ON GENERALISED EQUATIONS OF GEODESIC DEVIATION

Bernard F. Schutz
Department of Applied Mathematics and Astronomy,
University College, P.O. Box 78,
Cardiff CF1 IXL, Wales.

Abstract. The literature contains a number of different and inequivalent generalisations of the standard geodesic deviation equation to higher order in the separation of the geodesics and/or to large rates of separation. Motivated by applications to the motion of particles in gravitational waves, I adopt a definition of geodesic deviation based upon a geodesic curve leaving a reference geodesic orthogonally and connecting it to a target geodesic. Working in Riemann normal coordinates to higher order, I solve the geodesic equation and construct the appropriate geodesics. In the case of nearly parallel geodesics the resulting deviation equation to second order is derived and discussed. In the case of rapidly diverging geodesics I argue that no 'deviation equation' can substitute for a full solution of the geodesic equation.

Introduction

During the course of a study of the interaction of gravitational waves with matter, I went through the literature looking for extensions of the usual equation of geodesic deviation beyond first ('infinitesimal') order in the separation of the geodesics. To my initial surprise I found a number of inequivalent formulations claiming to be 'the' second-order geodesic-deviation equation, and it was not always clear what underlying assumptions caused them to differ. Reviews of the principal formulations and attempts to unify them into a common framework have been made by Manoff (1979) and Swaminarayan & Safko (1983). That there are inequivalent extensions is not really surprising. The usual equation of geodesic deviation (see Misner, *et al.*, 1973, whose conventions I adopt),

$$\nabla_u \nabla_u \xi = R(u,\xi)u, \tag{1}$$

applies not only when the connecting vector ξ is small but also (as has been stressed by Hodgkinson 1972) only to neighbouring geodesics that are nearly parallel. Any extension to geodesics which are not nearly parallel or to second order in ξ requires a definition of a *connecting curve* between the two geodesics, to which ξ is tangent. Such a curve may be defined in inequivalent ways. The easiest mathematically is to specify some initial connecting curve through a congruence of neighbouring geodesics and then to Lie drag it along the congruence. I will instead consider here the consequences of demanding that the connecting curve be a *geodesic*. This case has been treated before, but usually as a modification of the Lie-dragging definition (e.g. Bazanski 1977, but not Hodgkinson 1972). In order to clarify certain points, and because the problem of connecting up two separated curves is essentially nonlocal, I will adopt the 'sledgehammer' technique of explicitly solving the geodesic equation for the reference, target and connection geodesics. This can be done to the appropriate order in Riemann normal coordinates. I will initially not assume that the geodesics are almost parallel, specialising to that case later. In the non-parallel case I will argue that no extension of Eq.(1) is possible, in the sense of a second-order differential equation which may be integrated along one geodesic to find the deviation of the other: one must solve the 'global' problem for the two geodesics and their connecting geodesic.

There is an obvious drawback to defining a connecting curve by Lie dragging. Even if the initial connecting curve is arranged to be a geodesic, the dragged curves will not be geodesics. An example easy to visualise is that of a congruence of great circles of constant longitude on the sphere. Where they cross the equator, the equator itself is an obviously natural connecting curve. But dragging the equator along them means advancing it the same distance along each curve. This produces a circle of constant latitude; it still cuts the geodesics orthogonally but is not itself a geodesic. In particular, the length of the dragged connecting curve will not be a sensible measure of the distance from one great circle to another as measured by an observer located on one of them. This observer would do what we did at the equator: send out a connecting geodesic orthogonally from his own geodesic and use this to measure the rate of change of nearby geodesics as he moves along his own. The conventional textbook derivations of Eq.(1) do assume that ξ is Lie-dragged along u, but this is because they work only to first order in ξ; differences between the geodesic tangent to ξ and other connecting curves tangent to ξ are of second order in ξ.

