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Open string pair creation from worldsheet instantons

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Abstract

Worldline instantons provide a particularly elegant way to derive Schwinger's well-known formula for the pair creation rate due to a constant electric field in quantum electrodynamics. In this note, we show how to extend this method to the corresponding problem of open string pair creation.

1 Introduction: Schwinger's formula and its open string generalization

It was realized already in the early days of quantum electrodynamics that this theory implies the possibility of electron – positron pair production from the vacuum in a strong external electric field [1, 2, 3]. As shown by Schwinger [3], the existence of this process and the pair creation probability can be derived from the imaginary part of the effective Lagrangian. For the case of a constant electric field of magnitude E, he obtained the well-known formula (at the one-loop level)

$$\mathrm{Im}\mathcal{L}_{\mathrm{spin}}(E) = \frac{(eE)^2}{8\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left[-\frac{\pi k m^2}{eE}\right] \tag{1.1}$$

with m the electron mass. Schwinger also gave the corresponding formula for scalar quantum electrodynamics,

$$\mathrm{Im}\mathcal{L}_{\mathrm{scal}}(E) = \frac{(eE)^2}{16\pi^3}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}\exp\left[-\frac{\pi k m^2}{eE}\right] \tag{1.2}$$

The expressions (1.1),(1.2) are clearly nonperturbative in the field.

The corresponding problem for an open string moving in a constant electromagnetic background field was first considered by Burgess [4], who calculated the pair creation rate in the weak field limit. The full analogue of Schwinger's formulas was obtained, for both bosonic and supersymmetric open strings, by Bachas and Porrati [5]. For the bosonic open string, their result reads

$$\mathrm{Im}\mathcal{L}_{\mathrm{string}}(E) = \tag{1.3}$$
$$\frac{1}{4(2\pi)^{D-1}}\sum_{\mathrm{states\,S}}\frac{\beta_1+\beta_2}{\pi\epsilon}\sum_{k=1}^{\infty}(-)^{k+1}\Bigl(\frac{|\epsilon|}{k}\Bigr)^{D/2}\exp\Bigl(-\frac{\pi k}{|\epsilon|}(M_S^2+\epsilon^2)\Bigr)$$

Here the first sum is over the physical states of the bosonic string, with M_S the mass of the state. $D=26$ is the spacetime dimension. The parameters $\beta_{1,2}$ are defined as

$$\beta_{1,2} = \pi q_{1,2}E \tag{1.4}$$

where $q_{1,2}$ are the $U(1)$ charges at the string endpoints, and

$$\epsilon = \frac{1}{\pi}\Bigl(\mathrm{arctanh}\beta_1 + \mathrm{arctanh}\beta_2\Bigr) \tag{1.5}$$

The formula (1.3) reproduces in the weak – field limit Schwinger's formula for spin zero (1.2), as well as its generalizations to arbitrary integer spin J. For stronger fields it deviates from the field theory case, even qualitatively, since, due to the rapid growth of the density of string states the total rate for pair production derived from (1.3) diverges at a critical field strength [5]

$$E_{\rm cr} = \frac{1}{\pi\,|\max q_i|} \tag{1.6}$$

Heuristically, a field of this strength would break the string apart. However, overcritical fields probably do make sense physically as a mechanism for D-brane decay [6, 7, 8].

Nowadays, there are many methods available to obtain Schwinger's formulas (1.1), (1.2). Perhaps the most elegant one is the worldline instanton method, which was invented by Affleck et al. for the scalar QED case [9] and generalized to spinor QED in [10, 11]. It allows one to determine the kth Schwinger exponent through the calculation of a single periodic stationary trajectory. In the following, we will show how to extend this method to the bosonic string case.

2 The worldline instanton method

For easy reference, let us begin with sketching the worldline instanton calculation [9] of the spin zero Schwinger formula (1.2).

The (euclidean) one-loop effective action for scalar QED can be written in the following way [12]:

$$\begin{aligned}
\Gamma_{\rm scal}[A] &= \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(T)=x(0)}\mathcal{D}x\,\mathrm{e}^{-S[x(\tau)]}\\
S[x(\tau)] &= \int_0^T d\tau\left(\frac{\dot{x}^2}{4}+ieA\cdot\dot{x}\right)
\end{aligned} \tag{2.1}$$

Here m is the mass of the scalar particle, and the functional integral $\int\mathcal{D}x$ is over all closed spacetime paths $x^\mu(\tau)$ which are periodic in the proper-time parameter τ, with period T. Rescaling $\tau = Tu$, the effective action may be expressed as

$$\Gamma_{\rm scal}[A] = \int_0^\infty \frac{dT}{T}\,{\rm e}^{-m^2T}\int_{x(1)=x(0)}\mathcal{D}x\,\exp\left[-\left(\frac{1}{4T}\int_0^1 du\,\dot{x}^2 + ie\int_0^1 du\,A\cdot\dot{x}\right)\right] \tag{2.2}$$

where the functional integral $\int\mathcal{D}x$ is now over all closed spacetime paths $x^\mu(u)$ with period 1. After this rescaling we can perform the proper-time integral using the method of steepest descent. The T integral has a stationary point at

$$T_0 = \frac{1}{2m}\sqrt{\int_0^1 du\,\dot{x}^2} \tag{2.3}$$

leading to

$$\mathrm{Im}\,\Gamma_{\rm scal} = \frac{1}{m}\sqrt{\frac{\pi}{T_0}}\,\mathrm{Im}\int\mathcal{D}x\,{\rm e}^{-\left(m\sqrt{\int\dot{x}^2}+ie\int_0^1 du A\cdot\dot{x}\right)} \tag{2.4}$$

