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$E_{7(7)}$ constraints on counterterms in $\mathcal{N}=8$ supergravity

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ABSTRACT

We prove by explicit computation that 6-point matrix elements of D^4R^4 and D^6R^4 in $\mathcal{N}=8$ supergravity have non-vanishing single-soft scalar limits, and therefore these operators violate the continuous $E_{7(7)}$ symmetry. The soft limits precisely match automorphism constraints. Together with previous results for R^4, this provides a direct proof that no $E_{7(7)}$-invariant candidate counterterm exists below 7-loop order. At 7-loops, we characterize the infinite tower of independent supersymmetric operators D^4R^6, R^8, φ^2R^8, … with $n>4$ fields and prove that they all violate $E_{7(7)}$ symmetry. This means that the 4-graviton amplitude determines whether or not the theory is finite at 7-loop order. We show that the corresponding candidate counterterm D^8R^4 has a non-linear supersymmetrization such that its single- and double-soft scalar limits are compatible with $E_{7(7)}$ up to and including 6-points. At loop orders 7, 8, 9 we provide an exhaustive account of all independent candidate counterterms with up to 16, 14, 12 fields, respectively, together with their potential single-soft scalar limits.

1. Introduction

$\mathcal{N}=8$ supergravity has maximal supersymmetry, and the classical theory has global continuous $E_{7(7)}$ symmetry which is spontaneously broken to SU(8). Explicit calculations have demonstrated that the 4-graviton amplitude in $\mathcal{N}=8$ supergravity is finite up to 4-loop order [1]. Together with string- and superspace-based observations [2,3], this spurred a wave of renewed interest in the question of whether the loop computations based on generalized unitarity [4] could yield a UV finite result to all orders[1] — or at which loop order the first divergence might occur.

In gravity, logarithmic UV divergences in on-shell L-loop amplitudes are associated with local counterterm operators of mass dimension $\delta = 2L+2$ composed of fields from the classical theory. The counterterms must respect the non-anomalous symmetries of the theory. It was shown in [7–9] that below 7-loop order, there are only 3 independent operators consistent with linearized $\mathcal{N}=8$ supersymmetry and global SU(8) R-symmetry [10]. These are the 3-, 5- and 6-loop supersymmetric candidate counterterms R^4, D^4R^4, and D^6R^4.

The perturbative S-matrix of $\mathcal{N}=8$ supergravity should respect $E_{7(7)}$ symmetry [11], so one must subject R^4, D^4R^4, and D^6R^4 to this test. A necessary condition for a counterterm to be $E_{7(7)}$-compatible, is that its matrix elements vanish in the 'single-soft limit' $p^\mu \to 0$ for each external scalar line [12–14]. The scalars of $\mathcal{N}=8$ supergravity are the 'pions' of this soft-pion theorem since they are the 70 Goldstone bosons of the spontaneously broken generators of $E_{7(7)}$. It was recently proven [15] that the soft scalar property fails for 6-point matrix elements of the operator R^4 (see also [16]). Thus $E_{7(7)}$ excludes R^4 and explains the finite 3-loop result found in [1].

In the present Letter we show first that the 5- and 6-loop operators D^4R^4 and D^6R^4 are incompatible with $E_{7(7)}$ symmetry because their 6-point matrix elements have non-vanishing single-soft scalar limits. Previous string theory [17] and superspace [18] arguments suggested this $E_{7(7)}$-violation. Our results mean that no UV divergences occur in $\mathcal{N}=8$ supergravity below the 7-loop level.

We then survey the candidate counterterms for loop orders $L=7,8,9$ using two new algorithmic methods: one program counts monomials in the fields of $\mathcal{N}=8$ supergravity in representations of the superalgebra SU(2, 2|8), the other applies Gröbner basis methods to construct their explicit local matrix elements. Our

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[1] This question is well defined whether or not $\mathcal{N}=8$ supergravity is sensible as a full quantum theory [5,6].

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analysis shows that at each loop level 7, 8, 9, there is an infinite tower of independent n-point supersymmetric counterterms with $n \geqslant 4$. At 7-loop order we find that none of the n-field operators with $n > 4$ are $E_{7(7)}$-compatible. This leaves D^8R^4 as the only candidate counterterm at $L = 7$. We show that its matrix elements are $E_{7(7)}$-compatible at least up to 6-points. We observe that it requires remarkable cancellations for $E_{7(7)}$ to be satisfied to all orders for any $L \geqslant 7$ candidate counterterm.

2. $E_{7(7)}$-violation of D^4R^4 and D^6R^4

To investigate $E_{7(7)}$ we study the soft scalar limit of the 6-point NMHV matrix elements $\langle{+}{+}{-}{-}\varphi\bar{\varphi}\rangle_{D^{2k}R^4}$. The external states are two pairs of opposite helicity gravitons and two conjugate scalars. These matrix elements contain local terms from nth order field monomials in the non-linear SUSY completion of $D^{2k}R^4$ as well as non-local pole diagrams in which one or more lines of the operator are off-shell and communicate to tree vertices from the classical Lagrangian. It is practically impossible to calculate these matrix elements with either Feynman rules (because the non-linear supersymmetrizations of $D^{2k}R^4$ are unknown) or recursion relations (because the matrix elements do not fall off under standard complex deformations of their external momenta). Instead we use the α'-expansion of the closed string tree amplitude to obtain the desired matrix elements.

