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Motion of Primordial Black Holes in the Early Universe and Their Likely Distribution Today

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Abstract

We look in detail at those effects which slow down black holes of mass $\sim 10^{15}$ g and affect their spatial distribution today. In particular we treat effects caused by the charge fluctuations of the hole which result from quantum-mechanical processes. The dominant energy-loss mechanism for the holes is the expansion of the universe, which leaves them virtually at rest at the time of galaxy formation. The resultant violent relaxation should concentrate roughly half of them in present-day galaxies and their halos.

§(1): *Introduction*

Primordial black holes (pbh's) have attracted considerable attention ever since Hawking suggested that density fluctuations in the early universe should lead to their formation [1]. In a series of papers [2-4], Carr has examined their formation and likely mass spectrum, with many interesting conclusions. Any limits we can set today on the number of pbh's gives us important information about the big bang. One class of pbh—those with masses of order 10^{15} g—have attracted particular attention lately because they should be evaporating in the modern epoch due to their quantum mechanical emission of photons, neutrinos, and massive particles (the Hawking radiation [5-9]). These are doubly interesting: not only does their radiation make them easier to observe, but observing them would confirm the Hawking radiation itself. Many authors have studied the quantum mechanical aspects of these pbh's and many others have discussed the likelihood of our observing them [10-16]. This latter group typically is forced to make some assumption about their properties today,

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particularly their velocity and their spatial distribution, but there does not appear to be any thorough discussion of what spatial and velocity distributions may be expected. In particular, the detailed quantum mechanical behavior of pbh's of this size–especially their spontaneous charge fluctuations and their very different capture cross sections for protons and electrons–suggest effects on their motion that do not seem to have been considered. It is our purpose in this paper to study all these effects systematically. The study reveals several interesting points; that the capture of protons may slow a pbh down more rapidly in the early universe than the expansion of the universe itself does; that the rms charge on a pbh causes it to lose energy through Coulomb scattering as efficiently as through Newtonian gravitational scattering, but that neither effect is ever important; and that a naive application of the usual derivation of the Coulomb energy loss rate to the Newtonian gravitational problem leads–wrongly but instructively–to the conclusion that the expansion of the universe accelerates the pbh. But the main result of the study is to reinforce the usual picture of pbh's: the expansion of the universe red-shifts their velocities considerably, so that by the time of galaxy formation they are essentially at rest; then the formation of galaxies causes pbh's to cluster about the galaxies. We conclude that roughly half of all pbh's are within present day galaxies (or their massive halos). In an appendix we give a more rigorous derivation than we can find elsewhere of the energy loss rate of a massive particle due to gravitational scattering from the particles of whatever medium it is passing through.

Our main interest is in pbh's of mass about 10^{15} g, but many of our results apply to other masses as well. We are–except in the section on galaxy formation –mainly concerned with pbh's moving in homogeneous media, although again many of our formulas apply to pbh's in the galaxy today. We assume that the universe is the "standard" Friedmann model (Weinberg [17]); whether it is open or closed generally does not matter. We should also point out that most of the interactions of a pbh of this size with a surrounding medium do not depend on whether the medium is ionized or not; this is clear for gravitational scattering, but it is also true for its ability to capture electrons and protons, since its size (about 10^{-13} cm) is much smaller than that of an atom. In those sections in which we discuss the slowing down of a pbh by its interaction with a surrounding medium, we shall concentrate on pbh's moving supersonically. Not only is this case more straightforward to treat than the subsonic one; it is also more interesting, for a pbh that is traveling subsonically by the time of galaxy formation will be far more affected by galaxy formation than one which still has a considerable velocity. We shall always assume the pbh is nonrotating.

