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Pseudotensors in Asymptotically Curvilinear Coordinates

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We show how to calculate pseudotensor-based conserved quantities for isolated systems in general relativity, in a way which allows an arbitrary asymptotic behavior of the coordinate system used. Our method is a generalization of that given by Persides [1], and allows the asymptotic evaluation of energy, momentum, and angular momentum in any coordinate system. We carry out the calculation for the Schutz-Sorkin gravitational Noether operator, which is a pseudotensorial operator on vector fields that reduces to the familiar pseudotensors for particular choices of the fields.

1. INTRODUCTION

In the classical theories of physics (without gravity), conserved quantities such as energy, momentum, and angular momentum play a very important role: They provide us with first integrals of the equations of motion, enabling us to solve otherwise intractable problems (such as collisions, stability properties of physical systems, etc.). There is no doubt that for the study of gravitating systems one would also like to have these quantities at hand for the same purposes, but in the domain of general relativity they acquire a substantially different meaning. For example, while the energy density of the gravitational field should (by classical analogies) depend on the square of the first derivatives of the metric tensor $g_{\mu\nu}$, one can always transform this to the Minkowski flat form $\eta_{\mu\nu}$ at any point in space-time,

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and its first derivatives to zero, thus reducing the gravitational energy density to zero at that point.

That one cannot attach a precise meaning to the local distribution of energy and momentum is a consequence of the principle of equivalence: We know that conservation laws result from symmetries or invariances in the action principle (Noether's theorems). But a nonvanishing gravitational field corresponds to a nonflat Riemannian manifold, which in general possesses no symmetries. The equations of general relativity are invariant under an infinity of one-parameter groups, viz., the one-parameter subgroups of the full group of coordinate transformations: any vector field ξ^α generates the one-parameter group of transformation $x^\alpha \to x^\alpha + t\xi^\alpha$. Then any vector field gives rise to a conservation law. (Using this suggestion of Bergmann [2], Komar [3] arrived at his covariantly conserved quantities, certain of which, in some particular coordinate systems, may be identified as energy and momentum.) The problem is that for a general gravitational field, none of these groups has a natural intrinsic meaning.

For isolated systems (i.e., those containing matter fields of compact support) the field far away from the sources has the same form as for linearized theory, and so one has flux integrals for the total momentum and angular momentum *as long as they are evaluated at infinity in an asymptotically Lorentzian* coordinate system (ALCS). The Penrose [4] definition does not require an ALCS since it supplies its own Minkowski space. (For a modification to the original definition of the Penrose mass, which aparently does away with the problems it presented, but which is only valid in certain restricted cases, see [5].) The Hawking [6] definition, on the other hand, requires a Bondi-like coordinate system [7], which is not an ALCS. Apart from these, the usual way to evaluate energy and momentum at finite distances is to introduce a so-called pseudotensor.

While several choices of pseudotensors do give precisely the results that one expects at infinity, they need to be evaluated in an ALCS for this to be the case. This is a serious drawback, for ALCSs are not at all easy to construct except in very simple highly symmetric cases. What one would like is to extend the family of coordinate systems (possibly to cover all of them) in which one can evaluate conserved quantities in such a way that one obtains the same results as for ALCSs. This flexibility may then be exploited in various ways.

Two especially convenient choices of pseudotensors have been the Einstein pseudotensor $t_E^{\mu\nu}$ [8] and the Landau–Lifshitz pseudotensor $t_{LL}^{\mu\nu}$ [9]. Persides used the latter to show that it gives the Bondi four-momentum at null infinity [1]. Although he works in a particular coordinate system for simplicity, his result is covariant. But his method only allowed him to work asymptotically. In the rest of this paper we generalize his

procedure and show how to compute locally the conserved quantities in any coordinate system one chooses. The generalization is easy but powerful, and it does not seem to have appeared in the literature before.