In general relativity, a geodesic connecting curve has a useful physical interpretation. Consider a freely-falling local inertial observer keeping track of nearby free particles as a gravitational wave passes. If he tries to describe the resulting dynamics in special-relativistic language in which the Riemann tensor is a tidal gravitational 'force', then he will have to locate the particles' positions at any time in the flat three-space he carries along with him (the tangent space orthogonal to his four-velocity). The natural way to do this is to send out geodesics which are locally spatial, i.e. orthogonal to his world line. If the geodesic γ with tangent vector Λ^i intersects the world line of a particular particle at affine parameter μ, then the 'location' of the particle is $\mu\Lambda^i$ at that time. In this way, he uses the spatial section of the exponential map to provide spatial coordinates for his particles. This is the definition of geodesic deviation we shall explore. Given a reference geodesic Γ and a target geodesic Γ', we define the connecting geodesic from a point P on Γ to Γ' to be that geodesic γ_P which is orthogonal to Γ and intersects Γ' at unit affine parameter distance from P. (In a normal neighbourhood of P, γ_P will be unique.) We define the geodesic connecting vector Λ_P to be the tangent to γ_P at P. Geodesic deviation describes the evolution of Λ_P as P moves along Γ.

Intersection of Geodesics in Riemann Normal Coordinates

A particular point O on the reference geodesic Γ will serve as the origin of a Riemann normal coordinate system, with metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3}R_{\alpha\mu\beta\nu}x^\mu x^\nu - \frac{1}{6}R_{\alpha\mu\beta\nu;\sigma}x^\mu x^\nu x^\sigma + O(x^4), \qquad (2)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and the Riemann tensor and its derivative are evaluated at O, and are therefore constants in what follows. The Christoffel symbols are

$$\Gamma^\lambda{}_{\alpha\beta} = -\frac{2}{3}R^\lambda{}_{(\alpha\beta)\mu}x^\mu - \frac{1}{6}P^\lambda{}_{\alpha\beta\mu\nu}x^\mu x^\nu + O(x^3), \qquad (3)$$

where round brackets denote the symmetric part and I define

$$P^\lambda{}_{\alpha\beta\mu\nu} = \frac{5}{2}R^\lambda{}_{(\alpha\beta)(\mu;\nu)} - \frac{1}{2}R^\lambda{}_{(\mu\nu)(\alpha;\beta)}. \qquad (4)$$

We shall take the reference geodesic to be the time axis $x^0 = t$ of our coordinates, and we suppose that the target geodesic Γ' has affine parameter s and begins at $x^\alpha(s{=}0) = \xi^\alpha$ with derivative $dx^\alpha/ds(s{=}0) = \eta^\alpha$. It is straightforward to solve the geodesic equation

$$\frac{d^2x^\lambda}{ds^2} = -\Gamma^\lambda{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \qquad (5)$$

for the following power-series expression for x^α:

$$x^\alpha(s) = \xi^\alpha + s\eta^\alpha + s^2A^\alpha + s^3B^\alpha + s^4C^\alpha + O(s^5) \qquad (6)$$

where

$$A^\alpha = \frac{1}{3}R^\alpha{}_{\mu\nu\beta}\eta^\mu\eta^\nu\xi^\beta + \frac{1}{12}P^\alpha{}_{\mu\nu\beta\sigma}\eta^\mu\eta^\nu\xi^\beta\xi^\sigma + O_3 \qquad (7)$$

$$B^\alpha = \frac{1}{18}P^\alpha{}_{\mu\nu\beta\sigma}\eta^\mu\eta^\nu\eta^\beta\xi^\sigma + O_2 = \frac{1}{12}R^\alpha{}_{\mu\nu\sigma;\beta}\eta^\mu\eta^\nu\eta^\beta\xi^\sigma + O_2 \qquad (8)$$

$$C^\alpha = O_1, \qquad (9)$$

where from now on O_n means terms of n^{th} order in any small quantities: x^α, ξ^α, s, and any other affine parameters. In each case, the terms omitted in Eqs.(7)-(9) make O_5 errors in Eq.(6). Note that we do not assume that η^μ is small: the target geodesic need not move nearly parallel to the reference one. Notice also that ξ^μ and η^μ are coordinate-dependent representations of the initial data; we will return at the end to their description in invariant terms.

