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GRAVITATIONAL WAVES AND NUMERICAL RELATIVITY

BERNARD F. SCHUTZ

Department of Physics and Astronomy
University of Wales College of Cardiff,
Cardiff, *U.K.*

1. INTRODUCTION

The year 1991 marks a turning point in gravitational wave astronomy, with the approval by the U.S. Congress of funding for the first year of the LIGO project to build two large-scale (4 km) interferometric detectors for gravitational waves[1]. Although the opposition the project encountered from some astrophysicists in the U.S.A. may yet affect funding for it in future years, the project has passed its most important hurdle, and relativists can look forward with confidence to a time a few years from now when there will be anything from two to four (European plans for VIRGO and GEO detectors are also with the funding agencies of Italy, France, Germany, and Britain) and detectors making regular observations of Nature's most elusive form of radiation.

Relativists should not, however, simply sit back and await experimental developments. These detectors will be limited in their observing by internal detector noise, and that means that their sensitivity can be markedly improved by *pattern matching*, that is by looking through the noisy data for signals of a known (predicted) form. The firmer these predictions are, the more capable the detectors will be at finding them. There is currently a considerable amount of theoretical work going into the theory of gravitational wave sources, but more is needed. I review here two areas in which interesting work is being done.

The first is on the astrophysics of gravitational wave sources, making rough estimates of their frequency, strength, and spectrum. This is an area that has seen many thorough reviews recently[2, 3, 4], so I will confine myself to updating these reviews with what seem to me to be the most interesting recent developments.

Then I turn to what is emerging as the next big challenge in numerical relativity, namely the prediction of the waveform emitted by two black holes as they spiral together from a binary orbit. Such events are likely to be seen by interferometric detectors, and a detailed comparison of numerical predictions with observations could provide a unique strong-field test of general relativity.

2. RECENT DEVELOPMENTS IN THE ASTROPHYSICS OF SOURCES OF GRAVITATIONAL WAVES

The interferometric detectors will be built in roughly two stages, the first stage taking the sensitivity to about $h \sim 10^{-21}$ in about 5 years, and the second an improvement by a factor of about 10 to $h \sim 10^{-22}$ after a further 5 years or so. These, then, are the limits within which source calculations are made. (The numbers quoted are, of course, broadband sensitivities at a frequency of about 1 kHz. Sensitivities to other kinds of signals can be computed from the detailed predicted noise backgrounds in the detectors.[2, 3]) The question that is asked of the astrophysics is always, how often can waves of this amplitude be expected to pass through our detectors? Answers today are still patchy.

2.1 Supernovae

Although the development of gravitational wave detectors was largely motivated by the hope of detecting supernova explosions, supernovae remain among the most uncertain sources. We have a pretty good idea how many gravitational collapses occur in a given galaxy or cluster (perhaps once a week among the 2000 galaxies of the Virgo cluster), but since spherical collapse produces no radiation, we have little idea of how much radiation will come from most supernovae. It is generally felt that significant non-sphericity will arise only if the collapsing core preserves enough angular momentum to force it into non-axisymmetry by the development of rotation-driven instabilities.

Modern supercomputers can probably now simulate fully three-dimensional gravitational collapse with rotation, but reliable results are still a year or two away. Realistic calculations with neutrino transport and nuclear physics are probably a decade or more away. Recent axisymmetric collapse calculations[5, 6, 7] show, as expected, only small amounts of radiation. One typically gets only $10^{-8}\, M_{\odot}\, c^2$ of energy in gravitational waves, which would imply an amplitude of $h \sim 10^{-24}$ from such a collapse in Virgo. Such an event would still be detectable by second-stage interferometers if it occurred in our Galaxy, which is reassuring if one feels that axisymmetric collapse is the norm. These calculations also indicate that the dominant frequency of the emitted radiation may be lower than we have assumed up to now, perhaps below 500 Hz.

