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On duality symmetries of supergravity invariants

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Abstract

The role of duality symmetries in the construction of counterterms for maximal supergravity theories is discussed in a field-theoretic context from different points of view. These are: dimensional reduction, the question of whether appropriate superspace measures exist and information about non-linear invariants that can be gleaned from linearised ones. In $D = 4, N = 8$ supergravity we find that all three of these arguments suggest that F-term counterterms cannot be $E_{7(7)}$-invariant and that the theory should be finite up to seven loops as a consequence. We also argue that $N = 6$ supergravity is finite at three and four loops and that $N = 5$ supergravity should be three-loop finite.

Introduction

It has now been established that $D = 4, N = 8$ supergravity is finite at three loops [1], despite the existence of a linearised R^4 counterterm [2, 3], and that maximal supergravity is finite at four loops in $D = 5$ [4]. The only other candidate linearised short counterterms (i.e. F or BPS terms) in $D = 4$ occur at the five and six loop orders and are four-point terms of the form $\partial^{2k} R^4$ for $k = 2, 3$ [5, 6, 7]. The absence of the R^4 divergence can be seen from field-theoretic arguments [8, 9], including algebraic renormalisation theory, results that generalise those for the finiteness of one-half BPS counterterms in maximal super Yang-Mills theories in various dimensions [9]. However, even in the Yang-Mills case it does not seem easy to extend these results to the double-trace $\partial^2 F^4$ invariant [10, 11] which is known to be finite at three loops in $D = 6$ [12]. String theory provides an alternative approach to discussing field-theoretic finiteness issues and has been used to give arguments in favour of the known Yang-Mills results and also suggesting that $D = 4, N = 8$ supergravity should be finite at least up to six loops [13].

A key feature of supergravity theories which has no analogue in SYM is the existence of duality symmetries. It has recently been shown [14] that $E_{7(7)}$ can be maintained in perturbation theory in $D = 4$ (at the cost of manifest Lorentz invariance), and this suggests that these duality symmetries should be taken seriously in providing additional constraints on possible counterterms which might not be visible from a linearised analysis. For R^4 a scattering amplitude analysis supporting the idea that the full invariant is not compatible with $E_{7(7)}$ was given in [15], while in a recent paper this violation of $E_{7(7)}$ invariance was demonstrated by means of an argument based on dimensional reduction from type II string theory [16].

In this note we investigate this issue for $D = 4, N = 8$ supergravity in a field theory setting from three different points of view: dimensional reduction of higher-dimensional counterterms, the (non)-existence of appropriate superspace measures that generalise the linearised ones, and the use of the so-called ectoplasm formalism which allows one to write super-invariants in terms of closed superforms. Although our discussion is not completely rigourous, we find that it strongly suggests that $E_{7(7)}$ invariance would postpone the onset of UV divergences until at least seven loops.

Dimensional reduction

One way of stating the problem with duality invariance is to start from the R^4 counterterm in $D = 11$ and to reduce it to $D = 4$. This reduced invariant will only have the natural $SO(7)$ symmetry of a standard Kaluza-Klein reduction on T^7. However, the invariant may be promoted to a full $SU(8)$ invariant by first performing the necessary dualisations of higher form fields and then averaging, *i.e.* parametrising the embedding of $SO(7)$ into $SU(8)$ and integrating over the $SU(8)/SO(7)$ coset in a fashion similar to that employed in harmonic superspace constructions. Letting the volume modulus of a T^{11-D} reduction be $-(D-2)\vec{\alpha}\cdot\vec{\phi}$, where $\vec{\phi}$ are the Kaluza-Klein scalars emerging from the metric, the Kaluza-Klein reduction ansatz from 11 to D dimensions for the metric is $ds_{11}^2 = e^{2\vec{\alpha}\cdot\vec{\phi}} ds_D^2 + \ldots$. For the Einstein action, the T^{11-D} volume factor $e^{-(D-2)\vec{\alpha}\cdot\vec{\phi}}$ cancels against a factor $e^{D\vec{\alpha}\cdot\vec{\phi}} e^{-2\vec{\alpha}\cdot\vec{\phi}}$ coming from $\sqrt{-g_D} g_D^{\mu\nu}$. However, for the R^4 invariant in $D = 11$, the Einstein-frame reduction produces an extra factor of $e^{-6\vec{\alpha}\cdot\vec{\phi}}$ arising from the three additional inverse metrics present in the R^4 term as compared to the Einstein-Hilbert action. This dilatonic factor will then be promoted to an $SU(8)$ invariant by the $SU(8)/SO(7)$ integration. If we expand the exponentials in power series, the terms linear in the scalars vanish in such an averaging, but $SU(8)$ invariant quadratic terms survive. Such non-vanishing quadratic terms have been worked out explicitly for the $10 \to 4$ reduction of string/supergravity R^4 corrections in Ref. [16].

Such dilaton factors in front of purely gravitational terms containing only curvatures and their covariant derivatives prevent such terms from being constituent parts of duality invariants, since the lowest-order part of a duality transformation always involves a constant shift of the scalar fields. Of course, were there additional invariants arising in the lower dimension, without Kaluza-Klein origins, combinations of invariants could be capable of erasing the problematic dilatonic scalar prefactors and thus permitting a duality-invariant construction. This is precisely what must happen in $D = 8$, where an R^4 divergence in maximal supergravity occurs at the one-loop order. However, in $D = 4$, the available 1/2 BPS $SU(8)$ invariant R^4 counterterm [3] is *unique* at the 4-point level [5]. This counterterm develops a higher-point structure which is not currently known, but this higher-point structure must, once again, be unique. Were there alternative higher-point structures extending the 4-point linearised supersymmetry invariant, their differences would have to constitute new $D = 4$ invariants under $SU(8)$-covariant linearised supersymmetry, and these also do not exist [5]. Thus, the uniqueness of the $SU(8)$-symmetric R^4 invariant in $D = 4$ maximal supergravity shows that the $SU(8)$-symmetrised dimensional reduction of the $D = 11$ R^4 invariant is the only such supersymmetric candidate. Its ineligibility as an $E_{7(7)}$ duality invariant thus rules out this $D = 4$ R^4 candidate counterterm.

The above argument is a variant of the one given in Ref. [16] (where it was framed in terms of reduction from $D = 10$ type 2A superstring/supergravity amplitudes). It also gives a way to see that the maximal supergravity 1/4 BPS $\partial^4 R^4$ candidate counterterm at 5 loops and the 1/8 BPS $\partial^6 R^4$ candidate counterterm at 6 loops cannot be $E_{7(7)}$ duality invariants either. Once again, the argument hangs upon the uniqueness of the corresponding $D = 4$ $SU(8)$ symmetric BPS invariants, together with the inevitable dilaton factors that arise from dimensional reduction in front of the purely gravitational parts of the invariants.