§(2): *Velocity "Red-Shift"*

The most obvious and, as it happens, the most important influence on the velocity of a pbh is its slowing down due to the expansion of the universe. The equation governing this, (2.1) below, can be derived from the geodesic equations, but it is easier to use the homogeneity and isotropy of the universe directly. In a local inertial frame (indices $\hat{\alpha}, \hat{\beta}$) at rest with respect to the homogeneous hypersurfaces, the four-momentum of a particle can without loss of generality be chosen to have only the components $p^{\hat{0}}, p^{\hat{1}}$, where the $\chi^{\hat{1}}$ direction is a translation Killing vector ξ, and the dot product $p \cdot \xi = p^{\hat{1}} \xi^{\hat{1}}$ is constant along the particle's geodesic. But, because the spatial metric tensor is proportional to $R^2(t)$ (the square of the radial scale factor of the universe), the magnitude of the Killing vector ξ must be proportional to R: $\xi^{\hat{1}} \sim R(t)$. We conclude that

$$p^{\hat{1}} \sim R^{-1}(t)$$

Since the velocity of the particle is $V = cp^{\hat{1}}/p^{\hat{0}}$, and since $p^{\hat{1}}$ determines $p^{\hat{0}}$ from the equation $(p^{\hat{0}})^2 - (p^{\hat{1}})^2 = m^2c^4$, it is easy to show that

$$V(t_1) = cV_0 \Bigg/ \left\{ V_0{}^2 + (c^2 - V_0{}^2) \left[\frac{R(t_1)}{R(t_0)}\right]^2 \right\}^{1/2} \tag{2.1}$$

where $V_0 = V(t_0)$, and that

$$\frac{dV}{dt} = -V(1 - V^2/c^2)H \tag{2.2}$$

where

$$H = \frac{1}{R}\frac{dR}{dt}$$

is the Hubble "constant" of whatever epoch is being considered. Supposing that a pbh has a velocity $c/3$ at a time $t = 10^{-23}$ s (that is, a typical sound speed at a time well after it will have formed if it has a mass $\sim 10^{15}$ g) and that the universe from $t = 10^{-23}$ to $t = 10^{-3}$ sec had a relativistic equation of state ($p = \frac{1}{3}\rho$), we find that the pbh should have a velocity of order 10^{-18} c at the time of recombination. Only a pbh with an original velocity differing from c by one part in 10^{36} could have a substantial velocity at the time of recombination. Moreover, since the speed of sound stays at $\frac{1}{3}c$ for a very long time, it is reasonable to conclude that once a pbh's velocity is subsonic, it remains subsonic at least until the epoch of galaxy formation, which we shall discuss later. But first we shall compare the other influences on a pbh's motion with the cosmological red shift.

§(3): *Gravitational "Coulomb" Scattering*

It is usually supposed that, by analogy with the results for energy loss by a charged particle passing through matter, a pbh of mass M and velocity V_H will lose energy at a rate

$$\frac{dE}{dt} = -\frac{4\pi G^2 M^2}{V_H} \rho_m \tag{3.1}$$

in passing through matter of uniform rest-mass density ρ_m. This is, in fact, extremely small in any realistic situation [1]. However, the usual derivations of the charged particle analog of (3.1) are not adequate when the particle is in an expanding universe; in fact, a naive application of them could lead to the conclusion that the interactions of the pbh with the expanding gas around it *accelerates* the pbh. If this result were correct it would radically affect the pbh velocity spectrum today. We give this naive argument, show its flaw, and present a more careful justification of (3.1) in the Appendix. The result is that (3.1) should be multiplied by a logarithm, whose value in a homogeneous universe is

$$\ln (V_H{}^3/2GMH) \tag{3.2}$$

where H is Hubble's "constant" again. Like the analogous factor in the charged-particle case, this cannot get very large. For a 10^{15}-g pbh with speed c, and for $H = 10^2$ km sec^{-1} Mpc^{-1} (a considerable underestimate for early epochs) this factor is about 100. This does not, of course, make the energy loss for such a pbh significant.