If rotation dominates a collapse, then the classic "bar instability" of rotating stars may emit substantial radiation from a tumbling cigar-shaped core. The amount of radiation coming may be comparable to the radiation emitted when two neutron stars coalesce from a circular orbit: see the discussion of calculations by Nakamura and Oohara[8, 9] described in more detail in the following section. These suggest that up to $10^{-2}\, M_{\odot}\, c^2$ of energy can be carried away by gravitational waves, yielding an effective amplitude (after matched filtering) of about 10^{-21} for an event in the Virgo Cluster.

The uncertain question is whether collapse is dominated by rotation. Here, both theoretical arguments and observational data give support to slow and fast rotation. In favor of slow rotation and little non-axisymmetry, we have:

* Young pulsars seem to rotate rather slowly. The only rapid rotators are recycled pulsars spun up by accretion.
* Supernova simulations these days reproduce observations well, without any rotation at all.

On the other hand, one can argue for considerable non-axisymmetry:

* Almost all pulsars have high space velocities. Velocities of a few hundred km/s can easily arise in the break-up of the binary system in which the supernova occurred, but only approximately half of all pulsars were formed in binaries. Where did the others get their velocity? Perhaps all neutron stars are somehow given a kick from the explosion. A particularly striking example[10] of this is PSR 1758-24, whose apparent velocity of some 1600 km/s or more is implausibly large to be explained by the break-up of a binary system.
* Theoretically speaking, it is not hard to get strong rotation: if the pre-collapse core has as much angular momentum as the Sun, and if the angular momentum is preserved in the collapse, it will have more than enough to excite instabilities. However, there may be mechanisms to remove angular momentum efficiently from a collapsing core[11].

It is worth mentioning that the usual assumption is that all gravitational collapses lead to supernovae that leave behind a pulsar or (rarely) a black hole; but this assumption may be systematically wrong. A rotation-dominated collapse may, by virtue of its slower time-scale and lower peak density, produce a weaker shock than a spherical collapse does. Since the best recent spherical supernova calculations[12] produce weak shocks that are only marginally able to propel the outer envelope away, a rotation-dominated collapse may in fact not expel the envelope, and may lead to a black hole with little optical display. Therefore, neutron-star statistics may not give a good indication of what happens when black holes are produced.

But numerical simulations in this area are urgently needed, but the parameter space is so large that the uncertainty over the strength of gravitational waves from supernovae may only be resolved by observations with large-scale interferometric detectors. Indeed, there may not be a single simple answer: gravitational waves at detectable level may come from only a special or unusual fraction of gravitational collapses.

2.2 Coalescing Binaries

If two compact objects (neutron stars or black holes) find themselves in a sufficiently close binary system, they will spiral together under the action of gravitational radiation reaction until they merge. In the few seconds before merger, they are a powerful and predictable source of gravitational waves[2]. Since the gravitational radiation is predictable and carries considerable energy (about $5 \times 10^{-3} M_{\odot} c^2$ is emitted in the last two seconds of orbital motion), these are regarded as the most reliable source that second-stage interferometers can be expected to detect. They might be inaccessible to first-stage interferometers, because of their likely poor sensitivity at low frequencies (100 Hz or so).

There are a number of uncertainties in this picture that need to be cleared up: the number of coalescence events per galaxy per unit time is very uncertain[3]; the gravitational radiation given off after the stars begin to coalesce is much harder to predict; and there is uncertainty about whether such events can be (indeed, possibly have already been) seen through the electromagnetic energy they emit. I will address each in turn.

Estimates of the *event rate* depend on observations in our own Galaxy of *precursor systems*: compact-object binaries that will coalesce within a Hubble time. This must then be extrapolated to other galaxies. There are now three precursor systems known: PSR 1913 + 16 (the original "binary pulsar"), PSR 2127 + 11C (in the globular cluster M15, a binary system that is almost a clone of 1913 + 16), and PSR 1534 + 12 (a second "field" system). Incidentally, all now give individual masses of the component stars: they are all about $1.4 M_{\odot}$.

By careful consideration of the selection effects that govern the detection of such pulsars, and of current ideas about how they evolve, two groups[13, 14] have recently and independently reached similar estimates of the neutron-star–neutron-star coalescence rate. They conclude that there should at a minimum be a few coalescences per 10^6 years per galaxy, which extrapolates to several per year out to a distance of 200 Mpc. They estimate an uncertainty of a factor of 10 either way for this rate. This is consistent with the earlier, less certain estimates of Clark, *et al*[15] and of Schutz[16].