At tree level, the closed string effective action takes the form

$$\begin{aligned} S_{\text{eff}} = {} & S_{\text{SG}} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 \\ & + \frac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 - \frac{1}{2}\alpha'^7\zeta(7)e^{-14\phi}D^8R^4 \\ & + \cdots. \end{aligned} \tag{1}$$

All closed string amplitudes in this work are obtained via KLT [19] from the open string amplitudes of [20]. The amplitudes confirm the structure and coefficients of (1).

Couplings of the dilaton ϕ break the SU(8)-symmetry of the supergravity theory to $\mathrm{SU}(4) \times \mathrm{SU}(4)$ when $\alpha' > 0$, and thus the supersymmetric operators of S_{eff} are not the desired SU(8)-invariant operators. As explained in [15], an SU(8)-averaging procedure extracts the SU(8) singlet contribution from the string matrix elements. Specifically, the SU(8) average of the $\langle{+}{+}{-}{-}\varphi\bar{\varphi}\rangle_{e^{-(2k+6)\phi}D^{2k}R^4}$ matrix elements from string theory is

$$\begin{aligned} \langle{+}{+}{-}{-}\varphi\bar{\varphi}\rangle_{\text{avg}} = {} & \frac{1}{35}\langle{+}{+}{-}{-}\varphi^{1234}\varphi^{5678}\rangle \\ & - \frac{16}{35}\langle{+}{+}{-}{-}\varphi^{123|5}\varphi^{4|678}\rangle \\ & + \frac{18}{35}\langle{+}{+}{-}{-}\varphi^{12|56}\varphi^{34|78}\rangle. \end{aligned} \tag{2}$$

The 3 terms on the right side correspond to the 3 inequivalent ways to construct scalars from particles of the $\mathcal{N} = 4$ gauge theory, namely from gluons, gluinos, and $\mathcal{N} = 4$ scalars. There are 35 distinct embeddings of $\mathrm{SU}(4) \times \mathrm{SU}(4)$ in SU(8). Averaging is sufficient to give the matrix elements of the $\mathcal{N} = 8$ field theory operator R^4, as done in [15], and we extend it here to D^4R^4. For D^6R^4 a further correction is necessary and is discussed below.

Before proceeding, we note that the operators in the action (1) are normalized such that their 4-point matrix elements are $\langle{+}{+}{-}{-}\rangle = g(s,t,u)[12]^4\langle 34\rangle^4$ with

$$\begin{aligned} & g_{R^4} = 1, && g_{D^4R^4} = s^2 + t^2 + u^2, \\ & g_{D^6R^4} = s^3 + t^3 + u^3, && g_{D^8R^4} = \left(s^2 + t^2 + u^2\right)^2. \end{aligned} \tag{3}$$

2.1. 5-loop counterterm D^4R^4

At order α'^5, the SU(8)-average (2) of the string theory amplitudes directly gives the matrix elements of the unique SU(8)-invariant supersymmetrization of D^4R^4. The result is a complicated non-local expression, but its single-soft scalar limit is very simple and local, viz.

$$\lim_{p_6 \to 0}\langle{+}{+}{-}{-}\varphi\bar{\varphi}\rangle_{D^4R^4} = -\frac{6}{7}[12]^4\langle 34\rangle^4\sum_{i<j}s_{ij}^2. \tag{4}$$

Since this limit is non-vanishing, the operator D^4R^4 is incompatible with continuous $E_{7(7)}$ symmetry.

2.2. 6-loop counterterm D^6R^4

The single-soft scalar limit of the SU(8)-singlet part of the closed string matrix element at order α'^6, obtained by SU(8)-averaging, is

$$\lim_{p_6 \to 0}\langle{+}{+}{-}{-}\varphi\bar{\varphi}\rangle_{(e^{-12\phi}D^6R^4)_{\text{avg}}} = -\frac{33}{35}[12]^4\langle 34\rangle^4\sum_{i<j}s_{ij}^3. \tag{5}$$

It is important to realize that at order α'^6, the 6-point NMHV closed string amplitudes receive contributions from diagrams involving one vertex from $e^{-12\phi}D^6R^4$ (together with vertices from the supergravity Lagrangian) *and* from pole diagrams with two 4-point vertices of $e^{-6\phi}R^4$ (which coincides with R^4 at 4-points). Since R^4 is not present in $\mathcal{N} = 8$ supergravity, its contributions must be removed to extract the matrix elements of the supergravity operator D^6R^4. The removal process must be supersymmetric.