Although our formalism applies to any gravitational pseudotensor complex whatsoever, for concreteness we work with the class defined by the gravitational Noether operator $t_{\mathrm{N}}^{\mu\nu}$ of Schutz and Sorkin [10]. Its operation on any C^1 vector field ξ^ν is given by

$$8\pi t_{\mathrm{N}}^{\mu\nu}\cdot\xi^\nu := -(-g)^{1/2}\, G^\mu_\nu \xi^\nu + \partial_\alpha[h^{\mu\alpha\nu\beta}{}_{,\beta}\xi_\nu(-g)^{1/2}] \tag{1}$$

where

$$h^{\mu\alpha\nu\beta} = (-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\alpha\nu}g^{\mu\beta})$$

and G^μ_ν is the Einstein tensor.

This operator has the advantage that it does not depend on second derivatives of the metric, contains $t_{\mathrm{E}}^{\mu\nu}$ and $t_{\mathrm{LL}}^{\mu\nu}$ as particular cases (for particular choices of ξ^ν), and is rather insensitive to the asymptotic behavior of $g_{\mu\nu}$. Equation (1) is to be interpreted as the pseudotensorial momentum density associated with the vector field ξ^ν. Integrating this and the corresponding matter density allows the Schutz and Sorkin *ξ-momentum* to be defined

$$P[\xi, H] := \int_H (I_{\mathrm{N}}^{\mu\nu}\cdot\xi^\nu + t_{\mathrm{N}}^{\mu\nu}\cdot\xi^\nu)\, d\sigma_\mu \tag{2}$$

Here, $I_{\mathrm{N}}^{\mu\nu}$ is the Noether operator for matter: a covariant generalization of the so-called canonical stress-energy tensor, to which it reduces if ξ^ν has constant components; H is a spacelike hypersurface with boundary ∂H; $d\sigma_\mu$ is the *coordinate* volume element in an ALCS.

Furthermore, when the field and matter equations are satisfied, the ξ-momentum is conserved and may be written as

$$\begin{aligned} P[\xi, H] &= (1/16\pi)\int_H \partial_\alpha[h^{\mu\alpha\nu\beta}{}_{,\beta}\,\xi_\nu(-g)^{-1/2}]\, d\sigma_\mu \\ &= (1/16\pi)\oint_{\partial H} h^{\mu\alpha\nu\beta}{}_{,\beta}\xi_\nu(-g)^{-1/2}\, d\Sigma_{\mu\nu} \end{aligned} \tag{3}$$

where $d\Sigma_{\mu\alpha}$ is the coordinate two-surface element.

For example, if for the Schwarzschild geometry in isotropic coordinates we choose $H = \{t = \text{const.}\}$ and $\xi = \partial/\partial t$ (in which case $t_{\mathrm{N}}^{\mu\nu}$ reduces to $t_{\mathrm{E}}^{\mu\nu}$ thus giving the Schwarzschild mass at infinity), (3) gives a value of $M - M^2/(2\rho)$ for the energy inside a sphere of isotropic radius ρ. This is the

same as the result obtained by Lynden-Bell and Katz [11], who giv physical argument that spherically symmetric space-times have a w defined energy density. For a general space-time which is asymptotical flat near spacelike infinity, if ξ^ν is asymptotically a translation Killing ve tor field, then (3) gives the appropriate ADM four-momentum as ∂H i moved to infinity. If ∂H is kept at a finite distance then the exact expressior depends on the ALCS and on the way ξ^ν approaches a Killing field. This means that the terms like $-M^2/(2\rho)$ in the Schwarzschild example above will depend upon the ALCS. Therefore, one cannot place much emphasis on their physical interpretation. Pseudotensors are interesting, not so much for such localizations of energy, but for their role in conservation and variational arguments, such as those developed by Schutz and Sorkin, which we generalize to angular momentum in a future paper [12].

Section 2 below generalizes (3) to give *quasi-covariant* conserved quantities for the isolated system. Section 3 illustrates the formalism with several examples.

