (A.7)–(A.10) are BRST-closed.

B The 5-, 6- and 7-gluon amplitudes

The 5-gluon amplitude is easily obtained by using [8], and one can check that the first few terms are

$$\begin{aligned} 2880 \mathcal{A}_5(1, 2, 3, 4, 5) &= \tag{B.1} \\ &- (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(e^1 \cdot e^5) s_1^{-1} s_4^{-1} + (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^5)(e^1 \cdot e^4) s_1^{-1} s_4^{-1} \\ &- (k^1 \cdot e^2)(k^1 \cdot e^3)(k^2 \cdot e^4)(e^1 \cdot e^5) s_1^{-1} s_4^{-1} + (k^1 \cdot e^2)(k^1 \cdot e^3)(k^2 \cdot e^5)(e^1 \cdot e^4) s_1^{-1} s_4^{-1} \\ &- (k^1 \cdot e^2)(k^1 \cdot e^3)(k^3 \cdot e^4)(e^1 \cdot e^5) s_1^{-1} s_3^{-1} + \dots \end{aligned}$$

The 6-gluon component expansion from the ansatz (4.23) generates 6706 terms of which the first few are [8]

$$\begin{aligned} 2880 \mathcal{A}_6(1, 2, 3, 4, 5, 6) &= \tag{B.2} \\ &[(k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^6)(e^1 \cdot e^5) - (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^5)(e^1 \cdot e^6) \\ &- (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^2 \cdot e^5)(e^1 \cdot e^6) + (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^2 \cdot e^6)(e^1 \cdot e^5) \\ &- (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^3 \cdot e^5)(e^1 \cdot e^6) + (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^3 \cdot e^6)(e^1 \cdot e^5)] s_1^{-1} s_5^{-1} t_1^{-1} \\ &- (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^4 \cdot e^5)(e^1 \cdot e^6) s_1^{-1} s_4^{-1} t_1^{-1} + \dots \end{aligned}$$

Similarly, the 7-gluon component expansion of (4.24) has 134460 terms¹⁴ and the first ones are

$$\begin{aligned} 2880 \mathcal{A}_7(1, 2, 3, 4, 5, 6, 7) &= \tag{B.3} \\ &[+(k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^5)(k^1 \cdot e^6)(e^1 \cdot e^7) - (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^5)(k^1 \cdot e^7)(e^1 \cdot e^6) \\ &+ (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^5)(k^2 \cdot e^6)(e^1 \cdot e^7) - (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^5)(k^2 \cdot e^7)(e^1 \cdot e^6) \\ &+ (k^1 \cdot e^2)(k^1 \cdot e^3)(k^1 \cdot e^4)(k^1 \cdot e^5)(k^3 \cdot e^6)(e^1 \cdot e^7)] s_1^{-1} s_6^{-1} t_1^{-1} t_5^{-1} + \dots \end{aligned}$$

It is curious to note that the coefficient of $\pm 1/2880$ is the same for all the terms in the 5-, 6- and 7-gluon amplitudes alike. This is the same coefficient which was observed in [32] to be the conversion factor required to match the RNS amplitudes at tree-level.

¹⁴Some of those terms contain ϵ_{10} tensors and are expected to vanish once rules for the vanishing of things like $\epsilon_{10}^{[m_1 \dots m_{10}]}$ $\delta_n^{m_{11}}$ are implemented in [8].

C Shortcut to compute QL

There is a shortcut to compute QL 's for n -points using only the L 's appearing at $(n-1)$ -points. The definitions of \tilde{L}_{ij} and L_{jiki} are [12],

$$V^i(z_i)U^j(z_j) \rightarrow \frac{\tilde{L}_{ij}}{z_{ij}}, \quad \tilde{L}_{ij}(z_i)U^k(z_k) \rightarrow \frac{L_{jiki}}{z_{ik}}, \quad (\text{C.1})$$

so that $Q\tilde{L}_{ij} = \lim_{z_j \rightarrow z_i} z_{ij}Q(V^i(z_i)U^j(z_j))$ and $QL_{jiki} = \lim_{z_k \rightarrow z_i} z_{ik}Q(\tilde{L}_{ij}(z_i)U^k(z_k))$ leads to

$$\begin{aligned} Q\tilde{L}_{ij} &= \lim_{z_j \rightarrow z_i} z_{ij} \partial V^j(z_j) V^i(z_i) = -s_{ij} V^i V^j, \\ QL_{jiki} &= - \lim_{z_k \rightarrow z_i} z_{ik} (s_{ij} V^i(z_i) V^j(z_i) U^k(z_k) + \tilde{L}_{ij}(z_i) \partial V^k(z_k)) \\ &= -s_{ij} (\tilde{L}_{ik}(z_i) V^j(z_i) + V^i(z_i) \tilde{L}_{jk}(z_i)) + (s_{ik} + s_{jk}) V^k(z_i) \tilde{L}_{ij}(z_i), \end{aligned} \quad (\text{C.2})$$

which agree with (3.2) and (4.22), respectively. In the above we used $QU^i(z) = \partial V^i(z) = \Pi^m(z)k_m^i V^i(z) + \partial\theta^\alpha D_\alpha V^i(z) + \partial\lambda^\alpha A_\alpha^i$, which together with the OPE's of the conformal weight-one variables [18, 33] implies that

$$\lim_{z_i \rightarrow z_j} Q(U^i(z_i)V^j(z_j)) = \lim_{z_i \rightarrow z_j} \partial V^i(z_i)V^j(z_j) \rightarrow -s_{ij} \frac{V^i(z_i)V^j(z_i)}{z_{ij}}. \quad (\text{C.3})$$

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