Let the connecting geodesic γ originate at a point χ^α with tangent vector Λ^α. Calling its affine parameter μ, we have from Eqs.(6)-(9)

$$x^\alpha(\mu) = \chi^\alpha + \mu\Lambda^\alpha + \mu^2[\frac{1}{3}R^\alpha{}_{\mu\nu\beta}\Lambda^\mu\Lambda^\nu\chi^\beta + \frac{1}{12}P^\alpha{}_{\mu\nu\beta\sigma}\Lambda^\mu\Lambda^\nu\chi^\beta\chi^\sigma]$$

$$+ \frac{1}{12}R^\alpha{}_{\mu\nu\sigma;\beta}\Lambda^\mu\Lambda^\nu\Lambda^\beta\chi^\sigma\mu^3 + O_5. \qquad (10)$$

Now we suppose that the geodesics intersect. Let their intersection occur at parameter s_* on Γ' and $\mu = 1$ on γ. This choice of μ means that we can interpret Λ^α as the vector 'connecting' χ^α and $\Gamma'(s_*)$. Therefore we take Λ^α as a small quanitity of order s or x^α, The connecting vector Λ^α depends, of course, on the intersection point we choose on Γ', and hence depends on s_*. Expanding

$$\Lambda^\alpha(s_*) = {}_0\Lambda^\alpha + s_*\, {}_1\Lambda^\alpha + s_*^2\, {}_2\Lambda^\alpha + s_*^3\, {}_3\Lambda^\alpha + s_*^4\, {}_4\Lambda^\alpha + O_5, \tag{11}$$

we find from the equality of Eqs.(6) and (10)

$$\begin{aligned} {}_0\Lambda^\alpha = {} & \xi^\alpha - \chi^\alpha - \tfrac{1}{3} R^\alpha{}_{\mu\beta\nu}\, \chi^\nu (\xi^\mu-\chi^\mu)(\xi^\beta-\chi^\beta) \\ & - \tfrac{1}{12} P^\alpha{}_{\sigma\beta\mu\nu}\, \chi^\mu\chi^\nu (\xi^\sigma-\chi^\sigma)(\xi^\beta-\chi^\beta) \\ & - \tfrac{1}{12} R^\alpha{}_{\sigma\beta\nu;\mu}\, \chi^\nu (\xi^\sigma-\chi^\sigma)(\xi^\beta-\chi^\beta)(\xi^\mu-\chi^\mu) + O_5, \end{aligned} \tag{12}$$

$$\begin{aligned} {}_1\Lambda^\alpha = {} & \eta^\alpha - \tfrac{2}{3} R^\alpha{}_{(\mu\beta)\nu}\, \chi^\nu (\xi^\beta-\chi^\beta)\eta^\mu - \tfrac{1}{6} P^\alpha{}_{\sigma\beta\mu\nu}\, \chi^\mu\chi^\nu (\xi^\sigma-\chi^\sigma)\eta^\beta \\ & - \tfrac{1}{4} R^\alpha{}_{(\sigma\beta|\nu|;\mu)}\chi^\nu (\xi^\sigma-\chi^\sigma)(\xi^\beta-\chi^\beta)\eta^\mu + O_4 , \end{aligned} \tag{13}$$

$$\begin{aligned} {}_2\Lambda^\alpha = {} & \tfrac{1}{3} R^\alpha{}_{\mu\beta\nu}\, \xi^\nu\eta^\mu\eta^\beta - \tfrac{1}{3} R^\alpha{}_{\mu\beta\nu}\, \chi^\nu\eta^\mu\eta^\beta \\ & + \tfrac{1}{12} P^\alpha{}_{\sigma\beta\mu\nu} (\xi^\mu\xi^\nu-\chi^\mu\chi^\nu)\eta^\sigma\eta^\beta \\ & - \tfrac{1}{4} R^\alpha{}_{(\sigma\beta|\nu|;\mu)}\, \chi^\nu (\xi^\sigma-\chi^\sigma)\eta^\beta\eta^\mu + O_3 \end{aligned} \tag{14}$$

$${}_3\Lambda^\alpha = \tfrac{1}{12} R^\alpha{}_{\sigma\beta\nu;\mu}\, (\xi^\nu-\chi^\nu)\eta^\sigma\eta^\beta\eta^\mu + O_2 \tag{15}$$

$${}_4\Lambda^\alpha = O_1, \tag{16}$$

where vertical bars protect indices from symmetrisation.