In fact, it is precisely the known existence of the 1/2 BPS R^4 one-loop divergence of maximal supergravity in $D = 8$, the 1/4 BPS $\partial^4 R^4$ two-loop divergence in $D = 7$ and the 1/8 BPS $\partial^6 R^4$ three-loop divergence in $D = 6$ that permits us to rule out the descendants of these counterterms as $E_{7(7)}$ invariants in $D = 4$. The existence of these higher-dimensional divergences indicates the presence of counterterms without dilaton factors in the purely gravitational parts of the higher-dimensional versions of these counterterms. Indeed, the demonstration that $E_{7(7)}$ symmetry is preserved in perturbative theory for $N = 8$ supergravity [14], generalises straightforwardly to higher dimensions, provided there are no Lorentz $\times$ R-symmetry one-loop anomaly. The absence of such anomaly is trivial in odd dimensions, and there is none in six dimensions [17]. The $SL(2, \mathbb{R})$ symmetry is anomalous at one-loop in eight dimensions, however the latter does not affect the consequence of the tree level Ward identities for the one-loop divergence, and this one must therefore be associated to a duality invariant R^4 counterterm. Coupled with the $D = 4$ uniqueness of all these BPS counterterms [5], the inevitable appearance of dilaton factors in the corresponding $D = 4$ versions then rules out $E_{7(7)}$ invariance.

Consequently, the first $D = 4$ candidate counterterm with $E_{7(7)}$ invariance will be the non-BPS $\partial^8 R^4$ candidate at seven loops [18]. In superspace language, this first $E_{7(7)}$ invariant candidate is simply the volume of superspace, $\int d^4x d^{32}\theta \, \mathrm{det} E$. It remains to be verified whether this invariant is non-vanishing subject to the classical field equations of the $N = 8$ theory.

Harmonic measures

Another aspect of the difficulty in constructing non-linear invariants in maximal supergravity is that the necessary measures that generalise the linearised ones do not always exist. Here we discuss this issue in the case of $D = 4, N = 8$ supergravity. At the linearised level, there are three short invariants that can be written as integrals over certain harmonic superspaces [5]. We briefly review these and then discuss how one might try to generalise these integrals to the non-linear case.

We recall that harmonic superspace is the product of ordinary superspace with a coset of the R-symmetry group G which is always chosen to be a compact complex manifold, K [19, 20, 21]. Instead of working on K directly, it is convenient to work with fields that are defined on G and then demanding that their dependence on the isotropy group P defining K, $K = P\backslash G$, be fixed in such a way that they are equivalent to tensor fields on K [20]. We shall denote an element of G by $u_I{}^i$ where G (P) acts to the right (left) on the small (capital) index, and its inverse by $v_i{}^I$. In flat $D = 4$ superspace the derivatives are $(\partial_a, D_{\alpha i}\bar{D}^i_{\dot\alpha}),\ i = 1, \ldots N$. The introduction of the new variables allows us to define subsets of the odd derivatives that mutually anticommute without breaking the R-symmetry. Such a subset with p Ds and q $\bar{D}$s is called a Grassmann (G)-analytic structure of type (p, q), and a G-analytic field of type (p, q) is one that is annihilated by all of these derivatives.

For $N = 8$ we can take $P = S(U(p) \times U(8 - (p + q)) \times U(q)$ and set $u_I{}^i = (u_r{}^i, u_R{}^i, u_{r'}{}^i)$. The (p, q) mutually anticommuting derivatives are

$$D_{\alpha r} := u_r{}^i D_{\alpha i} \qquad \text{and} \qquad \bar{D}^{r'}_{\dot\alpha} := \bar{D}^i_{\dot\alpha} v_i{}^{r'} \ , \tag{1}$$

for $r = 1, \cdots p$ and $r' = (N - q), \cdots N$. As the superfields will also depend on u we need to introduce derivatives on $SU(8)$; they are the right-invariant vector fields $D_I{}^J$ and they satisfy the Lie algebra relations of $\mathfrak{su}(8)$. Their action on the u, v variables is given by

$$D_I{}^J u_K{}^k = \delta_K{}^J u_I{}^k - \frac{1}{8}\delta_I{}^J u_K{}^k; \qquad D_I{}^J v_k{}^K = -\delta_I{}^K v_k{}^J + \frac{1}{8}\delta_I{}^J v_k{}^K \ . \tag{2}$$

The derivatives split into subsets: $(D_r{}^s, D_R{}^S, D_{r'}{}^{s'})$ correspond to the isotropy subalgebra while $(D_r{}^S, D_r{}^{s'}, D_R{}^{s'})$ can be thought of as the components of the $\bar\partial$-operator on the complex manifold K. The remaining derivatives are the complex conjugates of these. This means that we can have superfields that are G-analytic (annihilated by $(D_{\alpha r}, \bar{D}^{r'}_{\dot\alpha})$), superfields that are harmonic, or H-analytic (annihilated by $(D_r{}^S, D_r{}^{s'}, D_R{}^{s'})$, and superfields that are annihilated by both sets since they are compatible in the sense that they are closed under graded commutation. We shall call such superfields analytic. They are the integrands for the short invariants. The fact that they are H-analytic implies that they have short expansions in u, because K is compact as well as complex, and the fact that they are G-analytic means that they can be integrated over $32 - 2(p + q)$ odd coordinates rather than the full 32.

The $N = 8$ field strength superfield W_{ijkl} is in the 70 of $SU(8)$; it is totally antisymmetric and self-dual on its $SU(8)$ indices and satisfies

$$\begin{aligned} D_{\alpha i} W_{jklm} &= D_{\alpha [i} W_{jklm]} \\ \bar{D}^i_{\dot\alpha} W_{jklm} &= -\frac{4}{5}\delta^i_{[j} \bar{D}^n_{\dot\alpha} W_{klm]n} \ . \end{aligned} \tag{3}$$

The R^4 invariant can be written in $(4,4)$ superspace. The field $W := \frac{1}{4!}\varepsilon^{rstu}u_r{}^i \ldots u_u{}^l W_{ijkl}$ is easily seen to be G-analytic and is also obviously H-analytic on the coset $S(U(4)\times U(4))\backslash SU(8)$. It is preferable to write W as W_{1234} as this exhibits the charges explicitly. The R^4 invariant is

$$I = \int d^4x\, du\, [D_5 \ldots D_8 \bar{D}^1 \ldots \bar{D}^4]^2\, (W_{1234})^4 \tag{4}$$

where du denotes the standard measure on the coset and the theta-integration is represented as differentiation with respect to all of the spinorial derivatives that do not annihilate W. It is easily seen to be unique as it makes use of the only dimension-zero analytic integrand with the right charges. The other two short invariants $\partial^4 R^4$ and $\partial^6 R^4$ can be written in a similarly unique fashion as integrals over $(2,2)$ and $(1,1)$ harmonic superspaces respectively.