§(4): *Effects Due to the Fluctuating Charge of the pbh*

The charge on a pbh whose mass is of the order of 10^{15} g fluctuates as it emits (by the Hawking process) and absorbs charged particles [9]. Any charge contributes to the pbh's energy loss rate. The energy loss rate due to the pbh's charge Ze' in the plasma era (before recombination) is

$$\frac{dE}{dt} = -\frac{4\pi Z^2 e^4 \rho_m}{V_H m_e m_p} \ln \left[\frac{V_H{}^3 m_e{}^{3/2} m_p{}^{1/2}}{Zc^3 (4\pi\rho_m)^{1/2}}\right] \tag{4.1}$$

where m_e and m_p are the masses of the electron and proton. The most optimistic value of the logarithm ($\rho_m/m_p = 10^4$ cm^{-3}, $Z = 1$, $V_H = c$) is about 40. So energy loss due to charge becomes comparable to that from gravitational scattering only for charges

$$Z \gtrsim 1.6 \frac{GM}{e^2} (m_e m_p)^{1/2} \tag{4.2}$$

the factor 1.6 coming from the ratio of the logarithms in (4.1) and (3.2). This can be written

$$Z \gtrsim 0.8 \frac{\hbar c}{e^2} R_H (\lambda_e \lambda_p)^{-1/2} \cong 9(M_H/10^{15}\text{ g}) \tag{4.3}$$

where R_H is the pbh's radius, λ_e and λ_p are the Compton wavelengths of the electron and proton, and $e^2/\hbar c$ is the fine-structure constant. We shall now consider the likely magnitude of the charge on a pbh.

4.1. Fluctuations Due to the Emission of Charged Particles. Page [9] has calculated detailed emission rates for charged leptons from a pbh. A neutral hole emits positrons and electrons with equal probability, but a positively charged hole will emit positrons more frequently than electrons. This leads to an rms charge on the hole which depends on the ratio R_H/λ_e. For a 10^{15}-g hole this ratio is 4×10^{-3}, and the rms charge is about $Z = 8$. So the effects of charge-particle scattering on the slowing down of a pbh of this size will be comparable to but no greater than those of gravitational scattering.

4.2. Fluctuations Due to Selective Absorption of Protons. Because, for a 10^{15}-g pbh, $R_H/\lambda_p \sim 3.5$ but $R_H/\lambda_e \sim 2 \times 10^{-3}$, quantum mechanical effects strongly discriminate against the absorption of electrons. The cross section for the absorption of protons will be the classical one,

$$\sigma_p \cong \frac{4\pi R_H^2}{V_H^2/c^2} \qquad (V_H \ll c) \tag{4.4}$$

While that for electrons will be less than this by a factor of roughly $\frac{1}{2} R_H/\lambda_e$ [9, 13]. If the pbh's charge could change only by this mechanism, this would lead to an "equilibrium" charge on the hole of order $2e\lambda_e/R_H \sim 500e$, far larger than one expects from Hawking fluctuations. But this can only happen if the rate of absorption far exceeds the rate of spontaneous emission. In fact, the reverse is true; a pbh of this size will change its charge by emission of leptons much more rapidly than by absorption of protons. Page's calculations [9] show that any excess charge on the pbh will be neutralized in a time $10^4\ GM_H/c$. The mean time to absorb a proton from the surrounding plasma of proton number density N_p is, from (4.4),

$$\tau = \frac{1}{\sigma_p n_p V_H} \sim \frac{V_H}{4\pi R_H^2 c^2 n_p} \tag{4.5}$$

For a pbh moving at speed $V_H \sim c$, absorption dominates emission if $n_p R_H^3 \gtrsim 2 \times 10^{-5}$. For a pbh of mass 10^{15} g this requires a proton number density of 10^{34} cm^{-3}! The universe had densities of this order only when protons and antiprotons were in equilibrium, in which situation absorption would not have led to a build up of charge anyway.