The Geodesic Connecting Vector

We now make some convenient specialisations. Let us take Γ' to originate at t = 0 and γ to originate on Γ at time t. These conditions imply

$$\xi^0 = 0, \quad \chi^i = 0,\ \chi^0 = t. \tag{17}$$

Now we impose our demand that γ be orthogonal to Γ. This fixes the intersection point $\Gamma'(s_*)$ of γ and Γ', and so it determines s_* as a function of the time t at which γ leaves Γ. In our coordinates, orthogonality simply means

$$\Lambda^0(s_*) = 0. \tag{18}$$

Solving this for $s_*(t)$ gives a power series of the form

$$s_*(t) = s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4 + O_5. \qquad (19)$$

There is no constant term s_0 because the coordinate line connecting the origin O with the point $\Gamma'(o)$ (which is just ξ^i) is already a geodesic orthogonal to the time axis. It is convenient to define

$$V^\alpha = \eta^\alpha/\eta^o \qquad (20)$$

and to rescale the coefficients in Eq.(19) by η^o:

$$\tilde{s}_1 = s_1\eta^o = 1 + \frac{1}{3} R^o{}_{jko}\, \xi^j\xi^k + \frac{1}{12} R^o{}_{jko;l}\, \xi^j\xi^k\xi^l + O_4, \qquad (21)$$

$$\tilde{s}_2 = s_2\eta^o = \frac{2}{3} R^o{}_{(jk)o}\, V^j\xi^k - \frac{1}{3} R^o{}_{j\beta k}\, V^jV^\beta\xi^k$$
$$- \frac{1}{12} P^o{}_{\sigma\beta jk}\, \xi^j\xi^k V^\sigma V^\beta + \frac{1}{12} P^o{}_{jkoo}\, \xi^j\xi^k$$
$$+ \frac{1}{6} R^o{}_{(jk)o;l}\, V^j\xi^k\xi^l + \frac{1}{12} R^o{}_{jko;l}\, V^l\xi^j\xi^k + O_3 \qquad (22)$$

$$\tilde{s}_3 = s_3\eta^o = \frac{1}{3} R^o{}_{jko}\, V^jV^k + \frac{1}{6} R^o{}_{(jk)o;l}\, \xi^jV^kV^l + \frac{1}{6} P^o{}_{jkoo}\, V^j\xi^k$$
$$+ \frac{1}{12} R^o{}_{jko;l}\, V^jV^k\xi^l - \frac{1}{12} R^o{}_{j\beta k;\mu}\, V^jV^\beta V^\mu\xi^k + O_2 , \qquad (23)$$

$$\tilde{s}_4 = s_4\eta^o = \frac{1}{12} P^o{}_{jkoo}\, V^jV^k + \frac{1}{12} R^o{}_{jko;l}\, V^jV^kV^l + O_1 . \qquad (24)$$

Rescaling the s_j's effectively removes the time dilation factor (and any other scale factors if η^α is not normalised) so that $\tilde{s}_j$ represents a time lapse as measured on Γ.

We can now use Eq.(19)-(24) in Eq.(11) to find the tangent to the connecting curve γ. If we define

$$Q^\beta{}_\alpha = \delta^\beta{}_\alpha - V^\beta\delta^o{}_\alpha, \qquad (25)$$

the parallel projection along V^μ onto hypersurfaces t = const., we find

$$\Lambda^i(s_*(t)) = \xi^i + tV^i + Q^i_\mu \{t[-\frac{1}{3} R^\mu{}_{jko}\xi^j\xi^k - \frac{1}{12} R^\mu{}_{jko;l}\xi^j\xi^k\xi^l]$$
$$+ t^2[-\frac{1}{3}R^\mu{}_{ojo}\, \xi^j - R^\mu{}_{kjo}\xi^jV^k + \frac{1}{3} R^\mu{}_{klj}\xi^jV^kV^l$$
$$- \frac{1}{4} R^\mu{}_{(jk|o|;l)}\, \xi^j\xi^kV^l - \frac{1}{12}(P^\mu{}_{jkoo} - P^\mu{}_{\alpha\beta jk}\, V^\alpha V^\beta)\xi^j\xi^k]$$

$$+ t^3[\frac{1}{3} R^\mu{}_{koj} V^j V^k - \frac{1}{6} P^\mu{}_{jkoo}\xi^j V^k + \frac{1}{12} R^\mu{}_{okj;o}\xi^j V^k$$

$$+ \frac{1}{12} R^\mu{}_{koj;o}\xi^j V^k - \frac{1}{4} R^\mu{}_{kjo;l}\xi^j V^k V^l - \frac{1}{12} R^\mu{}_{klo;j} \xi^j V^k V^l$$

$$+ \frac{1}{12} R^\mu{}_{klj;o} \xi^j V^k V^l + \frac{1}{12} R^\mu{}_{klj;m}\xi^j V^k V^l V^m - \frac{1}{12} R^\mu{}_{ojo;o} \xi^j]$$

$$+ t^4[- \frac{1}{12} P^\mu{}_{jkoo} V^j V^k - \frac{1}{12} R^\mu{}_{kjo;l} V^j V^k V^l]\} + O_5 \tag{26}$$

