

Normal modes of Bardeen discs – I. Uniformly rotating incompressible discs

Bernard F. Schutz and Alison M. Bowen *Department of Applied Mathematics and Astronomy, University College, PO Box 78, Cardiff CF1 1XL*

Received 1982 May 6; in original form 1981 March 3

Summary. We study the normal modes and stability of a sequence of perfect fluid discs in the Bardeen approximation, which enables systems in which rotation, pressure and self-gravitation are all of comparable importance to be studied with relative ease. The present paper treats uniformly rotating incompressible discs analytically as a prelude to the study of compressible polytropic discs in subsequent papers. The normal-mode eigenfunctions are found analytically, and the equation for the dynamical eigenfrequencies reduces to a cubic polynomial. The effects of gravitational radiation and viscosity on the spectrum are also studied, viscosity in some detail in order to show how to determine the left-eigenfunctions. The present sequence of discs may be regarded as an approximation to Maclaurin spheroids. In their structure the approximation leaves much to be desired, but in the onset of instability as a function of the Ostriker parameter $T/|W|$, agreement is remarkably good.

1 Introduction

Our present understanding of the perturbations and stability of rotating stellar models is rudimentary compared to that for non-rotating models (see Ledoux 1974 for a review of the non-rotating case). There has recently been considerable work on slowly-rotating models (e.g. Lebovitz 1970; Hansen, Cox & Carroll 1978; Berthomieu *et al.* 1978), in which rotational corrections to the star's structure are unimportant. At the other extreme, infinitesimally thin rapidly rotating fluid discs have been studied in some detail, particularly as simple models for galaxies (Hunter 1963, 1965), but these all have instabilities. The intermediate case, where rotational and pressure support are comparable, has remained largely unexplored. From the point of view of stability theory, this is the most interesting case, because it embraces the models which are marginally stable, secularly and dynamically. The considerable numerical difficulties of constructing such models and calculating their normal modes have only recently begun to be attacked. As Dyson & Schutz (1979) have emphasized, general stability criteria are of no help in pinpointing the important dynamical and secular instability points along sequences of differentially rotating perfect fluids: only the calculation of the normal modes of such stars gives the necessary information. Against such a background, it is helpful to have a relatively tractable sequence of systems in which rotation and

pressure are comparable and whose normal modes may be studied accurately without using huge amounts of computer time. The Bardeen 'warm' discs (Bardeen 1975) provided such a sequence. This and two subsequent papers (Schutz & Verdaguer 1983; Verdaguer 1983) study these discs with a view toward arriving at an understanding of the general features of normal modes of differentially rotating, self-gravitating perfect fluids.

The Bardeen discs are described below and in Bardeen (1975). Briefly, they are approximate solutions to the equilibrium equations for rotating bodies, in which the body is assumed to be extremely oblate and its structure is calculated only to first order in its polar thickness. This is an accurate approximation for very rapid rotation, which has led to its use as a model for investigating galactic spiral structure (Bardeen 1975, and in preparation). When the pressure is increased (the disc 'warmed up') the approximation becomes less realistic, but this does not diminish its usefulness for our purposes. The large pressure discs are still systems in which rotation, pressure and self-gravitation are all comparable, and we expect that their normal modes should exhibit most of the complicated behaviour of modes of more realistic systems. The discs' principal advantage is the simplicity of the numerical calculations. Finding the normal modes of realistic stars requires solving a set of two-dimensional partial differential equations, the third dimension — azimuth — being removed by Fourier analysis on the axially symmetric unperturbed background. The Bardeen approximation assumes in addition that the vertical direction (perpendicular to the equatorial plane) is always in equilibrium, and this reduces the normal mode problem to ordinary differential equations, which are much easier to solve numerically than the partial differential equations.

Other sequences of discs with pressure have recently been studied: Takahara (1976, 1978), Iye (1978) and Smith (1979). These authors investigated infinitesimally thin discs with two-dimensional pressure, generally taken to be a polytropic function of the surface density. These studies have mainly been interested in galactic structure and spiral density-wave theory, for which the two-dimensional pressure is meant to approximate random stellar motions. For studies of individual stars, the Bardeen discs are rather better approximations, since they solve the vertical structure, including corrections to the gravitational field due to the thickness. This leads to a somewhat different eigenfrequency equation, though of course the qualitative features and much of the mathematics is the same.

We will accordingly concentrate in this series on the questions that arise naturally from stellar stability theory and that were not studied in detail in previous papers, namely the behaviour of modes in the dynamically stable part of the sequences, and in particular the onset of secular and dynamical instability and, for differentially rotating discs, the nature of the continuous spectrum.

In this paper we introduce our study by considering the one sequence of discs which can be treated analytically: incompressible discs. This will guide us in interpreting the modes of the $n = 2$ polytropic discs (Schutz & Verdaguer 1983), and at the same time it permits us to make a comparison with the corresponding exact models, the Maclaurin spheroids, to judge the effect of the approximations on the various stability points.