§(5): *Effects of Accretion on pbh's*

We have already considered accretion in relation to the charge on the pbh in the previous section. But accretion can also slow the pbh down and increase its mass. If the pbh moves supersonically, as we have assumed all along, it will accrete mainly by direct capture of protons, with the cross section (4.4) above. From the discussion in the previous section, it is clear that the pbh will lose mass through emission of leptons much faster than it will gain mass through accretion of protons if the number density of protons is less than about 10^{30} cm^{-3}. Accretion will slow the hole down at the rate

$$M_H \dot{V}_H \simeq -(\sigma_p N_p V_H)(m_p V_H)$$
$$= -4\pi R_H{}^2 \rho c^2 \qquad (5.1)$$

where ρ is the mass density of the medium through which the pbh passes. (Note that the reemission of protons will not slow the pbh down because it is on average isotropic in the pbh's rest frame.) Comparison with (2.2) shows that accretion is important if

$$4\pi G\rho > \left(\frac{V_H}{c}\right)\left(\frac{cH}{4R_H}\right)$$

Using the relation $H^2 = \frac{8}{3}\pi G\rho_c$, where ρ_c is the density needed to close the universe, and denoting ρ/ρ_c by Ω and $2GM_H/c^2$ by R_H, we obtain

$$4\pi G\rho > \frac{V_H{}^2}{6R_H{}^2\Omega} \qquad (5.2)$$

For a hole of mass 10^{15} g and speed $V_H \sim c$ in a nearly closed universe ($\Omega \sim 1$) this gives $\rho > 5 \times 10^{-5}$ g cm^{-3} which, in the standard model of the universe, holds for $t \lesssim 2 \times 10^3$ sec, $T \gtrsim 3 \times 10^{8}$ °K, $z \gtrsim 10^8$, where z is the red shift. So in the early history of a supersonic pbh the dominant mechanism for its deceleration is accretion, at least until it becomes subsonic.

§(6): *Relaxation of the pbh Velocity Spectrum Produced by Galaxy Formation*

All the above calculations indicate that a pbh will reach the epoch of recombination essentially at rest in the cosmological frame. During the era of galaxy formation, pbh's will behave like a collisionless gas of free particles. They will be strongly affected by the time-dependent gravitational fields associated with the gravitational condensations, and can be expected to be concentrated in galactic halos [14]. It is not difficult to obtain an estimate of the degree of this concentration. We shall idealize the formation of a galaxy by considering the pressure-free collapse of a spherical cloud of initial radius R_0 and initial uniform density ρ_0. It is assumed that the cloud begins at rest and with its pbh's uniformly distributed throughout. At some radius R_f pressure is assumed suddenly to halt the cloud's collapse. The collapsing cloud obeys the parametric equations

$$R = R_0 \sin^2 \tfrac{1}{2}\Theta$$

$$t = \frac{3^{1/2}}{32\pi G\rho_0}(\Theta - \sin\Theta) \tag{6.1}$$

where the collapse begins at $\Theta = \pi$. Pressure halts the collapse of the cloud at $\Theta = \Theta_1$,

$$R_f = R_0 \sin \tfrac{1}{2}\Theta_1 \tag{6.2}$$

but pressure does not affect the pbh's. They therefore have at this time a velocity

$$V(\alpha) = \alpha R_0 (\tfrac{8}{3}\pi G\rho_0)^{1/2} \cot \tfrac{1}{2}\Theta_1 \tag{6.3}$$

where α is the ratio of the pbh's initial radial position to R_0. With this initial speed, all pbh's with α smaller than a critical value α_c will remain within the new radius R_f. A short calculation gives

$$\alpha_c = (2 + \cos\Theta_1)^{-1/2} = (3 - 2R_f^2/R_0^2)^{-1/2} \tag{6.4}$$

If the collapse proceeds reasonably far before halting, we will have

$$\alpha_c \approx 3^{-1/2} \tag{6.5}$$

In other words, a fraction $\alpha_c^3 \simeq 20\%$ of the original pbh's in the cloud will be permanently trapped within the collapsed cloud. The remaining pbh's will all spend some time inside the cloud, and so at any one time probably 40-50% of the original pbh's will be inside the collapsed cloud. Now, a more realistic collapse will be halted by pressure only gradually, and nonspherical effects, particularly violent relaxation, will be very important. But it may be expected that these will mainly affect the distribution of pbh's inside the collapsed cloud (turning it from uniform density into something more like isothermal), but the basic conclusion that a quarter to half of the pbh's are present within the cloud will not be changed.