We now want to try to generalise this picture to curved superspace.[1] In superspace the tangent spaces split invariantly into even and odd sectors (there is no supersymmetry in the tangent space) and for $N=8$ the structure group is $SL(2,\mathbb{C})\times SU(8)$. Because of the split structure, it is always best to work in a preferred basis. The preferred basis one-forms are related to the coordinate one-forms by the supervielbein, $E^A = dz^M E_M{}^A$; their duals are denoted E_A. We set $E^A = (E^a, E^{\alpha i}, \bar{E}_i^{\dot\alpha})$, where a is a vector index. $SL(2,\mathbb{C})$ acts on the spinor indices $\alpha, \dot\alpha$ and also on the vector index a via the corresponding element of the Lorentz group, while the local $SU(8)$ acts on i, j, etc. We also have a set of connection one-forms $\Omega_A{}^B$ with

$$\begin{aligned} \Omega_{\alpha i}{}^{\beta j} &= \delta_\alpha{}^\beta \Omega_i{}^j + \delta_i{}^j \Omega_\alpha{}^\beta \\ \Omega_{ab} \to \Omega_{\alpha\dot\alpha,\beta\dot\beta} &= \varepsilon_{\dot\alpha\dot\beta}\Omega_{\alpha\beta} + \varepsilon_{\alpha\beta}\bar{\Omega}_{\dot\alpha\dot\beta}\ , \end{aligned} \tag{5}$$

where we have used the usual relation between vector indices and pairs of spinor indices. $\Omega^{i\dot\beta}_{\dot\alpha j}$ is the complex conjugate of $\Omega_{\alpha i}{}^{\beta j}$ and the off-diagonal elements of $\Omega_A{}^B$ are zero. The torsion and curvature tensors are defined in the usual way, using the covariant exterior derivative D, by

$$T^A = DE^A; \qquad R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B\ . \tag{6}$$

In $N=8$ supergravity, the scalars are described by an element $\mathcal{V}$ of the group $E_{7(7)}$ [24], where the local $SU(8)$ acts from the left and the rigid $E_{7(7)}$ acts from the right. The Maurer-Cartan form Φ is

$$\Phi = d\mathcal{V}\mathcal{V}^{-1} = P + Q\ , \tag{7}$$

where P is in the 70 of $SU(8)$ and Q is the $\mathfrak{su}(8)$ connection which is to be identified with $\Omega_i{}^j$ above. In the geometrical quantities, the scalars only appear through the vector part of the one-form P, *i.e.* P_a, which one can think of as a suitably defined pullback of the covariant derivative for the scalar target manifold.

The constraints on the various tensors that need to be imposed in order to describe on-shell $N=8$ supergravity can be found in [25, 26]. At dimension zero, the only non-vanishing torsion is

[1] $N=2$ curved harmonic superspace was first studied in [22]; the sort of analysis given here was described for $N \leq 4$ conformal supergravity theories in [23].

$$T_{\alpha i,\dot{\beta}}{}^{j\,c} = -i\delta_i{}^j(\sigma^c)_{\alpha\dot{\beta}} \ , \tag{8}$$

and the only non-vanishing dimension one-half torsion is

$$T_{\alpha i,\beta j,k}{}^{\dot{\gamma}} = \varepsilon_{\alpha\beta}\bar{\Lambda}^{\dot{\gamma}}_{ijk} \tag{9}$$

and its conjugate, where Λ_α^{ijk} is the superfield whose leading component is the physical spinor field that transforms under the 56 of $SU(8)$.

This brief outline is enough to enable us to discuss whether there can be harmonic superspace measures of the required type in the non-linear theory where the $SU(8)$ R-symmetry becomes local. We need to enlarge the superspace by adjoining some group variables u; in fact, the resulting space is the principal bundle associated with the $SU(8)$ part of the structure group. The idea is to search for appropriate CR structures, that is, complex, involutive distributions which involve $2(p{+}q)$ odd directions and the accompanying holomorphic structures in the bundle coordinates. The way to do this is to introduce the horizontal lift basis in the total space of the bundle corresponding to a preferred basis in the base manifold. We have

$$\widetilde{E}_A := E_A - \Omega_{AI}{}^J D_J{}^I \ , \tag{10}$$

where one switches to an I index from an i index by means of $u_I{}^i$ and its inverse, as in the flat case. We then have

$$[\widetilde{E}_{\alpha I}, \widetilde{E}_{\beta J}] = -T_{\alpha I \beta J}{}^C \widetilde{E}_C + R_{\alpha I\beta JK}{}^L D_L{}^K \ + \ldots , \tag{11}$$

where the additional terms are irrelevant for this discussion, and similarly for the dotted and mixed commutators of the spinorial lifted bases. Now suppose that we require the CR structure to include $\widetilde{E}_{\alpha r}, \ r = 1\ldots p$ and $\widetilde{E}^{s'}_{\dot{\beta}}, \ s' = N-q, \ldots N$. We can see immediately that this leads to consistency conditions on the dimension one-half torsion, namely

$$T_{\alpha r,\beta s,\dot{\gamma} t} = T_{\alpha r,\beta s,\dot{\gamma} T} = 0 \ , \tag{12}$$

since otherwise these derivatives would not close among themselves. From the explicit form of the dimension one-half torsion (9) we can see that the only possibility is that r can only take on one value. A similar result holds for the dotted indices, and so we conclude that we can only have Grassmann analyticity of type $(1,1)$ in the full theory. (There are also conditions on the curvature but they are compatible with this.) As a CR structure is necessary in order that we can have harmonic superspaces with fewer odd coordinates (also called analytic superspaces) it follows that harmonic measures do not exist for $(p,q) = (4,4)$ and $(2,2)$ G-analyticity, and therefore that there can be no straightforward generalisation of the R^4 and $\partial^4 R^4$ invariants, expressed as harmonic superspace integrals, to the full non-linear theory that are compatible with local $SU(8)$ symmetry.

In the case of $(1,1)$ analyticity, relevant to the $\partial^6 R^4$ invariant, the measure should exist, which suggests that this invariant can be written as an harmonic superspace integral. However, the harmonic measure is definitely not R-symmetric, which implies that the integrand must be a non-trivial function of the scalars $\mathcal{V}$. In the formulation with gauged $SU(8)$ and linearly realised rigid $E_{7(7)}$, the measure will be $E_{7(7)}$ invariant whereas the integrand will necessarily transform non-trivially with respect to $E_{7(7)}$. It would then follow that the $\partial^6 R^4$ invariant is not $E_{7(7)}$ invariant, in agreement with the conclusion of the preceding section.