2 The equilibrium discs

Following Bardeen (1975) we here briefly sketch the 'warm disc approximation' in its simplest case, uniform density and rigid rotation. We begin by constructing an infinitesimally thin disc (the 'cold' disc, which has zero pressure) and then compute the first corrections when hydrodynamic pressure is taken into account.

The cold disc, supported entirely by its angular velocity, Ω_c , satisfies the equations, in

cylindrical coordinates (r, ϕ, z) ,

$$\Omega_c^2 r + \partial v_c / \partial r = 0, \tag{2.1}$$

$$\nabla^2 v_c = -4\pi G \rho = -4\pi G \sigma \delta(z), \tag{2.2}$$

where $\sigma(r)$ is the surface density of the disc. The solution (Hunter 1963) is most conveniently expressed in terms of the coordinate

$$\eta = (1 - r^2/R^2)^{1/2}, \tag{2.3}$$

where R is the radius of the disc:

$$\sigma(\eta) = \frac{2\Omega_c^2 R}{\pi^2 G} \eta \equiv \sigma_0 \eta, \tag{2.4}$$

$$v_c(\eta, z=0) = \frac{1}{2} \Omega_c^2 R^2 (1 + \eta^2), \tag{2.5}$$

$$M = 2\pi \int_0^R \sigma r dr = \frac{4\Omega_c^2 R^3}{3\pi G}. \tag{2.6}$$

By integrating (2.2) in z from $-\epsilon$ to $+\epsilon$ and taking the limit $\epsilon \rightarrow 0$ one finds that $\partial v_c / \partial z$ is discontinuous at the disc and has the limit from above

$$\frac{\partial v_c}{\partial z}(\eta, z=0^+) = -2\pi G \sigma(\eta) = -2\pi G \sigma_0 \eta, \tag{2.7}$$

so that near the disc we have (except at $\eta = 0$, the rim of the disc)

$$v_c(\eta, z) = \frac{1}{2} \Omega_c^2 R^2 (1 + \eta^2) - \frac{4\Omega_c^2 R}{\pi} \eta |z| + O(z^2). \tag{2.8}$$

The cold disc is the starting point for a family of discs of uniform density ρ and uniform angular velocity Ω_w , both of which change from one member of the family to another. The family may be defined by requiring all its members to have the same surface density $\sigma(\eta)$ as the cold disc (and therefore the same mass and radius). At any point the disc has thickness $h(\eta)$, found by

$$\sigma(\eta) = \int_{-h}^h \rho dz = 2\rho h(\eta), \tag{2.9}$$

$$h(\eta) = \frac{\Omega_c^2 R}{\pi^2 G \rho} \eta. \tag{2.10}$$

The surface of the disc, $|z|=h$, is an ellipsoid. Since the density is finite everywhere, the gravitational field and its first derivatives are continuous. By symmetry, the warm disc's gravitational field has the form

$$v_w(\eta, z) = v_w(\eta, 0) + A(\eta)z^2 + O(z^4),$$

where A is a function to be determined.

If $h \ll R$ then we should expect $|\partial v_w / \partial z| \gg |\partial v_w / \partial \eta|$ except near $\eta = 0$. We therefore approximate

$$\nabla^2 v_w \approx \frac{\partial^2}{\partial z^2} v_w = -4\pi G \rho \quad (|z| \leq h) \tag{2.11}$$

which implies

$$A = -2\pi G\rho,$$

or

$$\nu_w(\eta, z) = \nu_w(\eta, 0) - 2\pi G\rho z^2 + O(z^4). \quad (2.12)$$

The equatorial value $\nu_w(\eta, 0)$ is determined by the boundary condition that $\nu_w \rightarrow 0$ at infinity. But, within the approximation (2.11), ν_w and ν_c are identical for $|z| > h$ if

$$\nu_w(\eta, 0) = \nu_c(\eta, 0) - \frac{\pi G}{2\rho} \sigma^2(\eta). \quad (2.13)$$

[This is easily proved by showing that $\nu_w(\eta, h) = \nu_c(\eta, h)$ and $\partial\nu_w/\partial z(\eta, h) = \partial\nu_c/\partial z(\eta, h)$ provided (2.13) holds.]

Thus (2.12) and (2.13) are the unique solution for ν_w which vanishes at infinity. Again, this argument relies on neglecting $\partial\nu/\partial\eta$ (the plane-parallel assumption), so it will fail near the rim $\eta = 0$. But it should give an accurate approximation elsewhere.

The equilibrium angular velocity of the warm disc, Ω_w , can be found once we know the pressure, which is determined by the condition that it must support the disc vertically. (We are using isotropic hydrodynamic pressure, not the kinetic pressure of a collisionless gas.) Vertical equilibrium demands, for $z \leq h$,

$$\partial p/\partial z = \rho \partial\nu_w/\partial z = -4\pi G\rho^2 z.$$

The boundary condition that $p = 0$ at $z = h$ gives

$$p = 2\pi G\rho^2(h^2 - z^2), \quad (2.14)$$

$$p(\eta, z = 0) = \frac{1}{2} \pi G \sigma^2(\eta). \quad (2.15)$$

Then radial equilibrium gives (for $z = 0$)

$$\partial p/\partial r - \rho \partial\nu_w/\partial r = \rho r \Omega_w^2, \quad (2.16)$$

or

$$\Omega_w^2 = \Omega_c^2 \left(1 - \frac{8\Omega_c^2}{\pi^3 G\rho} \right). \quad (2.17)$$