The main problem in drawing any conclusions for our galaxy is that we do not know the final radius of the collapsing cloud. It is certainly at least as large as the radius of the visible disk of our galaxy, $R_f \geqslant 15$ kpc. But if the galaxy has a massive halo extending to 30 kpc or beyond, the pbh's would also be distributed over such a volume. If we adopt the conservative view, that the galaxy has a mass of about $10^{11} M_\odot$ concentrated within 15 kpc of its center, and if we accept Carr's limit [4] that the universe's average density of pbh's of mass $\sim 10^{15}$ g cannot exceed 10^{-8} of the closure density, we find that the density of pbh's near our galaxy should be less than $10^8\ \Omega^{-1}\ \text{pc}^{-3}$, where Ω is the ratio of the true mass density of the universe to the closure density. These pbh's should have velocities comparable to the rotational speeds of stars in orbit about the galactic center, $V \sim 200\ \text{km sec}^{-1}$.

Any observation which gives a limit on the density of pbh's within the disk of our galaxy (e.g., [15, 16]) sets a limit on the mean density of pbh's in the universe if one accepts the above estimates of the size and mass of our galaxy:

the mean density should be about $2 \times (15\ \text{kpc}/0.5\ \text{Mpc})^3 \sim 5 \times 10^{-5}$ times the local density, in close agreement with the estimate of Page and Hawking [14].

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Appendix: Energy Loss by Gravitational Scattering

The usual method of deriving the energy loss rate of a charged particle is to consider its interaction with each particle of the medium separately and then to integrate over all particles. (See Jackson [18], for example.) If the encounter is rapid, the impulse approximation can be used. For the gravitational case, in which a pbh of mass M_H and velocity V_H interacts via its Newtonian gravitational field with a "field" particle of mass m and impact parameter b, the reasoning is as follows. The encounter lasts a characteristic time $t_c = 2b/V_H$, during which time the field particle experiences a force $F = -GM_Hm/b^2$. Its momentum changes by $t_cF = \Delta p$, and its kinetic energy by $(\Delta p)^2/2m$. This produces a change in the pbh's kinetic energy of

$$E_H = \frac{(\Delta p)^2}{2m} = -\frac{2G^2 M_H{}^2 m}{V_H^2 b^2} \tag{A.1}$$

This equation is in fact exact for large b if the field particle starts out at rest and if m is negligible compared to M_H. (The overestimation of F above and the underestimation of t_c compensate one another.) But the argument is somewhat naive if one takes into account the initial velocity, $\mathbf{V}_0$, of the field particle. For then the change in its kinetic energy is

$$\Delta E_f = \mathbf{V}_0 \cdot \Delta\mathbf{p} + \frac{1}{2m} |\Delta\mathbf{p}|^2 \tag{A.2}$$

and the first term can overwhelm the second when $|\mathbf{V}_0| >> |\Delta\mathbf{V}|$. The reason that the first term can be neglected in the Coulomb case is the overall charge neutrality of the plasma: in the impulse approximation $\Delta\mathbf{p}$ has the same magnitude and opposite sign for particles of opposite charge, so the first term in (A.2) drops out when one averages over the pbh's interactions with several plasma particles. This leaves only (A.1). But in the gravitational case, $\Delta\mathbf{p}$ has the *same* sign for all particles. If the plasma is static in the mean, so that $\mathbf{V}_0$ is just a random thermal velocity, then the first term again averages to zero. But if, as in an expanding cosmology, there is a systematic $\mathbf{V}_0$ shared by all field particles, this term cannot be neglected. It would seem, then, that (A.1) should not apply to gravitational scattering in cosmologies.

