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Normal Modes of a Model Radiating System

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In order to gain insight into normal modes of realistic radiating systems, we study the simple model problem of a finite string and a semi-infinite string coupled by a spring. As expected there is a family of modes which are basically the modes of the finite string slowly damped by the "radiation" of energy to infinity on the semi-infinite string. But we also study another family of modes, found by Dyson in a different model problem, which are strongly damped modes of the semi-infinite string itself. These may be analogous to the modes of black holes, and they are likely to be present in relativistic stars as well. The question of whether the instability in these modes which Dyson found is present in realistic stars remains open.

1. INTRODUCTION

The dynamical oscillation or collapse of a physical system in general relativity is generally accompanied by the emission of gravitational radiation. One of the simplest ways to begin to study this problem is to calculate the normal modes of pulsation of nearly stationary bodies undergoing small-amplitude pulsation. Not only does this method yield important information about the stability of the system [1], it also serves as a test-bed for understanding the relationship between gravitational waves and their sources [2]. Even so, the problem is complicated and has to be solved by numerical methods. This has led some authors [3–6] to study model problems of simple wave fields interacting with simple oscillating sources, in order to be able to develop at least some ideas analytically. The models show what one might expect: if the source

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uncoupled from the wave field would have a normal mode with real eigenfrequency ω_0, then the coupled wave-source system has a normal mode with a complex eigenfrequency whose real part is close to ω_0 and whose small imaginary part represents the damping of the mode as the waves carry energy away. This is qualitatively similar to what is found in more realistic systems, like neutron stars [2, 7]. But Dyson [3] made two remarkable new discoveries in his model problem: (i) in addition to the eigenfrequencies referred to above, there was a set of frequencies with large imaginary part, which become infinitely strongly damped as the coupling between wave and source tended to zero; and (ii) an instability developed in these modes for large values of the coupling. It is possible that one or both of these features is present in realistic problems. To shed some light on this, we solve here an even simpler model system than Dyson dealt with. We find the eigenfrequencies with strong damping, but not the instability. We discuss why in the final section.

The model system consists of two strings, one finite with fastened ends (string 1) and the other seminfinite with one end fastened (string 2), and a massless spring connecting the two strings, as shown in Figure 1. The finite string represents the source (e.g., a star) whose oscillations will perturb the second string (the space-time). The outflow of energy will damp the motion of the first string. At first we take both strings to have the same local wave speed c. Later we relax this assumption.

2. NORMAL MODES AND EIGENFREQUENCIES

The local wave solution in each of the segments AB, BD, EZ, and ZH has the form

$$y = [A \exp(i\omega x/c) + B \exp(-i\omega x/c)] \exp(i\omega t) \tag{1}$$

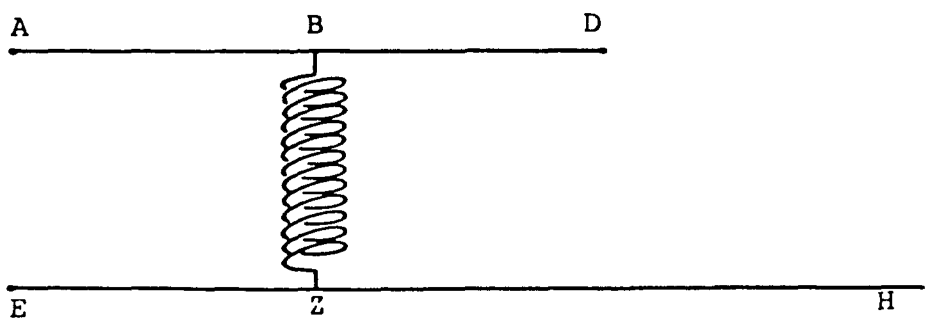

Fig. 1. The coupled system consists of a finite string of length $2l$ and a semi-infinite string, coupled as shown by a spring with spring constant k.

where A and B are the complex amplitudes that change from segment to segment. The boundary conditions are

$$y_1(A) = y_1(D) = y_2(E) = 0, \qquad y_{ZH} = C \exp[i\omega(t - x/c)] \tag{2}$$

where the second condition is the outgoing wave condition on the semi-infinite string, $\omega = \sigma + i/\tau$ is the complex frequency of the vibration, and c the wave speed.