The most interesting suggestion to emerge from the recent analysis[13, 14] is that the number of neutron-star–black-hole coalescences in any volume of space should be comparable to the number of neutron-star–neutron-star coalescences, because the supernova that leads to a black hole should be less likely to disrupt the binary system. Given that a network of 4 second-stage interferometers could detect neutron-star–neutron-star coalescences out to about 600 Mpc, its range for coalescences between a neutron star and a $10M_{\odot}$ black hole will be 1.4 Gpc, which is a redshift of between 0.25 and 0.5. It follows that detectors may see an event rate of something like one per day. This prediction is testable: pulsar observations should soon reveal a pulsar in orbit around a black hole in our Galaxy.

The predictable part of the gravitational wave-train from a coalescing binary system is the radiation that comes from the orbital motion as long as the stars are adequately approximated as point masses. Most quantitative estimates have used Newtonian point-mass orbits, shrinking due to quadrupolar gravitational radiation reaction. While this is adequate over much of the wave-train, when the stars get so close that Newtonian gravity is not adequate to describe their orbital motion, one must use *post-Newtonian corrections* to improve the pattern-matching and hence the detection rate. Calculation of this sort were first performed by Krolak[17], and recently they have been much improved by Lincoln and Will[18].

One post-Newtonian effect accumulates with time until it becomes quite large: there is a gradual change in the mean value of h about which the binary signal oscillates. This was first discovered by Christodoulou[19], and interpreted within the standard picture of gravitational radiation by Thorne[20] and by Wiseman & Will[21]. Although the magnitude of the shift can be a good fraction of the amplitude of the oscillating part of the wave, the time-scale for its growth is so slow that it will be below the low-frequency cut-off of the interferometers, so it will not be observable (nor will it affect observations).

The post-Newtonian approximations will break down when mass transfer or tidal effects begin to be important, and after this it is a problem for numerical simulation, although there have been attempts to construct models of the evolution of the binary during the mass-transfer stage[22].

The first supercomputer *simulations of the merger of two neutron stars* have been performed over the past couple of years by Nakamura & Oohara[8, 9] and very recently by Rasio & Shapiro[23]. These two independent calculations both suggest that considerable radiation will be emitted, perhaps as much as $0.01 M_{\odot} c^2$. Because this occurs after the orbital wave-train on which the detection of the system is based, this energy can be detected at relatively low signal-to-noise, and its characteristics will be very informative about the nature of the system.

These calculations are preliminary, however, and can only be suggestive of the real answer, because they are done in the context of Newtonian gravity with gravitational radiation reaction, and cannot represent adequately the effects of the formation of a black hole horizon, for example. There is the danger that some of the energy that the calculations find to be emitted by the system is radiated after it should have formed a black hole, which is unphysical. Nevertheless, these calculations show that we should not be too surprised to see the coalescence radiation from many binary systems.

When neutron stars merge, one ought to expect a considerable emission of electromagnetic radiation as well. This might come from the scattering of neutrinos to form electron-positron pairs, which subsequently decay to gamma rays. Just how much of this gamma radiation can get through whatever cloud of material is formed by the merger is difficult to calculate, but there have been some estimates[24, 25]. It is at least possible that coalescences—especially the "cleaner" black-hole—neutron-star kind—could be a strong source of gamma rays, and could explain the puzzling gamma-ray bursts seen regularly.

The recently launched Compton Observatory (formerly GRO) carries a gamma-ray burst detector called BATSE, which has been seeing about 1 burst per day[26]. These bursts are isotopically distributed to within the statistics of the number of events (more than 200), which appears to rule out the previously favored model that they were associated with some sort of accretion events on nearby neutron stars in our Galaxy The statistics of coalescing binaries, and the energy they make available, make them a strong candidate for the source of these bursts. Over the next few years, further observation and modelling may well resolve this question. If coalescences are responsible for bursts, then gravitational wave detectors operating in coincidence with gamma-ray detectors could be seen a much greater distances and could provide much more cosmological information.