Before showing its flaw, let us carry this interesting argument a little further. In an expanding cosmology, $\mathbf{V}_0 \cdot \Delta\mathbf{p}$ will be negative, because the field particle is attracted toward the pbh while its velocity is directed away from it. By conservation of energy, the pbh will *gain* energy: far from being slowed down by the collision, it will be accelerated!

This is easy to understand physically. Consider the two "halves" of the collision, before and after the pbh arrives at the point of closest approach. During the first half the pbh is accelerated and during the second decelerated. If the field particle is moving away during the collision, the force during the second half will be weaker than during the first half, so a net acceleration will result.

This physical picture contains both the key to the flaw in this argument and a pointer to a better derivation of (A.1) for the gravitational case. The acceleration of the pbh by the expanding gas is real, but it does not depend on any effect the pbh has on the gas: *it is produced by the unperturbed motion of the gas.* It should not be thought of as a "scattering" effect at all: if the zeroth-order motion of the pbh has been correctly calculated for the background gravitational field (i.e., if the pbh follows a geodesic in the cosmology) then this acceleration will already be included in its motion and should not be added in again. The only effects one should include in a scattering calculation are those effects on the pbh due directly to the changes in the motion of the field particle caused by the pbh: those effects which result from the fact the pbh is not a test particle but has an influence on its surroundings. It is safer, then, to avoid the energy-conservation argument of the standard Coulomb derivation and to look directly at the forces on the pbh. The situation is defined in Figure 1. We use subscript H to denote the pbh and f the field particle. The change in the energy of the pbh will be the line integral $\int \mathbf{F} \cdot d\mathbf{r}_H$ along the path. The force $\mathbf{F}$ has two pieces:

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1$$

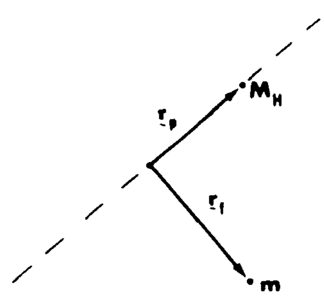

Fig. 1. Interaction of a pbh with a plasma particle. The origin of coordinates is taken to be the point of closest approach of the two bodies, which occurs at $t = 0$. The pbh (mass M_H) has, to this order of approximation, constant velocity $\mathbf{V}_p$ and position $\mathbf{r}_p = \mathbf{V}_p t$ (path indicated by dashed line). The plasma particle has position $\mathbf{r}_f$ and mass m.

where $\mathbf{F}_0$ is the force on the pbh due to the field particle in its unperturbed state of motion and $\mathbf{F}_1$ is the change in the force on the pbh due to the change in the position of the field particle produced by its interaction with the pbh. We will identify the change in the pbh's energy caused by Newtonian scattering as

$$\Delta E_H = \int \mathbf{F}_1 \cdot d\mathbf{r}_H \tag{A.3}$$

We suppose that the field particle does not move very far during the collision, i.e., that $|\mathbf{V}_H| >> |\mathbf{V}_f|$; that the collision makes a small change in the pbh's motion; and that the interaction is negligible at times t outside the range $-t_{\text{coll}} < t < t_{\text{coll}}$, where $t_{\text{coll}} = |\mathbf{r}_f|/|\mathbf{V}_H|$ is a typical collision time. In order to find $\mathbf{F}_1$ one must find $\Delta\mathbf{r}_f$, the change in the fluid particle's position. This one can find from the force on the plasma particle due to the unperturbed motion of the pbh; corrections to the motion of the pbh give higher-order changes in $\Delta\mathbf{r}_f$ and $\mathbf{F}_1$. On the impulse approximation, the field particle accelerates at the rate