A convenient parameter for describing this family of discs is the *aspect ratio*

$$\mu = \frac{h(r=0)}{R} = \frac{\Omega_c^2}{\pi^2 G\rho}, \quad (2.18)$$

in terms of which we have

$$\Omega_w^2 = \Omega_c^2 (1 - 8\mu/\pi), \quad (2.19)$$

$$\nu_w(\eta, z = 0) = \nu_c(\eta, z = 0) - \frac{2}{\pi} \mu \Omega_c^2 R^2 \eta^2. \quad (2.20)$$

It is not difficult to calculate that the ratio of kinetic to potential energy of such a disc is

$$\tau = \frac{T}{|W|} = \frac{1}{2} \frac{\pi - 8\mu}{\pi - 2\mu}. \quad (2.21)$$

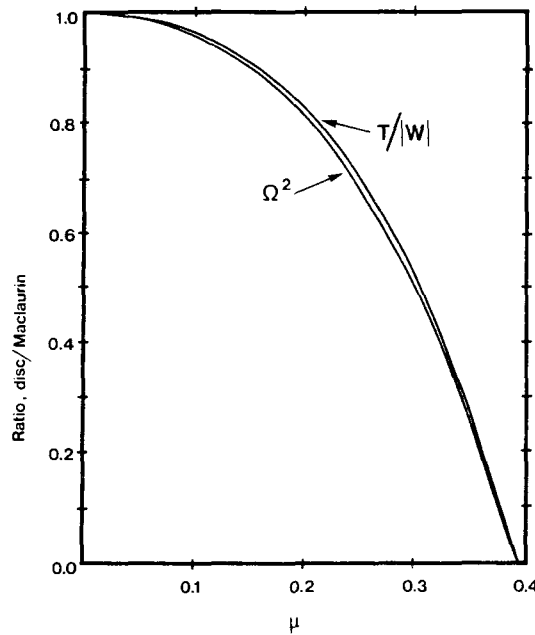


Figure 1. Comparison of Ω^2 and $T/|W|$ for the discs and the Maclaurin spheroids of the same aspect ratio [same eccentricity $e = \sqrt{1 - \mu^2}$]. Each curve is the ratio of the indicated quantity in the disc to that in the Maclaurin sequence. The limiting values of $\mu = 0$ (cold disc) and $\mu = \pi/8 \approx 0.3927$ ($\Omega_w = 0$) correspond to eccentricities of 1 and 0.9197, respectively.

These formulae allow us to compare the equilibrium discs with their Maclaurin counterparts of the same aspect ratio. In Fig. 1 we plot two curves: (i) the ratio of Ω_w^2 (equation 2.19) to the Maclaurin Ω^2 (Chandrasekhar 1969); and (ii) the ratio of $T/|W|$ for the discs (equation 2.21) to $T/|W|$ for the Maclaurins, given by

$$(T/|W|)_{\text{Maclaurin}} = (3/e^2 - 2 - 3\sqrt{1 - e^2})/e \sin^{-1} e/2.$$

The comparison shows that the discs are a good approximation to the Maclaurin spheroids of the same shape only for small μ , say $0 \leq \mu \leq 0.15$. We shall see later that the instability points of the two sequences occur at larger values of μ , so one would not necessarily expect any relation between them. Nevertheless, the values of $T/|W|$ for the marginally stable members of each sequence turn out to be remarkably close. We will discuss the significance of this coincidence later.

3 The normal mode equations

We can study perturbations of these discs within the same approximations by adding the assumption that vertical equilibrium applies even to the perturbation. This should be good provided the wavelength of the perturbation is large compared to h . The equation of motion for a first-order perturbation is

$$\delta \dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla) \delta \mathbf{v} + (\delta \mathbf{v} \cdot \nabla) \mathbf{v} - \nabla \delta \nu_{\text{eff}} = 0 \tag{3.1}$$

where the effective potential, $\delta \nu_{\text{eff}}$, comprises the gravitational and pressure effects:

$$\delta \nu_{\text{eff}} = \delta \nu - \delta p/\rho. \tag{3.2}$$

The assumption of vertical equilibrium means (i) that we can use equations (2.13) and (2.15) to calculate the perturbed effective potential, e.g.

$$\delta \nu_{\text{eff}}(\eta, z = 0) = \delta \nu_c(\eta, z = 0) - 2\pi G \sigma \delta \sigma/\rho \tag{3.3}$$

(the second term here contains equal contributions from the pressure and gravitational field); and (ii) that the horizontal components of $\delta \mathbf{v}$ are independent of z , so that we can integrate equation (3.1) over z to get a dynamical equation involving only components parallel to the equatorial plane. In fact, this yields equation (3.1) restricted to $z = 0$. This and the vertically integrated continuity equation

$$\delta \dot{\sigma} + \nabla \cdot (\sigma \delta \mathbf{v} + \mathbf{v} \delta \sigma) = 0 \quad (3.4)$$

form an independent system of equations for the functions of two variables $\delta \sigma(\eta, \phi)$ and $\delta \mathbf{v}(\eta, \phi)$. The vertical dependence in the problem can in principle be reconstructed later, but we shall not need to do so. From now on, *all* perturbed quantities will be interpreted as functions of η and ϕ only.