The coupling equations for the system are

$$T[(\partial y_{AB}/\partial x)_{x=l} - (\partial y_{BD}/\partial x)_{x=l}] = -k(y_B - y_Z) \tag{3}$$

$$T[(\partial y_{EZ}/\partial x)_{x=l} - (\partial y_{ZH}/\partial x)_{x=l}] = k(y_B - y_Z) \tag{4}$$

where y_B and y_Z are the displacements of the strings at the coupling points B and Z, k is the spring constant, T is the tension in the strings (assumed equal) and y_{AB}, y_{BD}, y_{EZ}, and y_{ZH} are the wave solutions for the corresponding parts of the strings.

If the term $\exp(-i\omega l/c) - \exp(-i\omega l/c)$ is zero, there is a solution which has no motion in the second string and a nodal point at B on the finite string. This means that there is a class of normal modes of the finite string which do not excite the second string into motion and therefore are identical to modes of the isolated string.

Except for these special modes the system has waves on both strings, for $\exp(-i\omega l/c) - \exp(i\omega l/c) \neq 0$. In this case, due to the boundary conditions, the complex amplitudes are connected by the relations

$$A_{AB} = -B_{AB} \qquad\qquad A_{EZ} = -B_{EZ} \tag{5}$$

$$A_{BD} = -A_{AB}e^{-2z} \qquad\qquad B_{BD} = A_{AB}e^{2z} \tag{6}$$

$$A_{EZ} = -A_{AB}(1 + e^{-2z}) \tag{7}$$

$$C = -A_{AB}(e^{2z} - e^{-2z}) \qquad \text{where} \tag{8}$$

$$z = i\omega l/c \tag{9}$$

Combining now the new forms of the wave solutions, which come from Equations (5), (6), (7), and (8) with the coupling eqs. (3) and (4), we can find the eigenfrequency equation for the system, which has the form

$$z(e^{-z} + e^{z}) = K(e^{-z} - e^{z})(2 + e^{-2z}) \qquad \text{where} \tag{10}$$

$$K = kl/(2T) \tag{11}$$

The eigenfrequency equation may be solved approximately for small K (i.e., small coupling of the two systems). This gives two different kinds of normal modes: modes with weak and modes with strong damping.

The normal modes for the weak damping case can be found by taking a "zero-th" approximation $K=0$, which implies the isolated-string eigenvalue equation

$$e^{-z}+e^{z}=0 \tag{12}$$

We call the solution of this equation ω_{0n}

$$\omega_{0n}=(2n+1)\,\pi c/2l \tag{13}$$

(We ignore here the modes with nodal points at B.) If K is small but nonzero, we expect a solution of equation (10) which is close to eq. (13), i.e., where $\exp(-z)+\exp(z)$ is small but nonzero (of order K). We can use the zero-th order solution for z on the right hand side of eq. (10), obtaining

$$\omega_n=\omega_{0n}+2Kc/[(2n+1)\,\pi l]+i8K^2c/[(2n+1)^2\,\pi^2 l]+0(K^3) \tag{14}$$

In this weak damping case the vibration of the system is concentrated in the finite string and the semi-infinite string vibrates with a small amplitude motion; from eqs. (6), (7), and (8) it can be proved that

$$A_{AB}=A_{BD}=-B_{BD} \tag{15}$$

$$|A_{AB}|\gg|A_{EZ}|>|C| \tag{16}$$

The vibrational pattern for the strongly damped normal modes is completely different. For these modes we solve eq. (10) for small K by assuming that $\exp(-z)$ is large, leading to the approximate relation