$$\Delta\ddot{\mathbf{r}}_f = -GM_H \frac{\mathbf{r}_f - \mathbf{r}_H}{|\mathbf{r}_f - \mathbf{r}_H|^3}$$

where in the right-hand side $\mathbf{r}_f = (\mathbf{r}_f)_0 = \text{const}$ and $\mathbf{r}_H = \mathbf{V}_H t$. Denoting the impact parameter by

$$b = |(\mathbf{r}_f)_0|$$

we have approximately

$$\Delta\ddot{\mathbf{r}}_f = -\frac{GM_H}{b^3}\left[(\mathbf{r}_f) - \mathbf{V}_H t\right] \tag{A.4}$$

(The replacement of $|\mathbf{r}_f - \mathbf{r}_H|$ by b in this expression overestimates the force and compensates the error introduced by underestimating the collision time, which is really infinite.) The solution to (A.4) for which $\Delta\mathbf{r}_f = 0$ and $\Delta\mathbf{V}_f = 0$ at $t = -t_{\text{coll}}$ is

$$\Delta\mathbf{r}_f = -\frac{GM_H}{2b^3}(t + t_{\text{coll}})^2\left[(\mathbf{r}_f)_0 - \tfrac{1}{3}(t - 2t_{\text{coll}})\,\mathbf{V}_H\right] \tag{A.5}$$

We find $\mathbf{F}_1$ by using

$$F = GmM_H \frac{\mathbf{r}_f - \mathbf{r}_H}{|\mathbf{r}_f - \mathbf{r}_H|^3}$$

replacing $\mathbf{r}_f$ by $(\mathbf{r}_f)_0 + \Delta\mathbf{r}_f$, and keeping terms first order in $\Delta\mathbf{r}_f$. This gives

$$\mathbf{F}_1 = \frac{GM_H m}{b^3}\left[\Delta\mathbf{r}_f - 3(\mathbf{r}_f - \mathbf{r}_H)\frac{(\mathbf{r}_f - \mathbf{r}_H)\cdot\Delta\mathbf{r}_f}{|\mathbf{r}_f - \mathbf{r}_H|^2}\right] \tag{A.6}$$

where now on the right-hand side $\mathbf{r}_f$ is $(\mathbf{r}_f)_0$. Rather than use (A.5) immediately in this, we note that we only want the component $F_1 \cdot \mathbf{V}_H$, which is slightly simpler to calculate since $\mathbf{V}_H \cdot (\mathbf{r}_f)_0 = 0$. We get

$$\mathbf{F}_1 \cdot V_H = -mV_H^2 \left(\frac{GM_H}{b^3}\right)^2 (t + t_{\text{coll}})^2 \left[\frac{3}{2}t + \frac{1}{6}(t - 2t_{\text{coll}})\left(\frac{3t^2}{t_{\text{coll}}^2} - 1\right)\right] \quad (A.7)$$

where for consistency we have set $|\mathbf{r}_f - \mathbf{r}_H|$ to b in the denominator of the second term in (A.6). This is readily integrated. The second term in (A.7) gives zero [no contribution from the sideways motion of the plasma particle in equation (A.5)] and the first term gives

$$\Delta E_H = -\frac{2G^2 M_H^2 m V_H^2 t_{\text{coll}}^4}{b^6} = -\frac{2G^2 M_H^2 m}{V_H^2 b^2} \quad (A.8)$$

This is identical to (A.1).