Note that this is not in contradiction with the existence of BPS duality invariants in higher dimensions (such as R^4 in $D = 8$, $\partial^4 R^4$ in $D = 7$ and $\partial^6 R^4$ in $D = 6$), since the BPS invariants are not unique in dimensions $D > 5$.

The non-existence of harmonic measures for the 1/2 and the 1/4 BPS invariants is not in contradiction with the existence of these non-linear invariants in the full non-linear theory. Indeed as we will discuss in the next section, not all supersymmetry invariants can be written as harmonic superspace integrals, and some are only described in terms of closed super-D-form.

Non-linear consequences of linear invariants

A more general approach to the construction of superinvariants is afforded by the ectoplasm formalism [27, 28, 29]. In D-dimensional spacetime, consider a closed super-D-form, L_D, in the corresponding superspace. The integral of the purely bosonic part of this form over spacetime is then guaranteed to be supersymmetric by virtue of the closure property. Moreover, if L_D is exact it will clearly give a total derivative so that we are really interested in the Dth superspace cohomology group. As we have seen in the preceding section, one cannot define a harmonic measure for every invariant, and in particular, not for the 1/2 and 1/4 BPS invariants in $N = 8$ supergravity. However, according to the algebraic Poincaré Lemma, any supersymmetry invariant necessarily defines a closed super-D-form.

In order to analyse superspace cohomology, it is convenient to split forms into their even and odd parts. Thus a (p, q)-form is a form with p even and q odd indices, totally antisymmetric on the former and totally symmetric on the latter. The exterior derivative can likewise be decomposed into parts with different bi-degrees,

$$d = d_0 + d_1 + t_0 + t_1 \ , \tag{13}$$

where the bi-degrees are $(1, 0)$, $(0, 1)$, $(-1, 2)$ and $(2, -1)$ respectively. So d_0 and d_1 are basically even and odd derivatives, while t_0 and t_1 are algebraic. The former acts by contracting an even index with the vector index on the dimension-zero torsion and then by symmetrising over all of the odd indices. The equation $d^2 = 0$ also splits into various parts of which the most relevant components are

$$t_0^2 = 0; \qquad d_1 t_0 + t_0 d_1 = 0; \qquad d_1^2 + t_0 d_0 + d_0 t_0 = 0 \ . \tag{14}$$

The first of these equations allows us to define t_0-cohomology groups, $H_t^{p,q}$ [30], and the other two allow us to introduce the spinorial derivative d_s which maps $H_t^{p,q}$ to $H_t^{p,q+1}$ by $d_s[\omega_{p,q}] = [d_1 \omega_{p,q}]$, where the brackets denote H_t cohomology classes, and which also squares to zero [31, 32]. The point of this is that one can often generate closed super-D-forms from elements of these cohomology groups.

In the context of curved superspace it is important to note that the invariant is constructed from the top component in a coordinate basis,

$$I = \frac{1}{D!} \int d^D x \, \varepsilon^{m_D \ldots m_1} \, E_{m_D}{}^{A_D} \cdots E_{m_1}{}^{A_1} \, L_{A_1 \ldots A_D}(x, \theta = 0) \ . \tag{15}$$

One transforms to a preferred basis by means of the supervielbein $E_M{}^A$. At $\theta = 0$ we can identify E_m^a with the spacetime vielbein $e_m{}^a$ and $E_m{}^{\underline{\alpha}}$ with the gravitino field $\psi_m{}^{\underline{\alpha}}$ (where $\underline{\alpha}$ includes both space-time α, $\dot{\alpha}$ and internal i indices for $N = 8$). In four dimensions, we therefore have

$$I = \frac{1}{24} \int \left(e^a_\wedge e^b_\wedge e^c_\wedge e^d \; L_{abcd} + 4 e^a_\wedge e^b_\wedge e^c_\wedge \psi^{\underline{\alpha}} L_{abc\underline{\alpha}} + 6 e^a_\wedge e^b_\wedge \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}} \; L_{ab\underline{\alpha}\underline{\beta}} \right.$$
$$\left. + 4 e^a_\wedge \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}_\wedge \psi^{\underline{\gamma}} L_{a\underline{\alpha}\underline{\beta}\underline{\gamma}} + \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}_\wedge \psi^{\underline{\gamma}}_\wedge \psi^{\underline{\delta}} L_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} \right) . \quad (16)$$

By definition, each component L_{abcd}, $L_{abc\underline{\alpha}}$, $L_{ab\underline{\alpha}\underline{\beta}}$, $L_{a\underline{\alpha}\underline{\beta}\underline{\gamma}}$, $L_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$ is supercovariant at $\theta = 0$. This is a useful formula because one can directly read off the invariant in components in this basis.

In $N = 8$ supergravity, all the non-trivial t_0-cohomology classes lie in ${H_t}^{0,4}$. Invariants are therefore completely determined by their $(0,4)$ components $L_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$, and all non-trivial $L_{0,4}$ satisfying $[d_1 L_{0,4}] = 0$ in t_0-cohomology define non-trivial invariants. ${H_t}^{0,4}$ is the set of functions of fields in the symmetric tensor product of four $\mathbf{2} \times \mathbf{8} \oplus \overline{\mathbf{2}} \times \overline{\mathbf{8}}$ of $SL(2,\mathbb{C}) \times SU(8)$ without $SU(8)$ contractions (since such functions would then be t_0-exact). Because of the reducibility of the representation, it will be convenient to decompose $L_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$ into components of degree $(0,p,q)$ $(p+q=4)$ with p $\mathbf{2} \times \mathbf{8}$ and q $\overline{\mathbf{2}} \times \overline{\mathbf{8}}$ symmetrised indices.

We will classify the elements of ${H_t}^{0,4}$ into three generations.[2] The first generation corresponds to elements that lie in the antisymmetric product of four $\mathbf{2} \times \mathbf{8} \oplus \overline{\mathbf{2}} \times \overline{\mathbf{8}}$ of $SL(2,\mathbb{C}) \times SU(8)$, and can therefore be directly related to the top component $L_{4,0}$ through the action of the superderivatives. We will write $M_{0,p,q}$ for the corresponding components of a given $L_{0,4}$. They lie in the following irreducible representations of $SL(2,\mathbb{C}) \times SU(8)$:

$$\begin{array}{ll} M_{0,4,0} : [0,0|0200000] & \bar{M}_{0,0,4} : [0,0|0000020] \\ M_{0,3,1} : [1,1|1100001] & \bar{M}_{0,1,3} : [1,1|1000011] \\ M_{0,2,2} : [2,0|2000010] & \bar{M}_{0,2,2} : [0,2|0100002] \; . \end{array} \quad (17)$$