Here we have implicitly used the large mass approximation

$$m\sqrt{\int_0^1 du\,\dot{x}^2} \gg 1 \quad . \tag{2.5}$$

The functional integral remaining in the effective action expression (2.4) may be approximated by a further, functional, stationary phase approximation. The new, nonlocal, worldline "action",

$$S_{\rm eff} = m\sqrt{\int_0^1 du\,\dot{x}^2} + ie\int_0^1 duA\cdot\dot{x} \tag{2.6}$$

is stationary if the path $x_\alpha(u)$ satisfies

$$m\frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du\,\dot{x}^2}} = ieF_{\mu\nu}\dot{x}_\nu \tag{2.7}$$

A periodic solution $x_\mu(u)$ to (2.7) is called a "worldline instanton". Further, contracting (2.7) with $\dot{x}_\mu$ shows that for such an instanton

$$\dot{x}^2 = \text{constant} \equiv a^2 \tag{2.8}$$

Generally, the existence of a worldline instanton for a background A leads to an imaginary part in the effective action $\Gamma_{\rm scal}[A]$, and the leading behavior is

$$\mathrm{Im}\Gamma_{\rm scal}[A] \sim e^{-S_0} \quad , \tag{2.9}$$

where S_0 is the worldline action (2.6) evaluated on the worldline instanton.

For a constant electric background of magnitude E, pointing in the z direction, the Euclidean gauge field is $A_3(x_4) = -iEx_4$. The instanton equation (2.7) for this case can be easily solved, and the solutions are simply circles in the $z-t$ plane of radius $\frac{m}{eE}$ [9]:

$$x_k^3(u) = \frac{m}{eE}\cos(2k\pi u) \quad , \quad x_k^4(u) = \frac{m}{eE}\sin(2k\pi u) \tag{2.10}$$

(with $x_{1,2}$ kept constant). The integer $k \in \mathbf{Z}^+$ counts the number of times the closed path is traversed, and the instanton action (2.6) becomes

$$S_0 := S_{\rm eff}[x_k^\mu] \quad = \quad 2k\,\frac{m^2\pi}{eE} - k\,\frac{m^2\pi}{eE} = k\,\frac{m^2\pi}{eE} \tag{2.11}$$

Thus in the large mass approximation (2.5) the contribution of the instanton with winding number k reproduces the exponent of the kth term of Schwinger's formula (1.2).

3 Generalization to the open string

The one-loop effective action for an open string in an electromagnetic background field A^μ with constant field strength tensor $F_{\mu\nu}$ has, in conformal gauge, the following path integral representation [13, 14, 4, 5],

$$\Gamma[A] \quad = \quad \frac{1}{2}\int_0^\infty \frac{dT}{T}(4\pi^2 T)^{-\frac{D}{2}} Z(T) \int \mathcal{D}x\, \mathrm{e}^{-S_E[x,A]} \tag{3.1}$$

Here T denotes the Teichmüller parameter of the annulus, and the path integral is over all the embeddings of the annulus at fixed T into $D = 26$ dimensional flat space. The worldsheet action is

$$\begin{aligned} S_E \;=\; & \frac{1}{4\pi\alpha'}\int d\sigma d\tau\, \partial_a x^\mu \partial^a x_\mu - i\frac{q_1}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=0} \\ & - i\frac{q_2}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=\frac{1}{2}} \end{aligned} \tag{3.2}$$

Here α' is the Regge slope, which will be set equal to $\frac{1}{2}$ in the following. The worldsheet is parameterized as a rectangle $\sigma \in [0, \frac{1}{2}]$ and $\tau \in [0, T]$ where $\tau = T$ is identified with $\tau = 0$. We use euclidean conventions where $\sigma^0 = -i\sigma^2 = -i\tau$, $x^0 = -ix^D$, and $A_D = -iA_0$. $q_{1,2}$ are the charges associated with the two boundaries. We will assume that $q_1 \neq q_2$, which eliminates the Möbius strip contribution to this amplitude. $Z(T)$ is the partition function of oriented open-string states, which in terms of the masses M_S of these states is given by

$$Z(T) = \sum_{\text{oriented states}} \mathrm{e}^{-\pi T M_S^2} \tag{3.3}$$

The equations of motion derived from (3.2) are

$$\begin{aligned}(\partial_\sigma^2 + \partial_\tau^2)\, x^\mu &= 0 \\ \partial_\sigma x^\mu &= i\pi q_2 F_{\mu\nu} \partial_\tau x^\nu \qquad (\sigma = \frac{1}{2}) \\ \partial_\sigma x^\mu &= -i\pi q_1 F_{\mu\nu} \partial_\tau x^\nu \qquad (\sigma = 0)\end{aligned} \tag{3.4}$$