We first compute the R^4–R^4 pole contributions to the 6-graviton NMHV matrix element $\langle{-}{-}{-}{+}{+}{+}\rangle$ as follows. This amplitude has dimension 14. Factorization at the pole determines the simple form

$$\langle 12\rangle^4[45]^4\langle 3|P_{126}|6]^4/P_{126}^2 + 8\ \text{permutations}, \tag{6}$$

up to a local polynomial. The 9 terms correspond to the 9 distinct 3-particle pole diagrams. The result (6) is then checked by computation of the Feynman diagrams from the R^4 vertex [21]. As the non-linear supersymmetrization of R^4 may contribute additional local terms, we also consider adding the most general gauge-invariant and Bose-symmetric polynomial of dimension 14 that can contribute to $\langle{-}{-}{-}{+}{+}{+}\rangle$, namely

$$\left(\langle 12\rangle\langle 23\rangle\langle 31\rangle[45][56][64]\right)^2 P_{123}^2. \tag{7}$$

To incorporate SUSY, we separately show that there is a basis for SU(8)-invariant 6-particle NMHV superamplitudes (an alternative to the basis in [22]) consisting of $\langle{-}{-}{-}{+}{+}{+}\rangle$ and 8 distinct permutations of the states. In this basis we write a superamplitude ansatz as the sum of the pole amplitude (6) plus a multiple of (7). We then impose full S_6 permutation symmetry on the ansatz. This fixes the coefficient of the polynomial (7) to vanish and determines the SUSY completion of the desired pole diagram uniquely!

Finally we project out the scalar-graviton matrix element from this superamplitude and take its single-soft scalar limit to find

$$\lim_{p_6 \to 0}\langle{+}{+}{-}{-}\varphi\bar{\varphi}\rangle_{(R^4)^2} = -\frac{1}{70}[12]^4\langle 34\rangle^4\sum_{i<j}s_{ij}^3. \tag{8}$$

It is this contribution that we need to subtract from (5) to obtain the single-soft scalar limits of the unique independent D^6R^4 operator in $\mathcal{N} = 8$ supergravity. Taking the relative normalization $[-2\alpha'^3\zeta(3)]^2/[\frac{2}{3}\alpha'^6\zeta(3)^2] = 6$ of operators in the string effective action (1) into account, we obtain

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{D^6R^4} = -\frac{30}{35}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^3. \quad (9)$$

This non-vanishing result shows that the operator D^6R^4 is also incompatible with continuous $E_{7(7)}$ symmetry.

R^4, D^4R^4 and D^6R^4 are the only local supersymmetric and SU(8)-symmetric operators for loop levels $L \leqslant 6$ [7–9]. Hence $\mathcal{N}=8$ supergravity has no potential counterterms that satisfy the continuous $E_{7(7)}$ symmetry for $L \leqslant 6$. We stress that string theory is used as a tool to extract SU(8)-invariant matrix elements that must agree with the matrix elements of the $\mathcal{N}=8$ supergravity operators R^4, D^4R^4 and D^6R^4 because each of these operators is unique. No remnant of string-specific dynamics remains in the final results.

2.3. Matching to automorphism analysis

The non-vanishing single-soft scalar limits found above have their origin in local 6-point interactions of the schematic form $\varphi^2D^{2k}R^4$ which appear in the non-linear completion of $D^{2k}R^4$. Let us encode this completion as $f(\varphi)D^{2k}R^4$, with

$$f(\varphi) = 1 - a\big[\varphi^{1234}\varphi^{5678} + 34 \text{ inequiv. perms}\big] + \cdots. \quad (10)$$

The "…" indicate higher order terms. The constant a depends on the operator; for example $a_{R^4} = \frac{6}{5}$ for R^4 [15]. We can determine a for D^4R^4 by taking a further single-soft limit $p_5 \to 0$ on (4) and comparing the resulting s,t,u-polynomial with the 4-point normalization of (3). The result is $a_{D^4R^4} = \frac{12}{7}$.

Ref. [17] used supersymmetry and duality considerations in d dimensions to constrain the moduli dependent functions $f(\varphi)$ of the BPS operators R^4, D^4R^4 and D^6R^4. Specifically, for R^4 and D^4R^4 in 4 dimensions, they found that $f(\varphi)$ should satisfy the Laplace equation

$$(\Delta+42)f_{R^4}(\varphi) = 0, \qquad (\Delta+60)f_{D^4R^4}(\varphi) = 0. \quad (11)$$

Here, Δ is the Laplacian on $E_{7(7)}/\mathrm{SU}(8)$; in terms of the scalars φ^{abcd} of $\mathcal{N}=8$ supergravity, its leading terms are

$$\Delta = \left[\frac{\partial}{\partial\varphi^{1234}}\frac{\partial}{\partial\varphi^{5678}} + 34 \text{ inequiv. perms}\right] + \cdots. \quad (12)$$

It is easy to see that the function (10) with the above values of a precisely satisfy the Laplace equations (11). This is a consistency check on our result for the single-soft scalar limits.

Let us now consider the function $f(\varphi)$ for D^6R^4. As explained, the quadratic order of R^4 interferes with D^6R^4 and it is therefore natural that the corresponding Laplace equation in [17] contains an inhomogeneous term that reflects the contribution from R^4–R^4. Adding a general linear combination $\lambda_{R^4}f_{R^4}R^4 + \lambda_{D^6R^4}f_{D^6R^4}D^6R^4$ to the effective action constrains the moduli-dependent functions to satisfy [17]

$$(\Delta+60)f_{D^6R^4}(\varphi) = -\frac{\lambda_{R^4}^2}{\lambda_{D^6R^4}}\big[f_{R^4}(\varphi)\big]^2. \quad (13)$$

From (5) and (9), we can reconstruct the coefficient a in (10) of the functions associated with the SU(8)-averaged string-theory operator $(e^{-12\phi}D^6R^4)_{\text{avg}}$ and with the supergravity operator D^6R^4. We find

$$a_{(e^{-12\phi}D^6R^4)_{\text{avg}}} = \frac{66}{35}, \qquad a_{D^6R^4} = \frac{60}{35}. \quad (14)$$