The significance of this derivation is that it shows that (A.1) is correct as long as the particles of the medium do not move much during the collision, independently of whether their velocities are random or systematic, provided that the zero-order motion of the pbh (that due to $\mathbf{F}_0$) is correctly calculated. It is easy to convert (A.8) into an expression for the energy loss per unit time in a plasma with rest-mass density ρ_m

$$\frac{dE_H}{dt} = -\frac{2G^2 M_H^2}{V_H} \int \frac{\rho_m}{b^2} 2\pi b \, db \quad (A.9)$$

where the integral represents a sum over all particles in a plane perpendicular to the pbh's velocity. (Naturally, one sums only over particles whose thermal speeds are less than V_H, or over fluids whose sound speed is less than V_H. This excludes photons.) If ρ_m is constant at small or large b the integral diverges logarithmically, and cutoffs are necessary. We shall now discuss the cutoffs. An inner cutoff can be justified by an argument similar to one used in the Coulomb case [18].

The impulse approximation fails when, as seen in the frame comoving with the pbh, the field particle's maximum potential energy $(GM_H m/b)$ becomes as large as its incident kinetic energy $(\frac{1}{2} m V_H^2)$. This gives

$$b_{\min} = 2GM_H/V_H^2. \quad (A.10)$$

Another inner cutoff is the effective radius of the capture cross section, but on dimensional grounds this must be of the order of (A.10). In fact, inspection of equation (4.4) in the body of the paper in the nonrelativistic limit gives a cutoff just twice equation (A.10). Since this cutoff affects the value of the integral only logarithmically, we shall use (A.10).

The outer cutoff is different from the ones used in the Coulomb case, essentially because of the equivalence principle. Since all particles at a given point in the field experience the same acceleration due to the pbh, there is no "polarization" and no Debye-type screening. Moreover, it does not matter whether individual particles in the field are truly independent or are bound into atoms, since their response to the pbh is independent of this. So the two principal Coulomb outer cutoffs [18]–(1) screening of the charge and (2) the collision time exceeding the orbital period of an electron bound in an atom–do not apply here. The only outer cutoff in the gravitational problem is the failure of the impulse approximation at large distances due to the expansion of the universe. If the Hubble rate is H, then the expansion speed reaches the pbh's speed at a distance

$$b_{\max} = V_H/H \tag{A.11}$$

This is always inside the cosmological horizon ($\sim c/H$) and so justifies our using Newtonian gravity in these calculations.

For a pbh in a homogeneous and isotropic cosmology, we obtain from (A.9)

$$\frac{dE_H}{dt} = -\frac{4G^2 M_H{}^2 \rho_m}{V_H} \ln\left(\frac{V_H{}^3}{2GM_H H}\right) \tag{A.12}$$

The logarithm factor, which is usually left out of these discussions, cannot be too large: for $V_H \sim c, M \sim 10^{15}$ g, and $H \sim 10^2$ km sec^{-1} Mpc^{-1},

$$\ln\left(\frac{V_H{}^3}{2GM_H H}\right) \sim 10^2$$

Note that only the density ρ_m enters equation (A.9); no other characteristic of the medium is important. Therefore (A.9) applies also in the present universe, ρ_m is taken to be the mean mass density of the universe despite the fact that the gas is clumped into stars and galaxies. As long as the pbh has a faster speed than the random stellar and galactic velocities, the impulse approximation will still apply.

It is interesting to ask what velocity a pbh would have to have in order to slow down due to gravitational scattering alone, in say, a Hubble time. For a nonrelativistic pbh, for which E_H is just its kinetic energy, we obtain from equations (A.10)-(A.12) the fractional energy loss rate, in units of the Hubble expansion rate:

$$\frac{1}{H}\frac{d}{dt}\ln\left(\tfrac{1}{2} M_H V_H{}^2\right) = -\frac{4\pi G\rho_m}{H^2}\frac{b_{\min}}{b_{\max}}\ln\left(\frac{b_{\max}}{b_{\min}}\right) \tag{A.13}$$

When the right-hand side of (A.13) is of order 1 the hole will slow down significantly. Since $3H^2/(8\pi G) = \rho_c$ = the critical density for closing the universe, and since it appears that our universe has $\rho_c > \rho_m$ (during the radiation-dominated era, $\rho_c \gg \rho_m$), the right-hand side of (A.13) can never in fact