2. THE QUASI-COVARIANT NOETHER OPERATOR FOR GRAVITY

The integrand in (3) is not a tensor density; thus, different ALCSs will yield different results. This is not surprising since it just reflects the non-localizability of energy momentum in general relativity as guaranteed by the principle of equivalence. The most serious drawback of this expression for momentum is the requirement of an ALCS, which is not readily available in most circumstances. When evaluated in, say, an asymptotically spherical coordinate system, (3) may give an infinite result. We here generalize (3) to make it workable in any coordinate system, in such a way that it gives the same results as one would get in some ALCS.

Let $(M, g_{\mu\nu})$ be the space-time manifold of an isolated gravitating system, together with an asymptotically Lorentzian coordinate system covering a neighborhood of infinity. From now on we restrict attention to this neighborhood. Associate to it a neighborhood of the flat space-time with Lorentz coordinates that corresponds to the original ALCS in M, and give it the Lorentz metric of special relativity in these coordinates: $(M^F, \eta_{\mu\nu})$. Such a manifold M^F always exists for an asymptotically flat manifold M. Let the $1-1$ map between these neighborhoods be called h.

Now work in M^F instead of in M, and bring down $g^{\mu\nu}$ and ξ_ν to M^F using h. Then, $g^{\mu\nu}$ is the tensor field in M^F with the same components in these coordinates as $g^{\mu\nu}$ in M, and similarly for ξ_ν. Also

will have the same expression as in M, and it is obviously a scalar density of weight 2 in M^F (just as in M).

It follows that $h^{\mu\alpha\nu\beta}{}_{,\beta}(-g)^{-1/2}\xi^\nu$ has weight 1 in M^F and this means that

$$\oint_{\partial H}(-\eta)^{-1/2}h^{\mu\alpha\nu\beta}{}_{,\beta}\xi_\nu(-g)^{-1/2}\,d\Sigma_{\mu\alpha} \tag{4}$$

is the integral of a scalar in M^F, where $d\Sigma_{\mu\alpha}$ is the *coordinate* volume element in M^F (which equals the proper volume element in our frame), and we have introduced a factor $(-\eta)^{-1/2}$ ($=1$ in our frame) to account for the weight (η is the determinant of $\eta_{\mu\nu}$). The integral (4) may therefore be evaluated in M^F, and will of course give the same result as its counterpart in M. It is easy to make this expression covariant in M^F: just replace commas by bars, which denote covariant derivatives with respect to $\eta_{\mu\nu}$, i.e.

$$P[\xi, H] = (1/16\pi)\oint_{\partial H}(-\eta)^{-1/2}h^{\mu\alpha\nu\beta}{}_{|\beta}\xi_\nu(\eta/g)^{1/2}\,d\Sigma_{\mu\alpha}$$

$$= (1/16\pi)\oint_{\partial H}h^{\mu\alpha\nu\beta}{}_{|\beta}\xi_\nu(-g)^{-1/2}\,d\Sigma_{\mu\alpha} \tag{5}$$

The question of convergence of this integral on space-like hypersurfaces, for the case of energy, was dealt with by Schutz and Sorkin [10]. (We deal with the more general case in another paper [12]. We note, however, that the convergence properties are the same in the general class of coordinate systems considered as in ALCSs.

Under a coordinate transformation in M^F the integral (5) is of course unaltered (we are just evaluating an integral in two coordinate frames in flat space). Any coordinate transformation in M^F induces a unique one in M, namely, the one which has the same functional form for M's coordinates as for M^F's. In the new coordinates of M, which need not be an ALCS, we *define* $P[\xi, H]$ by (5), i.e., to be the integral in M^F of the tensor density. This value is of course the same as the original integral in M using the original ALCS. The procedure, therefore, allows us to calculate the pseudotensors in any coordinates. We summarize this result as follows:

Proposition. Given any ALCS and a coordinate transformation from it to any curvilinear coordinate system, we can evaluate the ξ-momentum in this second system using (5), obtaining the same result as in the original ALCS.