For $(e^{-12\phi}D^6R^4)_{\text{avg}}$, the couplings λ in (13) must take their string theory values $\lambda_{R^4} = -2\alpha'^3\zeta(3)$ and $\lambda_{D^6R^4} = \frac{2}{3}\alpha'^6\zeta(3)^2$. The $\mathcal{N}=8$ operator D^6R^4, on the other hand, must satisfy (13) with $\lambda_{R^4}=0$ because the operator R^4 does not appear in the action of $\mathcal{N}=8$ supergravity. Indeed, our results for f for both operators satisfy the Laplace equation with the expected choice of λ's.

3. Construction and counting of counterterms

We now discuss the techniques used to classify and construct local supersymmetric operators, especially those needed for $L \geqslant 7$. We are interested in SU(8)-invariant operators, which are candidate counterterms, and in operators transforming in the **70** of SU(8). The latter are candidate operators for local single-soft scalar limits (SSL's) of the matrix elements of singlet counterterm operators. First we use representation theory of the superalgebra SU(2, 2|8) to determine the spectrum and multiplicity of these operators. The spectrum is classified by the number n of external fields, the scale dimension, and the order k of the N^{k}MHV type. Then we construct matrix elements of several operators explicitly using algorithms which incorporate Gröbner basis techniques.

3.1. Spectrum of local operators

Counterterms of $\mathcal{N}=8$ supergravity are supersymmetric, SU(8)-invariant, Lorentz scalar local operators C integrated over spacetime. These local operators involve n-fold products of the fundamental fields and their derivatives. We restrict to diffeomorphism–covariant combinations of the fields, such as the Riemann tensor R. In enumerating all local operators of a given order n (up to covariance), the equations of motion set the Ricci tensor equal to a combination of fields of quadratic order (and higher), which is automatically included at order $> n$ in the enumeration. The remaining 10 components of the on-shell Riemann tensor group into fields with Lorentz spin $(2,0)$ and $(0,2)$. The collection of all on-shell supergravity fields span a representation of the $\mathcal{N}=8$ super-Poincaré algebra as well as an ultrashort representation of $\mathcal{N}=8$ superconformal symmetry (see [9] for a recent discussion). Using the SU(2, 2|8) Dynkin diagram

$$\underset{\mathrm{SU(2)_L}}{\odot}\!-\!\otimes\!-\!\underbrace{\odot\!-\!\odot\!-\!\odot\!-\!\odot\!-\!\odot\!-\!\odot\!-\!\odot}_{\mathrm{SU(8)}}\!-\!\otimes\!-\!\underset{\mathrm{SU(2)_R}}{\odot}, \quad (15)$$

the Dynkin labels of this lowest-weight representation read [0, 0, 0001000, 0, 0], where the SU(8) labels [0001000] describe a **70** and the $\mathrm{SU(2)_L}\times\mathrm{SU(2)_R}$ Lorentz spins indicate a scalar.

The graded symmetric tensor product of n copies of the above multiplet provide all local operators with n fields. We are interested in supersymmetric operators: there is typically one such operator C in each irrep of the tensor product. For long supermultiplets it is the unique top component, obtained by acting with SUSY generators $Q^{16}\tilde{Q}^{16}$ on the lowest-weight component C_0. (In superspace approaches, this is equivalent to the full superspace measure $\int d^{32}\theta$.) For short or BPS supermultiplets fewer supersymmetries are needed to get from C_0 to the top component(s). Hence it is sufficient to enumerate the lowest superconformal weights C_0. Its superconformal transformation properties determine the spin, SU(8) representation as well as loop and N^{k}MHV level.

More concretely, Dynkin labels translate to scalar local operators as follows (assume $q \geqslant p$)

$$[0,p,0000000,q,0] \to D^{3p-q-n}\varphi^{n+p-q}R^{q-p} \quad (\textbf{singlet}),$$
$$[0,p,0001000,q,0] \to D^{3p-q-n+1}\varphi^{n+p-q}R^{q-p} \quad (\textbf{70}). \quad (16)$$

Note that we display only prototypical terms, mixture with other fields is implied: e.g. $D^4\varphi^2 \simeq R\bar{R}$. (Here we distinguish between the chiral and anti-chiral components of the Riemann tensor.) To get from C_0 to the supersymmetric C in long multiplets, apply

Table 1
Supersymmetric SU(8)-singlet L-loop counterterms and the SU(8) **70** operators which describe their potential single-soft scalar limits. When the singlet operator is in the N^kMHV classification, the single-soft scalar limit operator belongs to the $\text{N}^{(k-\frac{1}{2})}$MHV sector. For $L < 7$, there are no independent singlet operators with $n > 4$, but the non-vanishing single-soft scalar limits arise from the non-linear completions of the 4-point operators $D^{2k}R^4$.