Equations (3.1) and (3.4) must be solved under the boundary condition that $p = 0$ on the surface. Since the surface must follow the fluid flow on the boundary, we have Dp/Dt ($\partial/\partial t + \mathbf{v} \cdot \nabla$) $p = 0$ on the surface. The boundary condition is that the Lagrangian change in Dp/Dt should also vanish at the surface. But since $Dp/Dt \equiv 0$ everywhere in the unperturbed configuration, its Eulerian and Lagrangian changes are equal, and we conclude that the boundary condition for our problem is

$$\delta(Dp/Dt) = 0 = (\delta \dot{p} + \delta \mathbf{v} \cdot \nabla p + \mathbf{v} \cdot \nabla \delta p)_{\eta=0}. \quad (3.5)$$

We can use the continuity equation (3.4) to simplify this considerably, since by (2.15) we have $\delta p = \pi G \sigma \delta \sigma$. A straightforward calculation converts (3.5) into

$$0 = - [\eta(\nabla \cdot \mathbf{v} \delta \sigma + \sigma \nabla \cdot \delta \mathbf{v})]_{\eta=0}. \quad (3.6)$$

Since $\nabla \cdot \mathbf{v} = 0$ in the unperturbed configuration, and since $\sigma = \sigma_0 \eta$, this boundary condition will be satisfied if we only take $\nabla \cdot \delta \mathbf{v}$ to be finite at the surface.

In order to solve our equations it will be convenient to introduce a general potential representation for $\delta \mathbf{v}$:

$$\delta \mathbf{v} = -i \nabla \alpha - \nabla \times (\beta \mathbf{e}_z), \quad (3.7)$$

where α and β are arbitrary functions of η and ϕ , and \mathbf{e}_z is the unit vertical vector. Any vector field in the equatorial plane can be represented this way. Hunter (1963) showed that for the cold disc the following expansions in associated Legendre polynomials led to equations for the functions $[\alpha_l^m(t), \beta_l^m(t), \nu_l^m(t), \Sigma_l^m(t)]$ which completely decoupled for different values of l and m :*

$$\begin{bmatrix} \alpha \\ \beta \\ \delta \nu_c \\ \eta \delta \sigma \end{bmatrix} = R^2 \Omega_c \sum_{m=-\infty}^{\infty} \sum_{l=0}^{\infty} \begin{bmatrix} \alpha_l^m(t) \\ \beta_l^m(t) \\ \Omega_c \nu_l^m(t) \\ \frac{\Omega_c}{\pi^2 GR} \Sigma_l^m(t) \end{bmatrix} P_{2l+m}^m(\eta) \exp(im\phi). \quad (3.8)$$

We only need 'even' orders in the expansion (the use of $2l$ in the index for the associated Legendre polynomials) because by inspection the equations are invariant if $\eta \rightarrow -\eta$ everywhere. Moreover, (3.8) automatically satisfies the boundary condition. One crucial relation

* Note that these are *not* spherical harmonic expansions; in particular m is not restricted by l . The completeness of the expansions follows from Fourier analysis in ϕ followed by the completeness of the associated Legendre polynomials for functions of η .

which makes the equations separate when (3.8) is used is

$$\nu_l^m = 2\gamma_l^m \Sigma_l^m \quad (3.9)$$

with

$$\gamma_l^m = \frac{(2l+2m)!(2l)!}{2^{4l+2m+1} [(l+m)!l!]^2} \quad (3.10)$$

(Our notation γ_l^m is slightly at variance with Hunter's: he would call this γ_{2l+m}^m .)

Since our equations differ from Hunter's cold disc equations only in the replacement of δv_c by δv_{eff} , and since (3.3) shows that this change is proportional to $\eta\delta\sigma$, our equations also separate if we use the expansions (3.8), with (3.9) replaced by

$$\nu_l^m = 2(\gamma_l^m - 2\mu/\pi) \Sigma_l^m, \quad (3.11)$$

$$\equiv 2\Gamma_l^m \Sigma_l^m \quad (3.12)$$

where μ is defined in equation (2.18).