$$z=Ke^{-2z} \tag{17}$$

If $\exp(-z)$ is large then $\mathrm{Re}(z)$ is large and negative, so the real part of the right hand side of eq. (17) must be negative. This means that the imaginary part of z must be nearly an odd multiple of π. If we therefore take

$$z=-a+i(2n+1)\,\pi/2+ib \tag{18}$$

and take a large compared to $n\pi$, then the eigenfrequency of the oscillation is approximately

$$\omega_n=\omega_{0n}(1+1/2a)+iac/l \tag{19}$$

where a is determined by solving the real transcendental equation

$$a=Ke^{2a} \tag{20}$$

Table I. Some Representative Values of K for Given a[a]

a	K	a	K
5	2.27×10^{-4}	20	8.49×10^{-17}
6	3.68×10^{-5}	30	2.64×10^{-25}
7	5.82×10^{-6}	40	1.44×10^{-33}
8	9.00×10^{-7}	50	1.86×10^{-42}
9	1.37×10^{-7}	100	1.38×10^{-85}
10	2.06×10^{-8}	115	1.49×10^{-98}

[a] Using Eq. (20). One can easily see that the smaller the coupling constant K becomes, the larger is the damping part of the frequency.

Some representative values of K for given a are given in Table I. The energy of the vibration of these modes is mainly concentrated in the semi-infinite string and it is carried away very rapidly. Thus these modes, which do not exist if the strings are not coupled, damped out any initial excitation of the semi-infinite string. From eqs. (5), (6), (7), and (8) one can show that the amplitudes of this pattern of vibration obey

$$|C| > |A_{EH}| > |A_{AB}| \tag{21}$$

The important point in this case is that if K is sufficiently small (see Table I) the damping part of the frequency is independent of the index n of the normal mode as long as $n\pi \ll a$; it depends only on the coupling K.

We have so far studied the model system for the case in which the strings have the same propagation velocity c. This does not correspond well to any astrophysical analog: in stars the fluid waves propagate much slower than the velocity of the gravitational waves. If we let the strings have different wave speeds $c_j (j = 1, 2)$, then the previous results are still qualitatively valid, though more complicated. Thus the eigenfrequency equation for the two speeds is

$$z_1(e^{-z_1} + e^{z_1})[1 - K(e^{-z_2} - e^{z_2})/(z_2 e^{z_2})] = K(e^{-z_1} - e^{z_1}) \tag{22}$$

where

$$z_j = i\omega l/c_j \qquad (j = 1, 2) \tag{23}$$

In the following analysis we will assume that

$$r = c_1/c_2 \ll 1 \tag{24}$$

Following the same procedure that we developed for the weak and strong damping modes of the single wave speed system, we find that for the weakly damped case the frequency is approximately

$$\omega_n = \omega_{1n} + (rK/L)(c_1/l) + i(rMK^2/L^2)(c_1/l) + 0(K^3) \tag{25}$$

where

$$\omega_{jn} = (2n+1)(\pi c_j/2l) \tag{26}$$

$$L = (2n+1)\,\pi r/2 \tag{27}$$

$$M = 1 - \cos(2L) \tag{28}$$

The pattern of pulsation will be the same as that of small damping normal modes for the equal speed case; and the relation (16) between the amplitudes of the different parts of the system is still valid.

The strongly damped normal modes for the different wave speed case have the same pulsation pattern as that for the equal wave speed case. The eigenfrequency eq. (22) reduces to the analog of eq. (17), i.e.

$$z_2 = Ke^{-2z_2} \tag{29}$$

and the eigenfrequency has the form

$$\omega_n = \omega_{2n}(1 + 1/2a) + iac_2/l \tag{30}$$

where again a is the solution of the transcendental eq. (20). The amplitudes of the different parts of the system are governed by the relation (21) too; and the imaginary part of the frequency again does not depend on the index n provided n is small enough.