3-loop	4-pt	5-pt	6-pt
singlet	R^4 1×MHV	~~$\varphi^2 D^2 R^3$~~	~~$\varphi^2 R^4$~~ R^4 non-linear
			↙ soft
70		$1\times\varphi R^4$	

5-loop	4-pt	5-pt	6-pt
singlet	D^4R^4 1×MHV	~~$\varphi^2 D^6 R^3$~~	~~$\varphi^2 D^4 R^4$~~ D^4R^4 non-lin.
			↙ soft
70		$1\times\varphi D^4R^4$	

6-loop	4-pt	5-pt	6-pt
singlet	D^6R^4 1×MHV	~~$\varphi^2 D^8 R^3$~~	~~$\varphi^2 D^6 R^4$~~ D^6R^4 non-lin.
			↙ soft
70		$1\times\varphi D^6R^4$	

7-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt	13-pt	14-pt	15-pt	16-pt
singlet	D^8R^4 1×MHV	~~D^6R^5~~	D^4R^6 2×NMHV	~~D^2R^7~~	R^8 3×N²MHV	~~$\varphi^2 D^2R^7$~~	φ^2R^8 4×N³MHV	~~$\varphi^4 D^2R^7$~~	φ^4R^8 6×N⁴MHV	~~$\varphi^6 D^2R^7$~~	φ^6R^8 8×N⁵MHV	~~$\varphi^8 D^2R^7$~~	φ^8R^8 10×N⁶MHV
			↙ soft		↙ soft		↙ soft		↙ soft		↙ soft		↙ soft
70		φD^8R^4 2×		φD^4R^6 4×		φR^8 6×		φ^3R^8 9×		φ^5R^8 14×		φ^7R^8 19×	

8-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt	13-pt	14-pt
singlet	$D^{10}R^4$ 1×MHV	D^8R^5 1×MHV	D^6R^6 3×NMHV	D^4R^7 3×NMHV	D^2R^8 8×N²MHV	R^9 8×N²MHV	$\varphi^2D^2R^8$ 25×N³MHV	φ^2R^9 22×N³MHV	$\varphi^4D^2R^8$ 66×N⁴MHV	φ^4R^9 51×N⁴MHV	$\varphi^6D^2R^8$ 153×N⁵MHV
		↙	↙	↙	↙	↙	↙	↙	↙	↙	↙
70		$\varphi D^{10}R^4$ 3×	φD^8R^5 4×	φD^6R^6 17×	φD^4R^7 16×	φD^2R^8 81×	φR^9 63×	$\varphi^3D^2R^8$ 232×	φ^3R^9 211×	$\varphi^5D^2R^8$ 1033×	

9-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt
singlet	$D^{12}R^4$ 2×MHV	$D^{10}R^5$ 1×MHV	D^8R^6 12×NMHV 2×MHV	D^6R^7 14×NMHV	D^4R^8 117×N²MHV 7×NMHV	D^2R^9 123×N²MHV	R^{10} 780×N³MHV 36×N²MHV	$\varphi^2D^2R^9$ 783×N³MHV	φ^2R^{10} 4349×N⁴MHV 169×N³MHV
		↙	↙	↙	↙	↙	↙	↙	↙
70		$\varphi D^{12}R^4$ 5×N^0.5MHV	$\varphi D^{10}R^5$ 8×N^0.5MHV	φD^8R^6 122×N^1.5MHV 5×N^0.5MHV	φD^6R^7 194×N^1.5MHV	φD^4R^8 1814×N^2.5MHV 52×N^1.5MHV	φD^2R^9 2317×N^2.5MHV	φR^{10} 16485×N^3.5MHV 469×N^2.5MHV	

$Q^{16}\tilde{Q}^{16} \simeq D^{16}$. A lowest weight as in (16) then corresponds to a N^kMHV counterterm at L loops with

$$2k = n + p - q - 4, \qquad 2L = \begin{cases} 14 + p + q - n & (\textbf{singlet}), \\ 15 + p + q - n & (\textbf{70}). \end{cases} \tag{17}$$

Note that locality requires that the exponents in (16) are non-negative numbers. In particular, $3p - q - n \geqslant 0$ implies $2L \geqslant 6 + 2n - 4k$ using (17). This bound on the existence of local non-BPS operators was conjectured in [8] and confirmed very recently in [9].

As a simple illustration, consider (16) with $n = 4$ and $p = q$. We find $C_0 = D^{2p-4}\varphi^4$, and after application of D^{16} it becomes $D^{2p+4}R^2\bar{R}^2$; this is just the 4-point MHV local counterterm $D^{2p+4}R^4$. Locality of C_0 requires $p \geqslant 2$, so the first available non-BPS operator is D^8R^4.

In practice, we use a C++ program to enumerate all local operators with $2L \leqslant 30 - n$ amounting to $\sim 4.8 \cdot 10^{22}$ terms.[2] These are decomposed into irreps of SU(2, 2|8) by iteratively removing the lowest weights and their corresponding supermultiplets [23]. Special attention needs to be paid to BPS and short supermultiplets [24]. In total we obtained around $8.8 \cdot 10^5$ types of supermultiplets along with their multiplicities.[3] Finally we extract supermultiplets with scalar SU(8) singlets and **70**'s as top supersymmetry components. The results at $L \leqslant 9$ are presented in Table 1.