The divergence and curl of (3.1) now yield, respectively,

$$\dot{\alpha}_l^m + i(m\Omega_w \alpha_l^m + 2\Omega_w \beta_l^m - 2\Omega_c \Gamma_l^m \Sigma_l^m) = 0, \quad (3.13)$$

$$\dot{\beta}_l^m + i(m\Omega_w \beta_l^m - 2\Omega_w \alpha_l^m) = 0, \quad (3.14)$$

while the continuity equation gives

$$\dot{\Sigma}_l^m + i(m\Omega_w \Sigma_l^m + 2\Omega_c C_l^m \alpha_l^m + 2m\Omega_c \beta_l^m) = 0, \quad (3.15)$$

where we define

$$C_l^m = (2l+m)(2l+m+1) - m^2. \quad (3.16)$$

Normal modes are solutions of (3.13–3.15) with the time-dependence $\exp(i\omega t)$. If we define

$$\lambda = (\omega + m\Omega_w)/\Omega_c, \quad \Omega = \Omega_w/\Omega_c = (1 - 8\mu/\pi)^{1/2} \quad (3.17)$$

then we find the eigenvalue equation

$$\lambda^3 + 4(\Gamma_l^m C_l^m - \Omega^2)\lambda + 8m\Omega\Gamma_l^m = 0. \quad (3.18)$$

Equations (3.13–3.15) are valid for all (l, m) , except that for $l=0$ only the combination $\alpha_0^m + \beta_0^m$ is physically significant, as inspection of (3.7–3.8) shows. In this case, equation (3.15) and the sum of (3.13) and (3.14) give the quadratic eigenvalue equation

$$\lambda^2 - 2\Omega\lambda + 4m\Gamma_0^m = 0. \quad (3.19)$$

This quadratic is a factor of (3.18) for $l=0$: the extra root $\lambda = -2\Omega$ of (3.18) is spurious.

We shall discuss the stability of the discs in the next section. First we look at the general character of the spectrum of eigenfrequencies. If we fix attention on a fixed $m \neq 0$, then the limit $l \rightarrow \infty$ is the limit of short wavelength. In this limit,

$$\gamma_l^m \approx (2\pi l)^{-1}, \quad C_l^m \approx 4l^2. \quad (3.20)$$

Then equation (3.18) becomes

$$\lambda^3 - (32\mu l^2/\pi)\lambda - 16m\mu\Omega/\pi = 0. \quad (3.21)$$

This has two families of solutions,

$$\lambda \approx \pm (32\mu/\pi)^{1/2}l \quad (3.22)$$

and

$$\lambda \approx -\frac{1}{2}m\Omega/l^2. \quad (3.23)$$

The pattern speed (relative to the inertial frame) of a mode is $\omega_p = -\omega/m$, and for these families we have respectively

$$\omega_p \approx \pm \Omega_c(32\mu/\pi)^{1/2}l/m \quad (3.24)$$

and

$$\omega_p \approx \Omega_w(1 + 1/2l^2). \quad (3.25)$$

The first family are the analogues in these discs of the p -modes of non-rotating compressible stars. Although the discs are incompressible, their two-dimensional modes behave like compressible modes because the disc can change its thickness. Since the wavelength of these waves is roughly R/l , the wave speed is

$$c_s \approx R\Omega_c(32\mu/\pi m^2)^{1/2}. \quad (3.26)$$

This is independent of wavelength, as it is for p -waves.

The second family forms a sequence of pattern speeds which approaches Ω_w monotonically from above. These can be called Rossby modes, by analogy with the Rossby waves described by Greenspan (1968, p. 89).

4 Stability of the discs

Dynamical stability is assured if the roots of (3.18) are all real. The marginal stability point occurs when (3.18) has a double root, and this implies

$$4(\Omega^2 - C_l^m \Gamma_l^m)^3 - 27(m \Gamma_l^m \Omega)^2 = 0. \quad (4.1)$$

This is a cubic equation for μ . As we should expect from Hunter (1965), there are unstable modes for each m for sufficiently small μ . The eigenvalue which is hardest to stabilize is $m = 2, l = 0$: this is the familiar 'bar mode', and is stable for $\mu > \pi/16$. This is then the point of marginal dynamical stability.

As Friedman & Schutz (1978) have shown, the onset of secular instability to viscosity or gravitational radiation occurs when a mode has, respectively, $\lambda = 0$ or $\omega = 0$. As in the Maclaurin spheroids, the bar mode is the hardest to stabilize against viscosity and the easiest to stabilize against gravitational radiation. These transitional points both occur in the model with $\mu = 3\pi/32$. We have performed detailed calculations of the effects of both kinds of dissipation in these discs, and the results bear out the general picture presented by Friedman & Schutz (1978). Both require a knowledge of the left eigenfunctions associated with eigenfrequencies as well as the right eigenfunctions discussed above. This point is treated in the Appendix, and applied to viscosity. The gravitational radiation calculation is analogous and will not be commented on further.

The instability points are summarized in Table 1. All of them occur at values of the eccentricity too small for the discs to be a good approximation to the Maclaurins. Nevertheless the values of $T/|W|$ at which all the instabilities occur are remarkably similar in the two sequences.

Table 1. Critical values of the equilibria at the points of instability of the two sequences: aspect ratio μ , eccentricity e , ratio of kinetic to potential energy $T/|W|$ and angular velocity in units of $4\pi G\rho$.