Coalescing binaries are likely to be so important for detectors that much work recently has been done on studying how to extract information from the signals and how to enhance the sensitivity of detectors to them. Although their wave forms are predictable, they depend (in the Newtonian part of the orbit) on two unknown parameters (the so-called mass parameter[3] and the phase of the arriving wave-train), so one has to use a family of matched filters in order to detect them all. The uncertainties in the measurements of these parameters are not independent, and they also affect the accuracy with which one can determine the time-of-arrival of the signal. If there were no uncertainty in the parameters, the timing accuracy would be better than a millisecond[27]–but see Lobo[28] on the practical difficulty of achieving this. Errors in the parameters can degrade this.

A study now essentially completed by us at Cardiff with our collaborators at IUCAA in India and also in Warsaw, shows[29] that most of this degradation is in the absolute time of arrival of the waves at the Earth, and that the relative time of arrival of a signal at two different detectors can be determined almost as accurately as if there were no uncertainty in any parameters. This is important because the relative timing in a network of three or four detectors is the way that the position of the source on the sky will be measured. By measuring the position of nearby sources (closer than 100 Mpc) accurately, one can use coalescing binary observations to determine the Hubble constant with few systematic errors[30].

There has been interesting work recently on means of enhancing the performance of interferometers to detect coalescing binaries. These have grown out of Meers' idea[31] of "dual recycling". By adjusting the position of the signal-recycling mirror, one can tune the detector to a narrow bandwidth at any selected frequency. Meers, Lobo, and Krolak[32] have shown that by doing this tuning as the signal arrives, one can obtain an improvement of a factor of 3 or more in the signal-to-noise of a source that is strong enough to be detected early, so that the dynamical tuning can be turned on. This might allow, for example, the determination of the Hubble constant with just one event, by improving the positional accuracy of the source on the sky so that it can unambiguously be associated with a group or cluster of galaxies.

2.3 Stochastic Background

In addition to a thermal relic gravitational wave background arising from the big bang in much the same way that the microwave background radiation arises, there are a number of other possible sources of a stochastic background of radiation. Because all of them require some sort of new physics for their generation, a search for the stochastic background is one of the most interesting goals of interferometric detectors.

Radiation can be generated by transitions and by topological defects associated with gauge theorics for fundamental physics. Topological defects includes cosmic strings, which, if they seed galaxy formation, should also produce an observable background as they form loops and decay[33]. Cosmic texture[34] is another topological effect that has recently become interesting as a way of accounting for the observed large-scale structure of spacetime, and further work must be done to calculate the radiation that would be expected from it. A recent calculation of Turner and Wilczek[35] on bubble collisions suggested that this might be an additional source of observable radiation. Their calculation gives a relatively narrow band of radiation near 100-1000 Hz, with an energy density that should be easily detectable with second-stage interferometers.

Cross-correlation experiments between interferometers may be the only way to test many of the exotic theories of the early universe. Such experiments can make relatively broad spectral measurements at kiloHertz frequencies. A later generation of space-based detectors could probe the very interesting milliHertz region of the stochastic spectrum.

3. THE NUMERICAL SIMULATION OF COALESCING BLACK HOLES: A PERSONAL VIEW

The first really exciting numerical calculation in general relativity was the Eppley-Smarr[36] simulation of the head-on collision of two nonrotating black holes. Considering that this problem was done in the 1970s, it may be surprising that the generalization to an in-spiralling collision has not yet been done. But such a calculation requires a fully 3-dimensional spatial grid, and only now are supercomputers becoming fast enough for this to be a feasible calculation. Indeed, the two-black-hole problem is again at the top of many people's lists of exciting numerical problems in relativity.

3.1 Challenges Presented by the Two-black Hole Problem

The problem is attractive for a number of reasons. First, it makes predictions, about events that interferometers may well observe within the next 10 years. Second, it is challenging, for reasons that I will go into below. Third, it is just the first and easiest of a whole sequence of problems involving gravitational collapse, neutron star coalescence, and black hole accretion that are of great interest in astrophysics, but which involve much more complicated physics (hydrodynamic shocks, the nuclear equation of state, etc) than pure gravity does.