Our analysis shows that there are unique $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ BPS counterterms R^4, D^4R^4 and D^6R^4, in agreement with earlier results [7–9]. They correspond to the lowest weights

$$[0, 0, 0004000, 0, 0], \; [0, 0, 0200020, 0, 0], \; [0, 0, 2000002, 0, 0]. \tag{18}$$

In the previous section, we showed that their 6-point matrix elements have non-vanishing single-soft limits originating from the non-linear completion of the operators. The limits correspond to local **70** BPS operators φR^4, φD^4R^4 and φD^6R^4, which are descendants of the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ BPS superconformal primaries φ^5 with SU(8) Dynkin labels [0005000], [0201020], [2001002]. The relationship between BPS operators are illustrated in Table 1.

3.2. Explicit matrix elements and superamplitudes

The matrix elements of potential counterterms such as $D^{2k}R^n$ must be polynomials of degree $\delta = 2(k+n)$ in angle and square brackets $\langle ij\rangle$, $[kl]$ which satisfy several constraints. If a_i and s_i denote the number of angle $|i\rangle$ and square $|i]$ spinors for each particle $i = 1, 2, \ldots, n$, then the total number of spinors is fixed by the dimension of the operator to be $\sum_i (a_i + s_i) = 4(k+n)$. For each particle i, of helicity h_i, there is a helicity weight constraint $a_i - s_i = -2h_i$. We need polynomials which are independent under the constraints of momentum conservation and the Schouten identity,

$$\sum_j \langle ij\rangle[jk] = 0, \qquad \langle ij\rangle\langle kl\rangle + \langle jk\rangle\langle il\rangle + \langle ki\rangle\langle jl\rangle = 0, \tag{19}$$

and a similar Schouten identity for square brackets. These polynomials must satisfy Bose and Fermi symmetries when they contain identical particles. Finally, the polynomials must satisfy SUSY Ward identities. This was ensured in [8] by packaging n-point matrix elements into the manifestly SUSY- and SU(8)-invariant superamplitudes of [22].

In [8], Mathematica was used to construct the required independent polynomials. More efficient algorithms are needed for the higher dimension counterterms studied in this Letter. The constraints (19) define an ideal in a polynomial ring, and the Gröbner

[2] The computation took 3.5 hours on a desktop PC.
[3] The decomposition took 42 hours.

basis method [25,26] is well suited to choose a basis in the ideal and generate independent sets of polynomials in the quotient ring.

Given a (conventional) monomial ordering in the ring, a Gröbner basis is a subset of the ideal such that the leading term of any element of the ideal is divisible by a leading term of an element of the subset. Buchberger's algorithm generates the unique reduced Gröbner basis in which no monomial in a polynomial of this basis is divisible by a leading term of the other polynomials in the basis. For the ideal generated by (19), the reduced Gröbner basis is quite simple. By the theory of Gröbner bases, the monomials of degree δ (and specific helicity weights) that are not divisible by any leading term of the reduced Gröbner basis are a vector space basis of the quotient ring. This division test concerns only monomials and is computationally fast. (See Ch. 2, Sec. 7 and Prop. 5.3.1 of [25].)

We used the implementation of the Buchberger's algorithm in the algebraic software system Macaulay2 [27] to generate independent polynomials which satisfy dimension and helicity weight requirements and the constraints (19). These polynomials were then processed by computer programs similar to those used in [8] which imposed Bose symmetries. Among the resulting polynomials we select the ones that are independent under the conditions (19).

We have applied the Gröbner basis method to local counterterms with $n \leqslant 6$. The results are in perfect agreement with the multiplicities found from the enumeration of SU(2, 2|8) superconformal primary operators. In addition to an enumeration of independent operators, the explicit matrix elements allow us to test single-soft scalar limits. We discuss our $L = 7, 8, 9$ results below.

4. 7-loop counterterms: D^8R^4 and beyond

4.1. $E_{7(7)}$-compatibility of D^8R^4 at 6-points

The 6-point closed-string tree amplitude at order α'^7 only receives contributions from diagrams with one insertion of $e^{-14\phi} \times D^8R^4$. No lower-dimension operators in the closed string effective action (1) contribute. For the SU(8)-averaged single-soft scalar limits of $e^{-14\phi}D^8R^4$ we obtain

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{(e^{-14\phi}D^8R^4)_{\mathrm{avg}}} = -2[12]^4\langle 34\rangle^4\left[\frac{3}{4}\sum_{i<j}s_{ij}^4 + \frac{1}{16}\left(\sum_{i<j}s_{ij}^2\right)^2\right]. \tag{20}$$

However, we cannot conclude from this result that the operator D^8R^4 violates $E_{7(7)}$: contrary to the lower-loop cases we have studied, D^8R^4 is not unique. In fact, as we show later in this section, there is an infinite tower of supersymmetric operators of mass dimension 16. It is relevant for the 6-point matrix elements that there are two independent supersymmetrizations of D^4R^6. To any non-linear supersymmetrization of D^8R^4 we can add an arbitrary linear combination of these 6-point operators and obtain another valid supersymmetrization of D^8R^4. The SU(8)-averaged string amplitude picks out one particular such linear combination whose soft-limits (20) happen to be non-vanishing.