	μ	e	$T/ W $	$\Omega^2/4\pi G\rho$
Dynamical instability points:				
$m = 2, l = 0$ disc	0.1964	0.98053	0.2857	0.0771
First Maclaurin*	0.3033	0.95289	0.2738	0.1101
$m = 3, l = 0$ disc	0.1719	0.98512	0.3157	0.0759
Second Maclaurin†	0.2552	0.9669	0.3029	0.1049
$m = 0, l = 2$ disc	0.0399	0.99920	0.4609	0.0282
$m = 0, l = 3$ disc	0.0435	0.99905	0.4573	0.0304
Axisymmetric Maclaurin‡	0.0465	0.99892	0.4574	0.0324
Secular instability points:				
$m = 2, l = 0$ disc	0.2945	0.95564	0.1538	0.0578
Maclaurin bar mode*	0.5827	0.81267	0.1375	0.0936

Notes

* Chandrasekhar (1969).

† The third-harmonic instability, Chandrasekhar & Lebovitz (1963).

‡ The fourth-harmonic instability, Bardeen (1971). This corresponds to our $m = 0, l = 2$ eigenfunction. To our knowledge the axisymmetric Maclaurin sixth-harmonic instability has not been calculated. It corresponds to our $m = 0, l = 3$ mode, which is harder to stabilize than $m = 0, l = 2$.

This coincidence suggests that we can adopt the following point of view on the Bardeen sequences. Instead of being discouraged by Fig. (1), that the Bardeen approximation is not a good approximation to the Maclaurins when systems of the same eccentricity are compared, we can instead regard the Bardeen discs as an independent family of rotating systems, with a certain planar pressure which produces an effective potential given by equation (3.3). Along the sequence of these self-consistent dynamical systems the changes of stability occur in the same fashion as along the Maclaurin sequence: first the $m = 2$ secular, then the $m = 2$ dynamical, etc. And when the two sequences are compared according to the Ostriker parameter $T/|W|$, the places where these points of marginal stability occur are remarkably similar.

This point of view suggests that the polytropic Bardeen discs studied in the following paper, Schutz & Verdaguer (1982), will in fact be good guides to the qualitative and (in $T/|W|$) quantitative behaviour of stability along more realistic sequences.

Acknowledgments

Many of the results in this paper were originally obtained jointly by one of us (BFS) and James Bardeen some years ago. They have been gathered together here with considerable new material because they have not been published elsewhere and because they provide valuable insight into the compressible problem; but Bardeen's contribution is very large, both in devising the original approximation and in demonstrating its feasibility in the incompressible uniformly rotating case, and we gratefully acknowledge it. We have also benefited from our collaboration with E. Verdaguer who checked and often corrected many of our calculations, from conversations with C. Hunter, who brought several references to our attention, and from remarks by a referee that helped us to correct an error in an earlier version of this manuscript.

References

- Bardeen, J. M., 1971. *Astrophys. J.*, **167**, 425.
 Bardeen, J. M., 1975. In *Dynamics of Stellar Systems*, ed. Hayli, A., Reidel, Dordrecht, Holland.

- Berthomieu, G., Gonczi, G., Groff, Ph., Provost, J. & Rocca, A., 1978. *Astr. Astrophys.*, **70**, 597.
 Chandrasekhar, S., 1969. *Ellipsoidal Figures of Equilibrium*, Yale University Press, New Haven.
 Chandrasekhar, S. & Lebovitz, N. R., 1963. *Astrophys. J.*, **137**, 1162.
 Dyson, J. & Schutz, B. F., 1979. *Proc. R. Soc. (London) A*, **368**, 389.
 Friedman, J. L. & Schutz, B. F., 1978. *Astrophys. J.*, **222**, 281.
 Greenspan, H. P., 1968. *The Theory of Rotating Fluids*, Cambridge University Press.
 Hansen, C. J., Cox, J. P. & Carroll, B. W., 1978. *Astrophys. J.*, **226**, 210.
 Hunter, C., 1963. *Mon. Not. R. astr. Soc.*, **126**, 23.
 Hunter, C., 1965. *Mon. Not. R. astr. Soc.*, **129**, 321.
 Iye, M., 1978. *Publ. Astr. Soc. Japan*, **30**, 223.
 Lebovitz, N. R., 1970. *Astrophys. J.*, **160**, 701.
 Ledoux, P., 1974. In *Stellar Instability and Evolution*, IAU Symp. 59, eds Ledoux, P., Noels, A. & Rodgers, A. W., Reidel, Dordrecht, Holland.
 Schutz, B. F., 1980. *Mon. Not. R. astr. Soc.*, **190**, 7.
 Schutz, B. F. & Verdaguier, E., 1983. *Mon. Not. R. astr. Soc.*, **202**, 881.
 Smith, W. M., 1979. *Astr. J.*, **84**, 979.
 Takahara, F., 1976. *Prog. Theor. Phys.*, **56**, 1665.
 Takahara, F., 1978. *Publ. Astr. Soc. Japan*, **30**, 253.
 Verdaguier, E., 1983. *Mon. Not. R. astr. Soc.*, **202**, 903.