The problem now is that this procedure is still impractical, because we must know the ALCS from which we start in order to calculate the

Christoffel symbols in M^F to be used in (5). Given a particular curvilinear frame, we can arrive at it from different ALCSs via the appropriate transformations. In each case we obtain for $P[\xi, H]$ its value in the ALCS we choose to start with, i.e., we will have as many values for $P[\xi, H]$ as ALCSs we can construct for our space-time. This is just the freedom we had at the beginning, and is consistent with the nonlocalizability of the ξ-momentum.

It is easy to remove this problem by choosing, for a particular space-time, that coordinate system in M^F in which the connection coefficients are asymptotically equal to those in the coordinate system in M. For instance, if we want to evaluate $P[\xi, H]$ for Schwarzschild's space-time in Schwarzschild's coordinates, without bothering about finding an isotropic frame or any other ALCS, we can *decide* on the above choice and use spherical polar coordinates in M^F. This choice selects one particular value for the ξ-momentum, but allows us to compute it in the given coordinate system directly. In particular, it allows to calculate values for energy, momentum, and angular momentum in situations in which it was previously not possible by conventional pseudotensor methods, and then to use these values for the study of the properties of different configurations.

3. ILLUSTRATION OF THE FORMALISM

As a simple illustration of the formalism described above, we evaluate here the energy for Schwarzschild's and Kerr's geometries using different curvilinear coordinates and vector fields.

3.1. Schwarzschild Space-time in Schwarzschild Coordinates

We use $\eta_{\mu\nu}$ in flat spherical polar coordinates and $\xi^\mu = (\partial/\partial t)|_{\text{Schwarzschild}}$ so that $\xi_\nu = (1 - 2M/r)\, dt$. Also, $H = \{t = \text{const.}\} \cap [r_+, R]$, where $r = r_+$ defines the horizon of the hole and R is some number greater than r_+. Then

$$P[\xi, H]_{\text{field}} = \frac{1}{16\pi} \lim_{R \to \infty} \oint_{\substack{t = \text{const.} \\ r = R}} h^{\mu\alpha\nu\beta}{}_{|\beta} (-g)^{-1/2} \xi_\nu \, d\sigma_\mu$$

$$- \frac{1}{16\pi} \oint_{\substack{t = \text{const.} \\ r = r_+}} h^{\mu\alpha\nu\beta}{}_{|\beta} (-g)^{-1/2} \xi_\nu \, d\sigma_\mu$$

The first integral is to be interpreted as the energy contained inside a sphere of radius R (field + hole), the second integral as the energy of the

hole itself (i.e., that contained inside the horizon). If we want the energy of the complete system, hole plus field, we add that of the hole, i.e.

$$P[\xi, H] = (1/16\pi) \lim_{R \to \infty} \oint_{\substack{t = \text{const.} \\ r = R}} h^{\mu\alpha\nu\beta}{}_{|\beta} (-g)^{-1/2} \xi_\nu \, d\sigma_\mu$$

With the above choices for ξ and the coordinate system, this gives

$$P[\xi, H] \equiv M \text{ (independent of radius)} \tag{6}$$

It is interesting to see that even if we had evaluated the above integral for $r = R$ (with R some finite positive value), we would have obtained exactly the same result, i.e., $P[\xi, H] \equiv M$ independent of radius. This agrees with the Penrose mass for the Schwarzschild hole, as calculated by Tod [13], although the Penrose mass is defined in a completely different manner. We must stress again, however, that any value at finite distances obtained with our formalism is inherently coordinate-dependent. In fact, while this result ascribes all the energy to the region inside the horizon and no energy to the field outside, we could also obtain the result in Section 1 (the Lynden-Bell/Katz result) which does exactly the opposite, by using instead of the connection coefficients of spherical polar coordinates those of the coordinate system obtained from Cartesian coordinates under the same transformation which takes isotropic into Schwarzschild coordinates. Therefore, even though this may show that no direct physical interpretation can be given to results obtained at finite distances, it also shows that one can get sensible results even near black hole horizons (see results below for the Kerr metric) which may allow the extension of the variational methods of Schutz and Sorkin [10] to systems that include black holes.