I want to amplify on the challenges that this problem presents to the numerical analyst.

1. The problem should be solved on a grid that is not specially adapted to following two black holes: it should be done in such a way that a third hole could be added, or one hole replaced by a neutron star. This argues for a quasi-rectangular 3D grid, which can remain fixed at infinity, and through which the holes move. This makes stringent requirements of the gauge and slicing conditions, and these have not yet been satisfactorily solved.

2. If black holes move through the grid, then grid points go down into a hole and then pop back out the other side. This popping out requires that the grid move faster than light. This is, as we shall see, a big challenge for stable numerical integration.
3. Black hole integrations may begin with the holes relatively far apart, so their motion is slow compared to light. One might even want to adapt the grid to rotate with the basic orbital motion of the holes, in which case their effective speed would be very small. One would like in such a case to take relatively long time-steps in the numerical integration, limited only by considerations of accuracy. But conventional explicit integration schemes are restricted by the Courant condition, that the time-step be smaller than the time it takes light to cross the smallest grid cell. So one would like to use implicit methods, which are less well-known and potentially very intensive computationally.
4. A serious problem for previous numerical integrations involving black holes is that they tend to pull in the grid: points slide down the throat of the hole. This is an almost inevitable consequence of slicing spacetime with hypersurfaces that are complete in the centre: because they must avoid the singularity that develops inside the hole, they must hang back at early times inside while they proceed forwards in time outside. The stretching of the hypersurfaces caused by this tends to require more and more grid points inside the hole. One remedy is simply to cut the integration off somewhere at or just inside the horizon. What happens inside the hole is of no interest to the outside dynamics, so computational effort and inconvenience should not be wasted on it. This requires a new inner boundary condition.

While it is possible to avoid each of these difficulties in one way or another in the context of the two-black hole problem, I think these challenges are part of the attraction of the problem. They should be met and not skirted around. I see the two-black-hole problem as a driving force in the evolution of a new generation of numerical techniques that will open up many other problems in general relativity.

In Cardiff we have made substantial progress on three of the four challenges listed above. We do not yet have a gauge condition, although that is under investigation. However, for the rest of the points we have confidence that we know how to address them. The results I present here are mainly contained in a paper by Alcubierre & Schutz[37], but have been reviewed elsewhere as well[38, 39].

3.2 Superluminal Grid Shifts

We have investigated grids that move faster than light, in the context of integrations of the simple wave equation. All finite-difference integration schemes for the wave equation find the values of the field at a certain grid point at one time-step in terms of its values at a number of grid points at previous time-steps. The grid points thus linked are called the *computational molecule* of the grid point that we are solving for. The usual schemes implement fixed molecules, where the points that are linked together are determined by fixed relative index values. But when a grid is moving faster than light, such a fixed molecule can get stretched so that points at the desired time-step are outside the light cones of points in the molecule at previous time-steps. (This is illustrated in Fig. 1 for the case of a molecule that involves three grid points in one dimension at each of three time-steps, as would be the case for the standard implicit scheme described in the next section.) This is clearly an undesirable way to propagate initial data, and our numerical experiments show that the standard numerical schemes go unstable when this happens: the numerical solutions grow unboundedly, even when the correct analytic evolution is bounded.

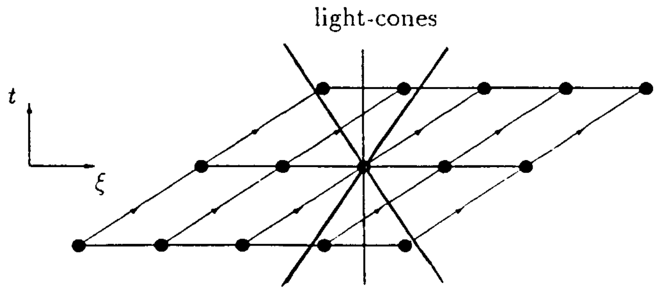

Fig. 1: Grid moving faster than light.

Our remedy for this is called *causal reconnection*: we simply reformulate the computational molecule so that its members are within each others light cones. This is shown in Fig. 2.