In order to understand the constraints that these functions must satisfy in order for $L_{0,4}$ to satisfy the descent equation

$$[d_1 L_{0,4}] = 0 \; , \quad (18)$$

it is useful to look at the possible representations of $d_1 L_{0,4}$ which define ${H_t}^{0,5}$ cohomology classes in general, without assuming any *à priori* constraint. We will split $d_1 = d_{1,0} + d_{0,1}$ according to the irreducible representations of $SL(2,\mathbb{C}) \times SU(8)$. One computes that

$$\begin{array}{ll} [d_{1,0} M_{0,4,0}] : [1,0|1200000] & [d_{0,1} \bar{M}_{0,0,4}] : [0,1|0000021] \\ [d_{0,1} M_{0,4,0}] : [0,1|0200001] & [d_{1,0} \bar{M}_{0,0,4}] : [1,0|1000020] \\ [d_{1,0} M_{0,3,1}] : [0,1|0200001] \oplus [2,0|2100001] & [d_{0,1} \bar{M}_{0,1,3}] : [1,0|1000020] \oplus [0,2|1000012] \\ [d_{0,1} M_{0,3,1}] : [1,0|1100010] \oplus [1,2|1100002] & [d_{1,0} \bar{M}_{0,3,1}] : [0,1|0100011] \oplus [2,1|2000011] \\ [d_{1,0} M_{0,2,2}] : [1,0|1100010] \oplus [3,0|3000010] & [d_{0,1} \bar{M}_{0,2,2}] : [0,1|0100011] \oplus [0,3|0100003] \\ [d_{0,1} M_{0,2,2}] : [2,1|2000011] & [d_{1,0} \bar{M}_{0,2,2}] : [1,2|1100002] \; . \end{array} \quad (19)$$

[2]We will avoid discussing the elements of ${H_t}^{0,4}$ of degree $(0,2,2)$ in the $[0,0|0200020]$ representation, which do not play any role.

In order for the component

$$L_{0,4} = \sum_{p=0}^{2} \big(M_{0,4-p,p} + \bar M_{0,p,4-p} \big) \tag{20}$$

to satisfy the descent equation (18), the components $d_1 M_{0,p,q}$ must individually vanish in the $[1,0|1200000]$, $[2,0|2100001]$, $[3,0|3000010]$ representations and their complex conjugates, and their components in the $[0,1|0200001]$, $[1,0|1100010]$, $[1,2|1100002]$ and their complex conjugates must cancel each other. This will indeed be the case if the invariant in question can be defined as a superaction and all the components of $L_{0,4}$ descend from a primary operator satisfying the appropriate constraint. However, as we have seen in the preceding section, there is no harmonic measure for the 1/2 and 1/4 BPS invariants, and this situation is therefore not the most general.

What will happen for these invariants is that, although the components of $d_1 M_{0,p,q}$ in the $[0,1|0200001]$, $[1,0|1100010]$ and their complex conjugate representations cancel each other, the components in the $[1,0|1200000]$, $[2,0|2100001]$, $[3,0|3000010]$, $[1,2|1100002]$ and the corresponding complex conjugates will not vanish. The latter will nevertheless be cancelled by the d_1 variation of a second generation of functions $N_{0,p,q}$ in ${H_t}^{0,4}$,

$$\begin{aligned} &N_{0,4,0} : [2,0|2100000] \qquad && \bar N_{0,0,4} : [0,2|0000012] \\ &N_{0,3,1} : [3,1|3000001] \qquad && \bar N_{0,1,3} : [1,3|1000003] \\ &\qquad N_{0,2,2} : [2,2|2000002]\ . \end{aligned} \tag{21}$$

Indeed, one computes that the components of $[d_1 N_{0,p,q}]$ lie in the following representations

$$\begin{aligned} &[d_{1,0} N_{0,4,0}] : [1,0|1200000] \oplus [3,0|3100000] \qquad && [d_{0,1} \bar N_{0,0,4}] : [0,1|0000021] \oplus [0,3|0000013] \\ &[d_{0,1} N_{0,4,0}] : [2,0|2100001] && [d_{1,0} \bar N_{0,0,4}] : [0,2|1000012] \\ &[d_{1,0} N_{0,3,1}] : [2,0|2100001] \oplus [4,1|4000001] && [d_{0,1} \bar N_{0,1,3}] : [0,2|1000012] \oplus [1,4|1000004] \\ &[d_{0,1} N_{0,3,1}] : [3,0|3000010] \oplus [3,2|3000002] && [d_{1,0} \bar N_{0,3,1}] : [0,3|0100003] \oplus [2,3|2000003] \\ &[d_{1,0} N_{0,2,2}] : [1,2|1100002] \oplus [3,2|3000002] && [d_{0,1} N_{0,2,2}] : [2,1|2000011] \oplus [2,3|2000003]\ . \end{aligned} \tag{22}$$

In addition to cancelling the components $[d_1 M_{0,p,q}]$, the components $[d_1 N_{p,q}]$ must cancel each other in the $[3,2|3000002]$ representation and its complex conjugate. Then there are two possibilities: either the components of $[d_1 N_{p,q}]$ identically vanish in the $[3,0|3100000]$, the $[4,1|4000001]$ and their complex conjugates, or a third generation of $O_{0,4,0}$ functions and their $\bar O_{0,0,4}$ complex conjugates in ${H_t}^{0,4}$ is required to cancel them,

$$O_{0,4,0} : [4,0|4000000] \qquad \bar O_{0,0,4} : [0,4|0000004]\ . \tag{23}$$

Now, $[d_1 O_{0,4,0}]$ lies in the following representations of ${H_t}^{0,5}$

$$\begin{aligned} &[d_{1,0} O_{0,4,0}] : [3,0|3100000] \oplus [5,0|5000000] \qquad && [d_{0,1} \bar O_{0,0,4}] : [0,3|0000013] \oplus [0,5|0000005] \\ &[d_{0,1} O_{0,4,0}] : [4,1|4000001] && [d_{1,0} \bar O_{0,0,4}] : [1,4|1000004]\ , \end{aligned} \tag{24}$$

and in addition to cancelling $[d_1 N_{p,q}]$ in the $[3,0|3100000]$, the $[4,1|4000001]$ and their complex conjugates, the components of $d_{1,0} O_{0,4,0}$ in the $[5,0|5000000]$ must identically vanish.