These are the components of Λ^α on the coordinate basis at time t on the t-axis. It is preferable to have them on the parallel-transported basis from the origin (which is equivalent to parallel-transporting Λ^α back to the origin). It is easy to calculate from Eqs.(2)-(4) that the basis one-forms $\tilde{\omega}^{\hat\mu}$ of the origin, which are an orthonormal tetrad, have the following components $(\tilde{\omega}^{\hat\mu})_\sigma$ when parallel-transported along Γ:

$$(\tilde{\omega}^{\hat\mu})_\sigma = \delta^\mu{}_\sigma - \frac{1}{6} t^2 R^\mu{}_{o\sigma o} - \frac{1}{12} t^3 R^\mu{}_{o\sigma o;o} + O(t^4). \tag{27}$$

This allows us to write down our main result:

$$\Lambda^{\hat o}(t) = (\tilde{\omega}^{\hat o})_\sigma \Lambda^\sigma(s_*(t)) = 0 \tag{28}$$

$$\Lambda^{\hat i}(t) = (\tilde{\omega}^{\hat i})_\sigma \Lambda^\sigma(s_*(t)) = \xi^i + tV^i + Q^i{}_\mu\{t[- \frac{1}{3}R^\mu{}_{jko}\xi^j\xi^k$$

$$- \frac{1}{12} R^\mu{}_{jko;l} \xi^j\xi^k\xi^l] + t^2[- \frac{1}{2} R^\mu{}_{ojo}\xi^j - R^\mu{}_{kjo}\xi^j V^k$$

$$+ \frac{1}{3} R^\mu{}_{klj}\xi^j V^k V^l - \frac{1}{4} R^\mu{}_{(jk|o|;l)}\xi^j\xi^k V^l - \frac{1}{12} (P^\mu{}_{jkoo}$$

$$- P^\mu{}_{\alpha\beta jk} V^\alpha V^\beta)\xi^j\xi^k] + t^3[- \frac{1}{6} R^\mu{}_{ojo}V^j + \frac{1}{3} R^\mu{}_{koj}V^j V^k$$

$$- \frac{1}{6} P^\mu{}_{jkoo}\xi^j V^k - \frac{1}{6} R^\mu{}_{0j0;o}\xi^j + \frac{1}{12} R^\mu{}_{okj;o}\xi^j V^k$$

$$+ \frac{1}{12} R^\mu{}_{koj;o}\xi^j V^k - \frac{1}{4} R^\mu{}_{kjo;l}\xi^j V^k V^l$$

$$- \frac{1}{12} R^\mu{}_{klo;j}\xi^j V^k V^l + \frac{1}{12} R^\mu{}_{klj;o}\xi^j V^k V^l$$

$$+ \frac{1}{12} R^{\mu}{}_{klj;m}\xi^j V^k V^l V^m] + t^4[- \frac{1}{12} P^{\mu}{}_{jkoo}V^j V^k$$

$$- \frac{1}{12} R^{\mu}{}_{kjo;l} V^j V^k V^l - \frac{1}{12} R^{\mu}{}_{ojo;o} V^j]\} + O_5. \tag{29}$$

Neglecting the $O(t^4)$ term in Eq.(27) does not destroy the $O(t^4)$ terms in Eq.(29) because it would only enter Eq.(29) multiplied by ξ^i, and would therefore be O_5.