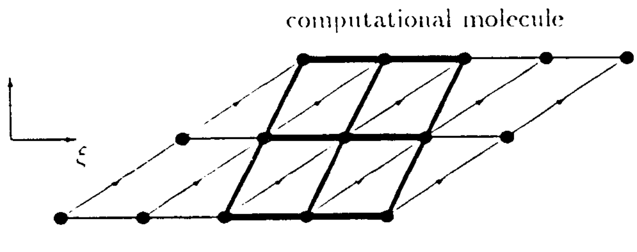

Fig. 2: Causal computational molecule.

3.3 Implicit Methods on Moving Grids

The use of implicit methods also presents problems on moving grids. These methods obtain their name from the fact that the computational molecule involves a number of points at the final time-step, rather than just one. Therefore the finite-difference equations cannot be solved explicitly for the value of the field at the final time-step; one has a set of simultaneous linear equations to solve, as many as there are grid points.

For the usual second-order methods in one spatial dimension, the matrix for this set of equations is tridiagonal, which means that it can be solved efficiently with a number of operations proportional to the number of grid points. In 2D or 3D, the matrix is no longer tridiagonal; it is sparse but has widely separated elements. Most methods for solving such matrices are very time-consuming. *Alternating Direction Implicit* (ADI) methods approximately factor the large sparse matrix into 2 or 3 tridiagonal ones, each of which has the form of the one-dimensional matrix for a different spatial direction. Such a product can be solved efficiently by successive application of the one-dimensional method, with a number of operations proportional to the number of grid points. That is why ADI methods are attractive, indeed almost essential, if one wants to use implicit methods in more than one spatial dimension.

However, because ADI methods replace the "correct sparse matrix with an approximation that factors, there is no unique ADI scheme: different styles of approximation lead to different schemes. We have found that *all the standard ADI schemes for the wave equation go unstable on grids that move*. This remarkable fact seems not to have been noticed before. We also noticed, however, that none of the standard methods preserve the fundamental time-symmetry of the original wave equation. In fact, requiring that time-symmetry maintained by the ADI approximation serves to determine a unique ADI scheme. *This scheme turns out to be stable for all grid speeds up to the speed of light in any direction.* For grids moving faster than light, we can marry this with causal reconnection. We have demonstrated the stability of the combination of these two methods on a rotating grid whose edge was moving at 15 times speed of light. The wave equation integrated without instability.

What is more, it turns out to be just as accurate (second order) as the fully implicit method. This is not a contradiction: the time-symmetric ADI scheme is indeed an approximation to the fully implicit scheme, but the errors in the approximation are only of the same order as the errors of the implicit scheme itself. It is clear therefore, that there is no treason to use fully implicit methods for the wave equation: time-symmetric ADI is just as accurate, is absolutely stable on shifting grids, and is computationally far more efficient.

Our work on ADI is all in the context of the wave equation. We believe it will generalize in a straightforward manner to Einstein's equations, and we are working on that now. However, the twin principles of time-symmetry and causality apply much more widely than in relativity, so it would seem that these methods could have much wider applications. However, their generalization of fluid dynamics, where they might be natural, is not straightforward: shocks are not time-symmetric. So there is much more work to be done on these methods.

3.4 The Inner Boundary Condition

Until one has a gauge condition that slices the horizon on a moving grid in a natural way, it will be impossible to study the form of a boundary condition near the horizon in great detail. But we have looked at the 1D wave equation again, in the context of placing a boundary condition at one end of the domain on a spacelike line. This models taking the boundary at, say, a marginally trapped surface inside the horizon. We have found that the solution of the 1D wave equation is unaffected by the form of the condition at the spacelike boundary. This is not surprising, since the boundary is moving away too fast: even a bad boundary condition cannot propagate information back into the rest of the spacetime.

Similarly in the black hole case, we expect that as long as the Einstein equations are represented adequately in the neighbourhood of the horizon, then even a very wrong condition on the marginally trapped surface will not cause difficulties outside the horizon. We would therefore expect that the next generation of calculations in numerical relativity will truncate their hypersurfaces at or inside the horizon.

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