To conclude this discussion, we have seen from the t_0-cohomology analysis that there exist more general cocycle structures than those associated to invariants that can be written as (harmonic) superspace integrals. The absence of harmonic measures for the 1/2 and 1/4 BPS invariants is therefore not in contradiction with the existence of such invariants. However, their cocycle structures involve two or three supermultiplets instead of only one, corresponding to the second generation of operators $N_{0,p,q}$, and possibly the third $O_{0,4,0}$. The expectation is that the 1/2 BPS invariant will admit a cocycle involving three generations,

$$L^{1/2}_{0,4} = \sum_{p=0}^{2} \bigl(M^{1/2}_{0,4-p,p} + \bar{M}^{1/2}_{0,p,4-p} \bigr) + \sum_{p=0}^{1} \bigl(N^{1/2}_{0,4-p,p} + \bar{N}^{1/2}_{0,p,4-p} \bigr) + N^{1/2}_{0,2,2} + O^{1/2}_{0,4,0} + \bar{O}^{1/2}_{0,0,4} \; , \tag{25}$$

and the 1/4 BPS invariant will admit a cocycle involving two generations,

$$L^{1/4}_{0,4} = \sum_{p=0}^{2} \bigl(M^{1/4}_{0,4-p,p} + \bar{M}^{1/4}_{0,p,4-p} \bigr) + \sum_{p=0}^{1} \bigl(N^{1/4}_{0,4-p,p} + \bar{N}^{1/4}_{0,p,4-p} \bigr) + N^{1/4}_{0,2,2} \; . \tag{26}$$

We have not derived the explicit functions which define these cocycles, but we would like to point out that the F^4 invariants in super Yang–Mills theory in ten dimensions define explicit example of such cocycles involving several generations of t_0-cohomology classes [11]. From this perspective, it seems that a careful study of the implications of supersymmetry Ward identities within the algebraic approach should rule out the possibility of both the 3 and 5-loop logarithmic divergences in $N = 8$ supergravity. (We recall also that the 4-loop divergence has no available on-shell nonvanishing counterterm [5].) However, the existence of a 1/8 BPS harmonic measure suggests that the 1/8 BPS cocycle has the same structure as the cocycle associated to full superspace integral invariants, and therefore that the supersymmetry Ward identities alone will be unable to rule out the corresponding 6-loop divergence within the algebraic approach. However, as we have discussed in the preceding section, the integrand in that case must be a function of the scalar superfield, which implies that it cannot be $E_{7(7)}$ invariant, and therefore that the $E_{7(7)}$ Ward identities nonetheless rule out this divergence.

The non-existence of a 1/2 BPS measure does not permit one to conclude directly that the R^4 invariant cannot be $E_{7(7)}$ invariant, without relying on the dimensional reduction argument outlined in the first section. Nevertheless, it follows from the structure of the invariant (16), that knowledge of the cocycle $L^{1/2}_4$ in the quartic field approximation provides information about terms of orders up to 8 in the invariant. If $I^{1/2}$ were invariant with respect to $E_{7(7)}$, then it would follow from the representation of $E_{7(7)}$ on the fields that each component $L^{1/2}_{abcd}$, $L^{1/2}_{abc\underline{\alpha}}$, $L^{1/2}_{ab\underline{\alpha}\,\underline{\beta}}$, $L^{1/2}_{a\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}}$, $L^{1/2}_{\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}\underline{\delta}}$ would independently have to be $E_{7(7)}$ invariant. In the linearised approximation, this means that each component would be invariant at lowest order with respect to a constant shift of the scalar superfield W^{ijkl}. It was pointed out in [3] that L_{abcd} is shift invariant, but we shall see that the last component $L^{1/2}_{\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}\underline{\delta}}$ is not, hence establishing that $I^{1/2}$ is not fully $E_{7(7)}$ invariant.

To start with, note that the 1/2 BPS invariant admits a superaction form in the linearised approximation. It follows that the second and third generations of $(0,4)$ components are not required in this approximation, and that $N^{1/2}_{0,p,q}$ and $O^{1/2}_{0,4,0}$ are at least quintic in fields. In order to establish the non-shift-invariance of $L^{1/2}_{0,4}$ in the quartic field approximation, it will be enough to consider its $M^{1/2}_{0,4,0}$ component. The latter can be obtained by acting on the 1/2 BPS primary operator defined by W^4 in the [0004000] of $SU(8)$ with the D^8 in the $[0,0|0002000]$, and $\bar{D}^4$ in the $[0,0|0000020]$. With the conventional notation

$$D_{\alpha p}W^{ijkl}=\delta^{[i}_p\chi^{jkl]}_\alpha\ ,\quad D_{\alpha l}\chi^{ijk}_\beta=\delta^{[i}_l F^{jk]}_{\alpha\beta}\ ,\quad D_{\alpha k}F^{ij}_{\beta\gamma}=\delta^{[i}_k\rho^{j]}_{\alpha\beta\gamma}\ ,\quad D_{\alpha j}\rho^i_{\beta\gamma\delta}=\delta^i_j C_{\alpha\beta\gamma\delta}\ ,\tag{27}$$

one obtains that D^8W^4 in the $[0,0|0002000]$ has the form

$$D^8W^4\sim W^2C^2+W\chi\rho C+WF^2C+WF\rho^2+\chi^2FC+\chi F^2\rho+F^4\ ,\tag{28}$$

where the index contractions and symmetrisations are unambiguously determined by the representation. Since we are interested in the shift invariance of $M^{1/2}_{0,4,0}$, we can already disregard the three last terms. Applying finally $\bar D^4$ to (28), one obtains various terms linear in W, terms in W^3C, W^2F^2, $W^2\chi\rho$ and $W\chi^2F$ involving four derivatives, terms in $\bar\chi W\chi C$ and $\bar\chi WF\rho$ involving three derivatives, and terms in $\bar FW\rho^2$ and $W\bar FFC$ involving two derivatives. They are clearly all independent, taking into account the equations of motion, and one can discuss them separately. The term in W^3C is, for example, of the form

$$W_{pqij}\partial^a\partial^bW^{pqrs}\partial^c\partial^dW_{klrs}C_{ac,bd}\ .\tag{29}$$

Although its shift variation is a total derivative, it is clearly non-vanishing. Similarly, the terms in W^2F^2 take the form

$$\frac{1}{2}W^2\partial^2F\partial^2F+W\partial W\partial F\partial^2F+W\partial\big(\partial^2W\partial F\partial F\big)\ ,\tag{30}$$

where the three terms involve one product of W^2 in the [0002000] with F^2 in the [0000020], one product of W^2 in the [0010100] with F^2 in the [0000101], while the third term moreover involves a product of W^2 in the [0100010] with F^2 in the [0001000]. Once again, the shift variation of this set of terms is a non-vanishing total derivative. Hence, the shift variation of $M^{1/2}_{0,4,0}$ can be shown to be a non-vanishing total derivative, whereas $E_{7(7)}$ invariance of the 1/2 BPS invariant would have required its strict invariance (total derivatives included). Moreover, the structure of the 1/2 BPS supermultiplet implies that $M^{1/2}_{0,4,0}$ is uniquely determined from the primary operator W^4, and the 1/2 BPS cocycle does not admit other representatives.