Notice that at t = 0 we have $\Lambda^{\hat{i}}(o) = \xi^i$. This is because, by definition of Riemann normal coordinates, the coordinates of a point are the components of the vector at the origin tangent to the geodesic joining the point to the origin with unit affine parameter interval. Therefore ξ^i is not just a coordinate quantity: it is the initial value of the geodesic connecting vector. On the other hand, we find

$$\frac{d}{dt} \Lambda^{\hat{i}}|(t)|_{t=0} = V^i + Q^i{}_{\mu} [- \frac{1}{3} R^{\mu}{}_{jko}\xi^j\xi^k - \frac{1}{12} R^{\mu}{}_{jko;l}\xi^j\xi^k\xi^l] + O_4. \tag{30}$$

The l.h.s. of this equation is the proper-time covariant derivative of Λ^{α} on Γ, so that we see that the initial rate of change of Λ^{α} is not simply related to the initial rate of change V^{α} of Γ'.

We can properly call $\Lambda^{\hat{\alpha}}$ the geodesic connecting vector, since it is the tangent vector at Γ to the geodesic γ that is orthogonal to Γ and connects Γ to Γ'. Under what circumstances Λ^{α} satisfies a differential equation is the subject of the next section.

Special Cases

Nearly parallel geodesics to first order

If Γ and Γ' are nearly parallel then V^i is O_1. Keeping all powers of t but working only to first order in ξ^i and V^i, we find

$$\Lambda^{\hat{i}}(t) = \xi^i + tV^i - \frac{1}{2} t^2 R^i{}_{ojo}\xi^j - \frac{1}{6} t^3[R^i{}_{ojo}V^j + R^i{}_{ojo;o}\xi^j] + O(t^5). \tag{31}$$

The second derivative of this is

$$\frac{d^2}{dt^2} \Lambda^{\hat{i}}|_{t=0} = - R^i{}_{ojo} \xi^j . \tag{32}$$

Given that $\Lambda^{\hat{i}}$ is the component on a parallel-transported basis, we can also write this in terms of the tangent U^{α} to Γ:

$$\nabla_U \nabla_U \Lambda^{\alpha}|_{t=0} = - R^{\alpha}{}_{\mu\beta\nu} U^{\mu}U^{\nu}\Lambda^{\beta}|_{t=0} . \tag{33}$$

This is just the usual first-order geodesic deviation equation, Eq.(1), at the origin. But can we infer from Eq.(31) that Eq.(33) will be valid at other times: does the geodesic deviation equation describe the evolution of Λ^α at other times? We can check consistency of this by asking if higher derivatives of Eq.(31) are consistent with what we would obtain from using Eq.(32) at other times, i.e. from the equation

$$\frac{d^2}{dt^2}\Lambda^{\hat{i}} = -R^{\hat{i}}{}_{o\hat{j}o}\Lambda^{\hat{j}}, \tag{34}$$

where hats over indices refer to the parallel-transported basis. Equation (34) implies

$$\frac{d^3}{dt^3}\Lambda^{\hat{i}}\Big|_{t=0} = -R^{\hat{i}}{}_{o\hat{j}o;o}\,\xi^j - R^{\hat{i}}{}_{o\hat{j}o}\,V^j, \tag{35}$$

which is the same as we obtain from Eq.(31), and

$$\frac{d^4}{dt^4}\Lambda^{\hat{i}}\Big|_{t=0} = O_1 \tag{36}$$

which is also (trivially)consistent with Eq.(31), since we have been able to keep only O_0 in the t^4 terms.

Nearly parallel geogesics to second order.
Keeping terms of second order in ξ^i and V^i in Eq.(29) gives

$$\begin{aligned}\Lambda^{\hat{i}}(t) &= \xi^i + t(V^i - \tfrac{1}{3}R^i{}_{jko}\,\xi^j\xi^k) + t^2[-\tfrac{1}{2}R^i{}_{ojo}\xi^j \\ &\quad + R^i{}_{kjo}\,\xi^j V^k + \tfrac{1}{4}(R^i{}_{ooj;k} - R^i{}_{jko;o})\xi^j\xi^k] \\ &\quad - \tfrac{1}{6}t^3(R^i{}_{ojo}\,V^j + R^i{}_{ojo;o}\,\xi^j) + O(t^5),\end{aligned} \tag{37}$$

where I have discarded t^3 terms quadratic in ξ^i and V^i since they are O_5 overall. By analogy with Eq.(34), the second order geodesic deviation equation for a geodesic connecting vector is