3.2. Kerr Space-time

For Kerr's space-time in Boyer–Linquist coordinates we may follow the steps above and obtain $P[\xi, H] = M$ for the total energy of the system (field plus hole), with $\xi = (\partial/\partial t)|_{\mathrm{BL}}$. Evaluating the ξ-momentum at finite distances, we obtain for the total energy inside a sphere of radius r, an exact expression in terms of hypergeometric functions

$$P[\xi, H]_r = M + \frac{1}{4\Delta} \left\{ \left(-a^2 r + 2Ma^2 - 4M^2 a^2 \frac{1}{r} + 2Ma^4 \frac{1}{r^2} \right) \right.$$
$$\times F\left(2, \frac{1}{2}; \frac{3}{2}; -\frac{a^2}{r^2}\right) + \left[a^2 r - \frac{14}{3} Ma^2 + \left(\frac{20}{3} M^2 a^2 + \frac{2}{3} a^4 \right) \right.$$

$$\times \frac{1}{r} - \frac{16Ma^4}{3r^2} + \left(\frac{4}{3}M^2a^4 + \frac{2}{3}a^6\right)\frac{1}{r^3} - \frac{2Ma^6}{3r^4}\Bigg]$$

$$\times F\left(2, \frac{3}{2}; \frac{5}{2}; -\frac{a^2}{r^2}\right)$$

$$+ \left(\frac{4}{5}a^4\frac{1}{r} - \frac{2Ma^4}{r^2} + \frac{12M^2a^4}{5r^3} + \frac{3a^6}{5r^3} - \frac{2Ma^6}{r^4} + \frac{2a^8}{5r^5}\right)$$

$$\times F\left(2, \frac{5}{2}; \frac{7}{2}; -\frac{a^2}{r^2}\right) + \left(\frac{1}{7}a^6 r\right) F\left(2, \frac{7}{2}; \frac{9}{2}; -\frac{a^2}{r^2}\right)\Bigg\} \tag{7}$$

where $\Delta = r^2 - 2Mr + a^2$.

For $a = 0$, $P[\xi, H] \equiv M$, in agreement with the result for Schwarzschild. Far away from the horizon, for small a, $P[\xi, H]_r = M - (2Ma^2/3\Delta) + (2M^2a^2/3\Delta r) + O(a^4)$ (note that the $O(1/r)$-term vanishes). However, the expression in (7) is singular at the horizon where $\Delta = 0$. This problem seems to arise from the combination of two facts: (1) The singularity of the coordinates themselves at the horizon, and (2) The fact that ξ^ν is space-like at and near the horizon. That this is so is suggested by the fact that nonsingular results are obtained either by using for ξ^ν the four-velocity of a locally nonrotating observer, properly "normalized," or by working in Kerr–Schild coordinates, which are the equivalent of the Eddington–Finkelstein coordinates for Schwarzschild's space-time and are obtained from the Boyer–Lindquist coordinates via the transformation

$$dT = dt - (2Mr/\Delta)\, dr, \qquad d\Phi = d\phi + (2Mar/\Gamma\Delta)\, dr \tag{8}$$

where $\Gamma = r^2 + a^2$.

CONCLUSIONS

The results of this paper give us considerably enhanced flexibility in using pseudotensors, since the natural coordinate systems for asymptotically flat space-times of isolated systems are spherical near infinity. We exploit this flexibility in future work on angular momentum and, in particular, on developing extremum theorems for the angular momentum of solutions of Einstein's equations, for nonstationary space-times.

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