We conclude that linearised analysis permits one to establish the $E_{7(7)}$ noninvariance of the full 1/2 BPS R^4 counterterm. However, this argument does not apply to the full 1/4 BPS ∂^4R^4 counterterm. Indeed, one can define the 1/4 BPS counterterm in the linear approximation by acting with the 1/2 BPS measure on the non-primary 1/2 BPS quartic term $\partial_aW\partial_bW\partial^aW\partial^bW$ in the [0004000] of $SU(8)$, which is manifestly shift invariant. But, of course, the shift invariance of the cocycle is a necessary but not sufficient condition for establishing $E_{7(7)}$ invariance of the corresponding supersymmetry invariant, and the dimensional reduction argument of the first section shows indeed that it is not $E_{7(7)}$ invariant.

$N=5$, 6 supergravity

Note that the demonstration that $E_{7(7)}$ symmetry is preserved in perturbative theory for $N=8$ supergravity [14], generalises straightforwardly to the $N=5$ and $N=6$ cases for the duality symmetries $SU(5,1)$ and $SO^*(12)$ respectively, because all the one-loop $SL(2,\mathbb{C})\times U(N)$ anomalies vanish [17]. Moreover, the linearised superalgebra in flat space can be embedded consistently into the corresponding superconformal algebra $\mathfrak{su}(2,2|N)$ similarly to the $N=8$ supergravity case, and one can again rely on superconformal representation analysis to prove that the BPS invariants are unique in these theories [5]. In this section, we will show that analysis of the linearised super 4-form associated to the corresponding R^4 invariants demonstrate that they also are not duality invariants, as in the $N = 8$ supergravity case. We will correspondingly prove the absence of logarithmic divergences at three loops in these theories.

In $N = 6$ supergravity, the complex scalar superfield W_{ij} and its complex conjugate W^{ij} define the following multiplets by the recursive action of $D_{\alpha i}$:

$$
\begin{aligned}
D_{\alpha k} W_{ij} &= \frac{1}{6}\varepsilon_{ijklmn}\chi_\alpha^{lmn} \ , \quad D_{\alpha l}\chi_\beta^{ijk} = \delta_l^{[i} F_{\alpha\beta}^{jk]} \ , \quad D_{\alpha k} F^{ij}_{\beta\gamma} = \delta_k^{[i}\rho^{j]}_{\alpha\beta\gamma} \ , \quad D_{\alpha j}\rho^i_{\beta\gamma\delta} = \delta^i_j C_{\alpha\beta\gamma\delta} \ , \\
D_{\alpha k} W^{ij} &= \delta_k^{[i}\chi_\alpha^{j]} \ , \quad D_{\alpha j}\chi^i_\beta = \delta^i_j F_{\alpha\beta} \ .
\end{aligned} \tag{31}
$$

The linearised R^4 invariant can be obtained by acting with $\bar{D}^8 D^8$ in the $[0,0|02020]$ representation of $SL(2,\mathbb{C}) \times SU(6)$ on the 1/3 BPS operator $W_{ij}W_{kl}W^{pq}W^{mn}$ in the $[0,0|02020]$ representation.[3] As for $N = 8$ supergravity, the cocycle's last components are $M_{0,p,q}$ with

$$
\begin{array}{ll}
M_{0,4,0} : [0,0|02000] & \bar{M}_{0,0,4} : [0,0|00020] \\
M_{0,3,1} : [1,1|11001] & \bar{M}_{0,1,3} : [1,1|10011] \\
M_{0,2,2} : [2,0|20010] & \bar{M}_{0,2,2} : [0,2|01002] \ ,
\end{array} \tag{32}
$$

and we will consider in particular the shift invariance of the $M_{0,4,0}$ component. The latter can be obtained by acting with $\bar{D}^4$ in the $[0,0|00020]$ and D^8 in the $[0,0|00020]$ on $W_{ij}W_{kl}W^{pq}W^{mn}$. $D^8 W^2 \bar{W}^2$ gives the $[0,0|00020]$ combination

$$
W^{ij}W^{kl}C^2 + W^{ij}\chi^{[k}\rho^{l]}C + W^{ij}FF^{kl}C + W^{ij}F\rho^{[k}\rho^{l]} + \dots \tag{33}
$$

where the dots stand for terms that are shift invariant. Applying then $\bar{D}^4$ to this expression, one obtains again various terms, including a single term in W^3C coming from WF^2C which reads

$$
\varepsilon_{ijpqrs} W^{pq}\partial^2 W^{rs}\partial^2 W_{kl} C \ , \tag{34}
$$

projected into the $[0,0|02000]$ representation. Similarly, one obtains various terms in $W^{ij}W^{kl}F^{mn}F^{pq}$ which appear in combinations similar to (30) in $N = 8$; as well as one term in $W^{ij}W_{kl}FF^{pq}$ coming from $W^{ij}FF^{kl}C$,

$$
\varepsilon_{ijpqrs} W^{pq} F \partial^2 W_{kl} \partial^2 F^{rs} \tag{35}
$$

projected into the $[0,0|02000]$ representation. It follows that the result of a shift of the scalar field W^{ij} in $M_{0,4,0}$ is non-vanishing, and not even a total derivative. We therefore conclude that the unique R^4 invariant in $N = 6$ supergravity is not $SO^*(12)$ invariant.

The $\partial^2 R^4$ counterterm can be obtained in a similar way from the 1/6 BPS operator $W^{ip}W^{jq}W_{kp}W_{lq}$ in the $[0,0|20002]$ representation, or from the non-primary 1/3 BPS operator $W_{ij}W^{pq}\partial^a W_{kl}\partial_a W^{mn}$ in the $[0,0|02020]$. Note that any combination with two derivatives would necessarily be a total derivative in $N = 8$ because the scalar field is then real, which explains why there is no $\partial^2 R^4$ invariant in that case. All the possible ways of adding two derivatives to $W_{ij}W_{kl}W^{pq}W^{mn}$ are in fact equivalent, up to a total derivative. One can easily see that one cannot adjust the derivatives such that both $M_{0,4,0}$ and $\bar{M}_{0,0,4}$ are shift invariant. However, one must also consider the possibility of defining the cocycle directly from the 1/6 BPS operator $W^{ip}W^{jq}W_{kp}W_{lq}$. In that case $M_{0,4,0}$ is obtained by acting with $\bar{D}^6$ in the $[0,0|00200]$ and D^{10} in the $[0,0|00002]$ on $W^{ip}W^{jq}W_{kp}W_{lq}$. Applying D^{10}, one already obtains an operator that does not depend on the scalars, so $M_{0,4,0}$ will be trivially shift invariant in this case. We have not checked if the other components are actually shift invariant as well, but they might well be.

[3] We will not write explicitly the $U(1)$ weight, which is zero for both the measure and the integrand.