We construct the matrix elements of D^4R^6 explicitly with Gröbner basis techniques and find that they have non-vanishing SSL's; specifically we find that the SSL's of the 6-point matrix elements of the operators D^4R^6 span the 2-parameter space

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{D^4R^6} = [12]^4\langle 34\rangle^4\left[c_1\sum_{i<j}s_{ij}^4 + c_2\left(\sum_{i<j}s_{ij}^2\right)^2\right]. \tag{21}$$

It follows from (20) and (21) that we can choose a suitable linear combination of the two D^4R^6 operators to make the SSL of the 6-point matrix elements of the resulting non-linear supersymmetrization of D^8R^4 vanish: thus *there exists a supersymmetrization of D^8R^4 that satisfies*

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{D^8R^4} = 0. \tag{22}$$

Since this particular D^8R^4 satisfies the single-soft scalar theorems up to 6 points, it is important to also analyze the *double-soft* limit constraints of [13] that probe the structure of the coset $E_{7(7)}/\mathrm{SU}(8)$. We numerically verified that various non-trivial double-soft limits [16] of the 6-point matrix elements of D^8R^4 behave precisely as required for $E_{7(7)}$-invariance. Therefore the matrix elements of D^8R^4 are compatible with continuous $E_{7(7)}$ up to 6 points.

We would like to alert the reader to an alternative construction of the full n-point superamplitudes for the matrix elements of n-point 7-loop counterterms. Once the counting of the operator's multiplicity has been established by other means (as described above), it is easy to write down a corresponding set of superamplitudes. For the two 6-point superamplitudes of D^4R^6, for example, one can choose the basis

$$\begin{aligned}\mathcal{A}_{D^4R^6} &= \delta(\tilde{Q})\delta(Q)\big[(\varphi_1,\varphi_2)(\varphi_3,\varphi_4)(\varphi_5,\varphi_6) + \text{perms}\big],\\ \mathcal{A}_{D^4R^{6\prime}} &= \delta(\tilde{Q})\delta(Q)\big[(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5,\varphi_6) + \text{perms}\big].\end{aligned} \tag{23}$$

Here Q, $\tilde{Q}$ are the usual supercharges that act on the Grassmann η-variables of the superamplitude as differentiation and multiplication, respectively, and thus

$$\delta(Q) = \prod_{a=1}^{8}\sum_{i<j}[ij]\frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ja}}, \qquad \delta(\tilde{Q}) = \prod_{a=1}^{8}\sum_{i<j}\langle ij\rangle\eta_{ia}\eta_{ja}. \tag{24}$$

The sums in (23) run over all inequivalent permutations of the external state labels i of the φ_i, and the φ-products are defined as

$$\begin{aligned}(\varphi_i,\varphi_j) &\equiv \textstyle\prod_{t=1}^4\eta_{ia_t}\eta_{jb_t} \times \epsilon^{a_1a_2a_3a_4b_1b_2b_3b_4},\\ (\varphi_i,\varphi_j,\varphi_k,\varphi_l,\varphi_m,\varphi_n) &\equiv \prod_{t=1}^{4}\eta_{ia_t}\eta_{jb_t}\eta_{kc_t}\eta_{ld_t}\eta_{me_t}\eta_{nf_t}\\ &\quad\times \epsilon^{a_1a_2b_1b_2b_3b_4c_1c_2}\epsilon^{c_3c_4d_1d_2d_3d_4e_1e_2}\epsilon^{e_3e_4f_1f_2f_3f_4a_3a_4}.\end{aligned} \tag{25}$$

Of course, the choice of contractions is not unique, and only through the previously established multiplicity count do we know that it is sufficient to consider the two contractions given in (23). A similar construction can be carried out for the three 8-point $\mathrm{N}^2\mathrm{MHV}$ superamplitudes of R^8. Again, one can immediately propose three superamplitudes that span the space of R^8 counterterms, for example by considering three order-8 contractions involving the φ-products (25) and their 8-scalar generalizations.

4.2. The infinite tower of 7-loop counterterms

We now examine the multiplicity of potential 7-loop n-point counterterms [8]

$$D^8R^4,\ D^4R^6,\ R^8,\ \varphi^2R^8,\ \varphi^4R^8,\ldots. \tag{26}$$

7-loop operators correspond to long multiplets, and are thus supersymmetric descendants of local operators composed from only scalars with no derivatives. This follows from setting $L = 7$ in (16) and (17). SU(8)-singlet operators C only exist for even n at $L = 7$, and we write them schematically as

$$C \simeq Q^{16}\tilde{Q}^{16}\varphi^{2q}. \tag{27}$$

The lowest weight φ^{2q} must be in an SU(8)-singlet combination, and every such singlet gives rise to a long supermultiplet. Hence there is one 7-loop n-point counterterm for each singlet in the decomposition of the symmetric tensor product of $n = 2q$ **70**'s. With increasing q there is a (swiftly) increasing number of singlets, as illustrated by the explicit multiplicities up to $n = 16$ in Table 1. Consequently, there is an infinite 'tower' of independent 7-loop operators that are potential counterterms. Operators corresponding to their SSL are also listed in Table 1. Their construction is similar, and their multiplicities is the number of **70**'s in a product of $(2q-1)$ **70**'s.