$$\begin{aligned}\frac{d^2}{dt^2}\Lambda^{\hat{i}} &= -R^{\hat{i}}{}_{o\hat{j}o}\Lambda^{\hat{j}} - 2R^{\hat{i}}{}_{\hat{k}\hat{j}o}\Lambda^{\hat{j}}\frac{d\Lambda^{\hat{k}}}{dt} + \frac{1}{2}(R^{\hat{i}}{}_{oo\hat{j};\hat{k}} \\ &\quad - R^{\hat{i}}{}_{\hat{j}\hat{k}o;o})\Lambda^{\hat{j}}\Lambda^{\hat{k}},\end{aligned} \tag{38}$$

or in terms of the unit vector U^α tangent to Γ,

$$\nabla_U \nabla_U \Lambda^\alpha = [- R^\gamma{}_{\mu\beta\nu} U^\mu U^\nu \Lambda^\beta + 2R^\gamma{}_{\beta\sigma\mu} \Lambda^\sigma \nabla_U \Lambda^\beta U^\mu$$

$$+ \frac{1}{2}(R^\gamma{}_{\mu\nu\beta;\sigma} - R^\gamma{}_{\beta\sigma\mu;\nu} \Lambda^\beta \Lambda^\sigma U^\mu U^\nu] (\delta^\alpha{}_\gamma + U^\alpha U_\gamma) \tag{39}$$

The second-order part of this is equivalent to Eq.(4.2) of Bazanski when we realise that his $r_\perp^\alpha$ and s^α is related to our Λ^α by $\Lambda^\alpha = r_\perp^\alpha + \frac{1}{2}s^\alpha$. It is also equivalent to the second-order part of Eq.(2.51) of Hodgkinson (1972), although he does not make a clear statement that he is using a geodesic connecting curve. Our O_5 accuracy does not permit us to test consistency of the higher time derivatives as we did in the first-order case.

Rapidly diverging geodesics

If V^i is not small then Γ and Γ' are close but rapidly diverging. The origin O is therefore a special point, and it cannot be expected that an equation derived near it will hold everywhere along Γ. Indeed, to first order in ξ Eq.(29) becomes

$$\Lambda^{\hat{i}}(t) = \xi^i + tV^i + Q^i_\mu\{t^2[- \frac{1}{2} R^\mu{}_{ojo}\, \xi^j - R^\mu{}_{kjo}\, \xi^j V^k$$

$$+ \frac{1}{3} R^\mu{}_{klj}\, \xi^j V^k V^l] + O(t^3), \tag{40}$$

from which one might postulate the deviation equation

$$\frac{d^2}{dt^2} \Lambda^{\hat{i}} = Q^{\hat{i}}{}_\mu(- R^\mu{}_{o\hat{j}o}\, \Lambda^{\hat{j}} - 2R^\mu{}_{\hat{k}\hat{j}o}\, \Lambda^{\hat{j}} V^{\hat{k}}$$

$$+ \frac{2}{3} R^\mu{}_{\hat{k}\hat{l}\hat{j}}\, \Lambda^{\hat{j}} V^{\hat{k}} V^{\hat{l}}), \tag{41}$$

which is essentially Eq.(2.52) of Hodgkinson (1972). But since the r.h.s. of Eq.(41) is of order Λ^j, its time-derivative is of order 1, and therefore Eq.(41) cannot be integrated for as long as Eq.(34) can before the assumptions under which it was derived break down. Indeed, inspection of Eq.(29) at order t^3 reveals terms that involve derivatives of the Riemann tensor along V^j and ξ^j, which would not be produced by differentiating Eq.(41) along Γ. Therefore there is no real 'equation of geodesic deviation' in the rapidly diverging case. We have only power-series solutions like Eq.(29), or relations among derivatives, such as Eq.(41), which are valid only asymptotically near the origin O.

Acknowledgement

The calculations carried out here were inspired by many stimulating discussions with T. Futamase and K. Krishnasamy.

References

Bazanski, S.L. (1977). Ann.Inst.H.Poincare, 27, 115-144.
Hodgkinson, D.E. (1972). Gen.Rel. and Grav., 3, 351-375.
Manoff, S. (1979). Gen.Rel. and Grav., 11, 189-204.
Misner, C.W., Thorne, K.S., & Wheeler, J.A. (1973). Gravitation. San Francisco: Freeman.
Swaminarayan, N.S. & Safko, J.L. (1983). J.Math.Phys., 24 883-885.