In $N = 5$ supergravity, the complex scalar superfield W_i and its complex conjugate W^i define the following multiplets by the recursive action of $D_{\alpha i}$:

$$\begin{aligned} D_{\alpha i} W_j &= \chi_{\alpha ij} \ , \quad D_{\alpha k} \chi_{\beta ij} = \frac{1}{6} \varepsilon_{ijklp} F^{lp}_{\alpha\beta} \ , \quad D_{\alpha k} F^{ij}_{\beta\gamma} = \delta^{[i}_k \rho^{j]}_{\alpha\beta\gamma} \ , \quad D_{\alpha j} \rho^i_{\beta\gamma\delta} = \delta^i_j C_{\alpha\beta\gamma\delta} \ , \\ D_{\alpha j} W^i &= \delta^i_j \chi_\alpha \ . \end{aligned} \tag{36}$$

The linearised R^4 invariant can be obtained by acting with $\bar{D}^8 D^8$ in the $[0,0|2002]$ representation of $SL(2,\mathbb{C}) \times SU(5)$ on the 1/5 BPS operator $W_i W_j W^k W^l$ in the $[0,0|2002]$ representation. As for $N = 8$ supergravity, the cocycle's last components are $M_{0,p,q}$ with

$$\begin{aligned} &M_{0,4,0} : [0,0|0200] && \bar{M}_{0,0,4} : [0,0|0020] \\ &M_{0,3,1} : [1,1|1101] && \bar{M}_{0,1,3} : [1,1|1011] \\ &M_{0,2,2} : [2,0|2010] && \bar{M}_{0,2,2} : [0,2|0102] \ , \end{aligned} \tag{37}$$

and we will consider in particular the shift invariance of the $M_{0,4,0}$ component. The latter can be obtained by acting with $\bar{D}^4$ in the $[0,0|0020]$ and D^8 in the $[0,0|0002]$ on $W_i W_j W^k W^l$. $D^8 W^2 \bar{W}^2$ gives the $[0,0|0002]$ combination

$$W^i W^j C^2 + W^{(i} \chi \rho^{j)} C + \dots \tag{38}$$

where the dots stand for terms that are shift invariant. Applying then $\bar{D}^4$ to this expression, one again obtains various terms. Although there is no $W^3 C$ term, the terms in $W^2 F^2$ combine into

$$\frac{1}{2} \varepsilon_{ijpqr} \varepsilon_{klstu} W^p W^s \partial^2 F^{qr} \partial^2 F^{tu} + \varepsilon_{ijpqr} \varepsilon_{klstu} W^p \partial W^s \partial F^{qr} \partial^2 F^{tu} \ ,$$

plus a term

$$\varepsilon_{ijpqr} \varepsilon_{klstu} W^p \partial W^q \partial F^{rs} \partial^2 F^{tu} \ ,$$

both being projected into the $[0,0|0200]$. The term $W \chi \rho C$ also produces a term

$$\varepsilon_{ijpqr} W^p \chi \partial \chi_{kl} \partial^2 F^{qr} \ , \tag{39}$$

projected into the $[0,0|0200]$. Once again, the shift variation of $M_{0,4,0}$ does not vanish, and is not a total derivative either. We therefore conclude that the unique R^4 invariant in $N = 5$ supergravity is not $SU(5,1)$ invariant.

To conclude this section, we have shown that duality invariance implies the absence of 3-loop divergences in $N = 5, 6$ supergravity. It is likely that there is also a supersymmetry non-renormalisation theorem for $N = 6$ supergravity, but further investigation would be required to establish this. Indeed, one expects only measures corresponding to $(1,1)$ Grassmann-analyticity to exist in $N = 5$ and $N = 6$. It is therefore likely that one could define the unique $\partial^2 R^4$ invariant by a manifestly $SO^*(12)$ noninvariant harmonic superspace integral, which would imply the finiteness of $N = 6$ supergravity at four loops.

Concluding remarks

In this note, we have advanced field-theoretic arguments in favour of the idea that the short BPS invariants in $N = 8$ supergravity fail to be $E_{7(7)}$ invariant. From this, one concludes that the onset of divergences should be postponed to at least seven loops, where there is a candidate $E_{7(7)}$ invariant counterterm, namely the volume of superspace. For the short invariants, we have presented arguments based on the impossibility of achieving a trivial scalar factor in front of the purely gravitational R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms because a non-trivial scalar factor is required by dimensional reduction and because the uniqueness of the linearised $D = 4$ counterterms at the 3, 5 and 6 loop orders rules out the possibility of a cancellation between inequivalent terms coming from higher dimensions. We have also demonstrated that the R^4 invariant is indeed not $E_{7(7)}$ invariant by establishing the non-invariance of the last component of the associated linearised closed super four-form under constant shifts of the scalar fields, *i.e.* under linearised $\mathfrak{e}_7$ transformations. This comes about because this linearised term will affect the four-gravitino term (an eight-point contribution) in the non-linear spacetime invariant.

In addition, we have investigated the question of whether appropriate measures exist in curved $N = 8$ superspace. In two cases, corresponding to the R^4 and $\partial^4 R^4$ invariants, the answer is no, whereas for the $\partial^6 R^4$ invariant a measure seems to be available. However, even in this case, there is no available integrand that could be $E_{7(7)}$ invariant as such an integrand would have to be constructed from the undifferentiated scalars. We stress that the non-existence of harmonic measures for the R^4 and $\partial^4 R^4$ invariants does not imply that there are no such invariants in the full theory. Indeed, our analysis of the t_0-cohomology in $N = 8$ supergravity demonstrates that there exist in principle super-4-forms whose structure is incompatible with the possibility of writing them as harmonic superspace integrals. This translates in components into the property that such invariants admit terms quartic in undifferentiated gravitino fields with a tensor structure that cannot appear in harmonic superspace integrals (at least without introducing a prepotential). This suggests that the non-linear R^4 and $\partial^4 R^4$ invariants are associated to super-4-forms with a structure different from that of the other invariants, so that the supersymmetry Ward identities within the algebraic approach would by themselves be sufficient to rule out the possibility of the corresponding logarithmic divergences at 3 or 5-loops.

A further aspect of this purely field-theoretic analysis is that there are UV divergence implications for supergravity theories with fewer supersymmetries. The R^4 counterterm is a BPS invariant for $N = 5$ and $N = 6$ and the superspace arguments given above adapt to these cases straightforwardly. Indeed, we have shown that the closed super 4-forms associated to these counterterms are not invariant under constant shifts of the scalar fields, hence establishing that they are not duality invariant. Moreover, one only expects measures corresponding to $(1, 1)$ Grassmann-analyticity to exist in each of these cases. It therefore follows that there are non-renormalisation theorems at three loops for $N = 5, 6$ and also at four loops for $N = 6$ (note that there is a linearised four-loop invariant in $N = 6$, unlike the case of $N = 8$).

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