Let us now consider the constant electric field case, $F_{D,D-1} = -F_{D-1,D} = iE$. We use (3.3) to rewrite

$$\Gamma[F] = \frac{1}{2} \sum_{\text{oriented states}} \int_0^\infty \frac{dT}{T} (4\pi^2 T)^{-\frac{D}{2}} \mathrm{e}^{-\pi T M_S^2} \int \mathcal{D}x\, \mathrm{e}^{-S_E[x,F]} \tag{3.5}$$

We rescale $\tau = Tu$ and do the T - integral by the method of steepest descent. The stationary point is

$$T_0 = \sqrt{\frac{I_u}{I_\sigma + 2\pi^2 M_S^2}} \tag{3.6}$$

where we have abbreviated

$$\begin{aligned}I_\sigma &:= \int_0^1 du \int_0^{\frac{1}{2}} d\sigma\, \partial_\sigma x^\mu \partial_\sigma x_\mu \\ I_u &:= \int_0^1 du \int_0^{\frac{1}{2}} d\sigma\, \partial_u x^\mu \partial_u x_\mu\end{aligned} \tag{3.7}$$

The new worldsheet action is

$$
\begin{aligned}
S_{\text{eff}} &= \frac{1}{\pi}\sqrt{I_u}\sqrt{I_\sigma + 2\pi^2 M_S^2} - i\frac{q_1}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=0} \\
&\quad -i\frac{q_2}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=\frac{1}{2}}
\end{aligned} \tag{3.8}
$$

It leads to the equations of motions (compare (3.4))

$$
\Big[I_u \partial_\sigma^2 + (I_\sigma + 2\pi^2 M_S^2)\partial_u^2\Big] x^\mu = 0 \tag{3.9}
$$

$$
T_0 \partial_\sigma x^\mu = i\pi q_2 F_{\mu\nu}\partial_u x^\nu \qquad (\sigma = \frac{1}{2}) \tag{3.10}
$$

$$
T_0 \partial_\sigma x^\mu = -i\pi q_1 F_{\mu\nu}\partial_u x^\nu \qquad (\sigma = 0) \tag{3.11}
$$

The kth worldsheet instanton solving these equations is obtained by the following ansatz:

$$
\begin{aligned}
x_k^{D-1} &= N\cos(2\pi k u)\cosh(b - a\sigma) \\
x_k^{D} &= N\sin(2\pi k u)\cosh(b - a\sigma)
\end{aligned} \tag{3.12}
$$

with the remaining coordinates constants. We take equal signs for k and a. Then (3.9) and (3.6) imply that

$$
T_0 = \frac{2\pi k}{a} \tag{3.13}
$$

and eqs. (3.10), (3.11) give

$$
\begin{aligned}
\sinh b &= \pi q_1 E \cosh b \\
\sinh\Big(b - \frac{a}{2}\Big) &= -\pi q_2 E \cosh\Big(b - \frac{a}{2}\Big)
\end{aligned} \tag{3.14}
$$

Eqs. (3.14) determine the parameters a, b as

$$
b = \text{arctanh}\beta_1 \tag{3.15}
$$

$$
a = 2\Big(\text{arctanh}\beta_1 + \text{arctanh}\beta_2\Big) \tag{3.16}
$$

Calculating I_u, I_σ we find

$$
\begin{aligned}
I_u &= N^2 \frac{(2\pi k)^2}{2a} \Big[\frac{a}{2} + \beta_1 \cosh^2 b + \beta_2 \cosh^2 (b - a/2) \Big] \\
I_\sigma &= \frac{a^2}{(2\pi k)^2} I_u - \frac{1}{2} N^2 a^2
\end{aligned} \tag{3.17}
$$

Finally, the combination of (3.17) with (3.6) and (3.13) fixes the normalization of the instanton:

$$
N = \frac{2\pi M_S}{|a|} \tag{3.18}
$$

We can then evaluate the stationary action:

$$
S_0 = S_{\mathrm{eff}}[x_k^\mu] = 2\pi^2 M_S^2 \frac{k}{a} \tag{3.19}
$$

Noting that $a = 2\pi\epsilon$ this correctly reproduces the exponent in (1.3) in the large M_S limit.

4 Discussion

Although our calculation does not provide new information on the string pair creation problem, we consider it worth presenting nonetheless. This is because, in the QED case, the worldline instanton approach has turned out to offer a relatively easy route to obtain pair creation rates for certain classes of non-constant fields [10, 11, 15]. Moreover, the form of the critical trajectories may also provide new physical insights. It would be interesting to extend this calculation to the prefactor determinant, as well as to the superstring case. The method may possibly also generalize to the problem of D-brane decay into open strings.

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