Appendix: Left-eigenvectors and the effect of viscosity

If we take the point of view that viscosity is small and will therefore make a small change in the eigenfrequencies, then it is possible to find this change in terms of the inviscid eigenfunctions. This is of course familiar to students of quantum mechanics, but in our case a complication is that the inviscid eigenvalue problem is non-selfadjoint. As was pointed out in an earlier paper (Schutz 1980) we must in this case solve the adjoint of the inviscid system of equations, or in other words find their left-eigenvectors. There are two possible approaches to this. Since our Legendre-polynomial expansions have reduced our differential equations to algebraic ones (equations 3.13–3.15), we could similarly cast the viscid equations in matrix form and use the left-eigenvectors of the inviscid matrix to find the frequency change to first order. This method depends, however, on the separation in Legendre polynomials peculiar to our present family of discs. Instead, we shall adopt a method which generalizes to any uniformly rotating fluid, compressible or not. This is to find an operator S which transforms a right-eigenfunction directly into a left-eigenfunction. One of us (Schutz 1980) discovered such a transformation in the Lagrangian representation of the perturbation equations, and in this appendix we shall find its Eulerian equivalent and use it to find the viscous eigenfrequencies.

A.1 THE ADJOINT EQUATIONS

Equations (3.1) and (3.4), the vertically integrated inviscid equations, can be written in the eigenvalue form

$$2\Omega\epsilon_{jk}\delta v^k + \nabla_j\delta\Phi[\delta\sigma] = i\lambda\delta v_j, \quad (\text{A.1})$$

$$-\Omega_c^{-1}\nabla_j(\sigma\delta v^j) = i\lambda\delta\sigma \quad (\text{A.2})$$

where $\delta \Phi$ is a self-adjoint operator on $\delta \sigma$ which gives $\sigma v_{\text{eff}}/\Omega_c$,

$$\delta \Phi [\delta \sigma] (\mathbf{r}) = - \frac{2\pi G}{\rho \Omega_c} \sigma \delta \sigma(\mathbf{r}) + \frac{G}{\Omega_c} \int \frac{\delta \sigma(\mathbf{r}') d^2 x'}{|\mathbf{r} - \mathbf{r}'|}, \tag{A.3}$$

where λ is defined in equation (3.17), and where ϵ_{jk} is the anti-symmetric tensor whose Cartesian components are $\epsilon_{xy} = 1, \epsilon_{yx} = -1, \epsilon_{xx} = \epsilon_{yy} = 0$. We define a Hilbert space for this problem to be the vector space whose elements are pairs $(\delta \mathbf{v}, \delta \sigma)$ and whose inner product between two pairs $y_1 = (\delta \mathbf{v}_1, \delta \sigma_1)$ and $y_2 = (\delta \mathbf{v}_2, \delta \sigma_2)$ is

$$\langle y_1, y_2 \rangle = \int \sigma \delta \mathbf{v}_1^* \cdot \delta \mathbf{v}_2 d^2 x + \int \delta \sigma_1^* \delta \sigma_2 d^2 x, \tag{A.4}$$

where a^* denotes complex conjugation. (The inner product we choose here is to a large extent arbitrary: it does not even have consistent dimensions. The only role it plays in the end is to provide a topology for the Hilbert space.) If we summarize the equations (A.1–A.2) as $L[y] = i\lambda y$, then the adjoint operator L^* is defined by demanding that

$$\langle y_1, L[y_2] \rangle = \langle L^*[y_1], y_2 \rangle \tag{A.5}$$

for all y_1 and y_2 obeying suitable boundary and differentiability conditions. This operator may be found by integrating by parts. If L has an eigenvalue $i\lambda$ then L^* has an eigenvalue $-i\lambda^*$, and if we denote the eigenvector by $y = (\mathbf{p}, q)$ then the eigenequation $L^*[y] = -i\lambda^* y$ becomes

$$-2\Omega \epsilon_{kj} p^j + \Omega_c^{-1} \nabla_k q = -i\lambda^* p_k \tag{A.6}$$

$$-\delta \Phi [\nabla_j (\sigma p^j)] = -i\lambda^* q. \tag{A.7}$$

The boundary condition is that p^k and q be finite. The pair (\mathbf{p}, q) , or more properly its adjoint, is called the left-eigenvector of L for the eigenvalue λ .

A.2 TRANSFORMATION OF RIGHT-EIGENVECTORS TO LEFT-EIGENVECTORS

The adjoint equations could be solved in a manner analogous to that used for the original system, but it is quicker and more instructive to obtain the solution directly from the original eigensolution. Schutz (1980) has shown that the Lagrangian form of the perturbation equations has a remarkable symmetry operator which, when applied to a right-eigenvector, produces the left-eigenvector for the same eigenvalue. It is not hard to discover the corresponding transformation here. Let S be an operator which combines complex conjugations with ϕ -reflection:

$$S\delta \mathbf{v} = (\delta v^r, -\delta v^\phi), \quad S^2 = 1. \tag{A.8}$$

If $\delta \mathbf{v}$ has axial eigenvalue m , so does $S\delta \mathbf{v}$. The operator S commutes with ∇ and with metric operations, such as raising and lowering indices, but it anti-commutes with ϵ_{kj} (i.e. with the curl operator) because ϕ -reflection changes the handedness of the coordinates. These facts enable one to prove that if $(\delta \mathbf{v}, \delta \sigma)$ is a right-eigensolution of the original equations with eigenvalue λ , then the associated eigensolution of equation (A.6) and (A.7) is

$$\mathbf{p} = S\delta \mathbf{v}, \quad q = \Omega_c \delta \Phi [S\delta \sigma]. \tag{A.9}$$

The left-eigensolution is the adjoint of this.