4.3. $E_{7(7)}$ violation of higher-point 7-loop operators

Consider the leading n-point matrix elements of a local counterterm C. If non-vanishing, the SSL produces a local $(n-1)$-matrix element, which can be generated by an $(n-1)$-point local operator dC in the **70**. Locality of C ensures that the SSL operation $C \to dC$ commutes with the SUSY generators Q and $\tilde{Q}$. For a long-multiplet ($L \geqslant 7$) counterterm $C = Q^{16}\tilde{Q}^{16}C_0$ we can therefore write

$$dC = Q^{16}\tilde{Q}^{16}dC_0. \tag{28}$$

$E_{7(7)}$ requires that the SSL vanishes. Now there are two ways to obtain $dC = 0$: either the single-soft limit of C_0 vanishes ($dC_0 = 0$), or dC_0 is annihilated by $Q^{16}\tilde{Q}^{16}$. At the seven-loop level, $C_0 = \varphi^{2q}$ consists of only scalars with no derivatives, and consequently $dC_0 \neq 0$. C_0 is also not annihilated by $Q^{16}\tilde{Q}^{16}$ because $dC_0 = \varphi^{2q-1}$ in a **70** satisfies a shortening condition only for $n = 2q \leqslant 4$ [24]. Therefore, all 7-loop linearized counterterms with $n > 4$ have non-vanishing single-soft scalar limits and thus violate $E_{7(7)}$. (This was also observed in [28]; see [29] for discussion of non-perturbative aspects.) These operators may, however, play an important role as dependent terms in the non-linear completion of the D^8R^4 operator, as we demonstrated above at the 6-point level.

5. SSL structure: 7-, 8- and 9-loops

We now show that all our findings on $E_{7(7)}$-(in)compatibility of operators have a natural explanation in terms of the multiplicities of SSL operators in the **70** that is displayed in Table 1.[4] Let us first revisit the case of D^4R^4 and D^6R^4. At the 5- and 6-loop level, there are no potential 3-point or 4-point SSL operators available. Therefore the matrix elements of D^4R^4 and D^6R^4 must have vanishing soft limits at 4- and 5-points, and this is indeed the case. There exists, however, one potential 5-point SSL operator at $L = 5$ and $L = 6$. Generically, one expects the soft-limits of the 6-point matrix elements of D^4R^4 and D^6R^4 to be proportional to this operator with some non-vanishing coefficient. This is precisely what happens.

At 7 loops with $n > 4$ points, the number n_S of SSL operators $D^{16}\varphi^{n-1}$ is always at least as large as the number n_C of potential counterterms $D^{16}\varphi^n$. Generically, one therefore expects the soft limits of the potential counterterms to span an n_C-dimensional subspace in the n_S-dimensional space of SSL operators. It would follow that all potential counterterms with $n > 4$ violate $E_{7(7)}$. Indeed, this is what we explicitly proved above for the 7-loop case.

[4] Throughout this section we are only concerned with the lowest-point non-vanishing SSL's of an operator. If an operator has non-vanishing SSL's at n-point, its higher-point matrix elements will generically have *non-local* SSL's, which are not classified by our analysis.

By the same logic, it is not at all surprising that there is a non-linear supersymmetrization of D^8R^4 that preserves $E_{7(7)}$ at the 6-point level. The number of 5-point SSL operators precisely matches the number of D^4R^6 operators. Therefore, the 6-point soft limit of D^8R^4 can be made to vanish after adding an appropriate combination of the two D^4R^6 operators, just as we found above. For the 8-point soft limits of D^8R^4, however, there are 4 SSL operators available; more than the 3 potential 8-point counterterms R^8. If the 8-point soft limits of D^8R^4 take a generic value in the 4-dimensional space of SSL operators, no linear combination of R^8 operators can be chosen to give an $E_{7(7)}$-preserving supersymmetrization of D^8R^4; a remarkable cancellation is thus required for D^8R^4 to be compatible with $E_{7(7)}$.

As Table 1 illustrates, $E_{7(7)}$ becomes more and more constraining as we increase the number of points and loops. For example, the 14-point soft limits of $D^{10}R^4$ have to lie in a specific 153-dimensional subspace of the 1033-dimensional space of SSL operators in order for $D^{10}R^4$ to satisfy $E_{7(7)}$ after an appropriate addition of independent 14-point operators. It follows that $E_{7(7)}$ is a very constraining symmetry even for $L = 7$ and beyond. Although there is an infinite tower of independent counterterms at each of loop $L \geqslant 7$, we cannot expect any of these operators to preserve $E_{7(7)}$ 'accidentally'. There may, however, be a very good reason for the cancellations of terms that is needed for $E_{7(7)}$-invariant operators to exist for $L \geqslant 7$; namely, when there is a construction of a manifestly $E_{7(7)}$-invariant supersymmetric operator [30,31]. At the 7-loop level, for example, we can only expect the 8-point single-soft limits of D^8R^4 to vanish after an appropriate addition of R^8, if the manifestly $E_{7(7)}$-invariant superspace integral that was proposed as a candidate counterterm in [30] is indeed non-vanishing.

One new feature that emerges at $L = 8, 9$ is the existence of $n > 4$ operators that have vanishing soft-limits at the linearized level. This holds for the MHV operators D^8R^5, $D^{10}R^5$ and $2 \times D^8R^6$ as well as for at least 7 of the $12 \times D^8R^6$ NMHV operators. The latter follows from the multiplicity 5 of 5-point SSL operators at 9 loops. $E_{7(7)}$-invariance beyond the linearized level, however, is a highly non-trivial constraint on all of these operators.

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