3. DISCUSSION

The simple model presented here mimics the most basic properties of pulsating stellar systems: a wave system of compact support (the finite string) coupled (through the spring) to a wave system on an infinite domain. It shows the normal modes we expect from numerical studies of stellar pulsation [2, 7], namely, the weakly damped modes whose frequencies are close to those the finite wave system would have on its own. The eigenfunctions of these modes have larger amplitudes in the finite string: the energy is located there and it only leaks out slowly and is radiated away. If a star is set in pulsation, say by being formed in a collapse, then the energy in the fluid's pulsations will be radiated away by the analogs of these modes.

Among the weakly damped modes of our model are some that do not damp at all. These have a nodal point at the attachment point of the spring. They are rather special, almost accidental, because they depend upon the special placement of the attachment point. They have analogs in real systems: spherical stars have spherical, dipole and odd parity nonradial pulsations that do not couple to gravitational radiation [8]. These are special, too; a rotating nonspherical star will not have any modes uncoupled from radiation, except for those which change its rotation rate to that of a nearby equilibrium.

The model also shows the strongly damped modes that Dyson [3] discovered. Notice that these modes have no counterparts in the uncoupled strings: as $K \to 0$ these eigenfrequencies go to infinity. Their energy is predominantly in the semi-infinite string, but the coupling to the finite string is essential: a semi-infinite string on its own has no normal modes satisfying an outgoing-wave boundary condition. It seems clear that the physical role of these modes is to carry away the initial excitation energy of the semi-infinite string. The weakly damped modes cannot do this: given some initial excitation of the finite string, these modes can be superposed in a unique way to achieve that amplitude. But then their amplitudes in the semi-infinite string are determined, and cannot be adjusted to represent any additional independent initial excitation of that string. They damp quickly because the energy they contain is already in the semi-infinite string. It simply moves out along the string.

These strongly damped modes may already have been found in a more realistic system: the normal modes of the Schwarzschild metric [9, 10]. This is best compared to an isolated semi-infinite string, with no coupling to another dynamical system. However, unlike the string on its own, the Schwarzschild metric has nontrivial normal modes because the gravitational waves back-scatter off the curvature of space-time, so the waves that are outgoing at infinity can be ingoing at the horizon. Indeed if we were to put a point mass on our semi-infinite string in place of the coupling to the other string, we would also have nontrivial normal modes here, because waves could reflect off the mass. (This explains "why" the coupled system we have studied here has strongly-damped modes: they are waves in the semi-infinite string which reflect off the attachment point of the string.) We suggest that the same modes probably exist in real stars, and it might be possible to find at least a first approximation to them by studying the modes of the gravitational field in the curved background geometry of the stars, perhaps by WKB techniques like those of Schutz and Will [10]. In this context we note that Comins and Schutz [11] showed that modes do exist if the background geometry has, or nearly has, an ergoregion. But these were weakly damped modes and may not be related to the modes we are seeking here.

What of the instability found by Dyson for large values of the coupling? We have found no evidence of it here, and can in fact exclude instabilities from the fact that the total energy of the system is positive-definite and decreasing because of the outgoing-wave boundary condition. (The method of Friedman and Schutz [12] can be used to give a rigorous proof of stability.) Dyson's model does not have a positive-definite energy, so such instabilities are allowed. Realistic systems may also fall prey to these instabilities: the total gravitational energy is not positive-definite [13]. The analog of the coupling constant would be the compactness GM/Rc^2 of the star, so it would be interesting to see if along a sequence of increasing compactness such instability set in. Again, there is no evidence either for or against them from existing numerical calculations. There is a class of gravitational-wave induced instabilities in stars, discovered by Chandrasekhar [14] and studied in detail by Friedman and Schutz [13, 15] and Comins [16, 17], which arises through the coupling of the matter system to the wave system [1]. But the analog with Dyson's instability is poor: the mode which goes unstable is a weakly damped mode. There is another sort of instability in stars which depends on the compactness: Chandrasekhar's [18] post-Newtonian instability. But this is an instability of a nonradiative mode (a spherical pulsation), and again it is not the analog of Dyson's instability. The question of the existence of Dyson's instability in real systems remains open.

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