A.3 THE CHANGE IN THE EIGENFREQUENCY

The Navier–Stokes equation for incompressible viscous flow can be integrated vertically to give the following eigenvalue problem

$$2\Omega\epsilon_{jk}\delta v^k + \nabla_j\delta\Phi[\delta\sigma] - (\sigma\Omega_c)^{-1}\nabla_k t_j^k = i\lambda\delta v_j \quad (\text{A.10})$$

$$-\Omega_c^{-1}\nabla_j(\sigma\delta v^j) = i\lambda\delta\sigma, \quad (\text{A.11})$$

where the viscous stress tensor is (for incompressible flow)

$$t_{kj} = -\sigma\nu(\nabla_k\delta v_j + \nabla_j\delta v_k), \quad (\text{A.12})$$

ν being the kinematic viscosity. We may write the viscid system as:

$$L[y] + L'[y] = i\lambda y, \quad (\text{A.13})$$

generalizing our earlier notation. If we now take the inner product of this with our left-eigenvector for the eigenvalue λ_0 , which we call y_L , then we have

$$\langle y_L, L[y] \rangle + \langle y_L, L'[y] \rangle = i\lambda \langle y_L, y \rangle. \quad (\text{A.14})$$

The first term, by virtue of equation (A.5), becomes

$$\langle y_L, L[y] \rangle = \langle L^*[y_L], y \rangle = \langle -i\lambda_0^* y_L, y \rangle = i\lambda_0 \langle y_L, y \rangle. \quad (\text{A.15})$$

So far we have made no approximations. Now we assume that λ is close to λ_0 and y close to y_R , the right-eigenvector for λ_0 . Then the dominant terms in (A.19) give

$$\lambda - \lambda_0 = -i \langle y_L, L'[y_R] \rangle / \langle y_L, y_R \rangle. \quad (\text{A.16})$$

This is the standard result. We have all we need to evaluate these integrals: y_L from equation (A.9), y_R in Section 3. After an integration by parts, the numerator of (A.16) becomes

$$\langle y_L, L'[y_R] \rangle = -\frac{1}{2\Omega_c} \int \nu\sigma [\nabla_k(S\delta v_j)^* + \nabla_j(S\delta v_k)^*] \times (\nabla^k\delta v^j + \nabla^j\delta v^k) d^2x. \quad (\text{A.17})$$

It is useful to note that if λ is *real*, then applying the operator S to equation (A.1–A.2) shows that we can choose the complex phase of δv^j and $\delta\sigma$ so that $S\delta v^j = -\delta v^j$ and $S\delta\sigma = \delta\sigma$. For such modes, equation (A.17) is manifestly positive-definite.

The denominator of (A.16) is

$$\langle y_L, y_R \rangle = \int \{ \sigma(S\delta v_j)^* \delta v^j + \Omega_c \delta\sigma \delta\Phi[S\delta\sigma] \} d^2x. \quad (\text{A.18})$$

Again when λ is real, equations (A.1–A.2) imply

$$\langle y_L, y_R \rangle = -\frac{2i\Omega}{\lambda} \int \sigma\epsilon_{jk}(\delta v^j)^* \delta v^k d^2x. \quad (\text{A.19})$$

This is real but of indefinite sign. The place where it changes sign locates the onset of viscous instability. By using the representation (3.7) of the velocity, plus the inviscid eigenequations, one can cast this in the form (again for real λ)

$$\langle y_L, y_R \rangle = \left(\frac{R^5 \Omega_c^4}{\pi^2 G} N_I^m \right) \frac{\lambda(\lambda^2 - 4\Omega^2) [m(\lambda^2 + 4\Omega^2) + 4C_I^m \Omega \lambda]}{(C_I^m \lambda + 2m\Omega)^2} (\Sigma_I^m)^2 \quad (\text{A.20})$$

where

$$N_l^m = \int [P_{2l+m}^m(\eta)]^2 d\eta d\phi = \frac{2\pi}{4l+2m+1} \frac{(2l+2m)!}{(2l)!}. \quad (\text{A.21})$$

Thus $\langle y_L, y_R \rangle$ changes sign as λ passes through zero, i.e. the secular instability sets in through a mode whose frequency in the rotating frame is zero, as expected. Moreover, the zero of $\langle y_L, y_R \rangle$ at $\lambda = 0$ does not cause $\lambda - \lambda_0$ to diverge in equation (A.16), because equations (3.13–3.15) show that α and β also vanish there, so that (A.17) goes to zero quadratically in λ . The points of marginal secular stability to viscosity, $\lambda = 0$, are the models in which $\Gamma_l^m = 0$, or from equation (3.11)

$$\mu = \frac{\pi}{2} \gamma_l^m.$$

For the bar mode, $m = 2$ and $l = 0$, this model is also marginally stable to gravitational radiation, just as in the Maclaurins.