THE ASTROPHYSICAL JOURNAL, 161: 1173–1176, September 1970


# TAYLOR INSTABILITIES IN RELATIVISTIC STARS*

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*Received 1970 April 2*

## ABSTRACT

Using the relativistic Schwarzschild criterion for the onset of convection, we show that the density of total mass-energy inside a spherically symmetric relativistic star must decrease monotonically outward from the center to the surface. Otherwise a Taylor-type instability will develop.

In the construction of spherically symmetric relativistic stellar models, one usually assumes that the density of total mass-energy, $\rho$, decreases monotonically outward from the center of a star to its surface. This assumption is justified by analogy with the Newtonian case, in which a star is known to be unstable against convective-type motions ("Taylor instability") if $\rho$ increases outward in some region (Lebovitz 1966). The demand for decreasing $\rho$ has been fruitful in general relativity: Bondi (1964) used it to place an upper limit of 1.13 on the gravitational redshift of light coming from the surface of a supermassive star. This precluded the possibility that QSOs (whose redshifts go beyond 2) could be "local" supermassive stars.

Recently, however, Geroch (1969) has asked whether one should require only that $(\rho + p)$, not $\rho$, decrease monotonically outward, where $p$ is the ordinary scalar pressure.[1] This is a weaker condition, and might increase Bondi's redshift limit beyond 2. Geroch reasoned as follows: The instability is characterized by convective-type motions of the fluid; any motion is regulated in part by the inertia of the fluid; and the inertial mass per unit volume is $(\rho + p)$, not $\rho$ (see, e.g., Thorne 1967). Consequently, *perhaps* $(\rho + p)$, not $\rho$, is the key to the Taylor instability.

The purpose of this Note is to point out that Bondi was right: it is $\rho$ and *not* $(\rho + p)$ which must decrease monotonically outward, if the star's fluid is to avoid the Taylor instability.

The name "Taylor instability" comes from the situation in Newtonian hydrodynamics in which a fluid of a certain density lies above a fluid of a lower density in an external gravitational field (see Chandrasekhar 1961). This configuration is said to be Taylor (or Rayleigh-Taylor) unstable against perturbations that would cause mixing of the two layers.

In stars the analogous situation is a density that increases outward in some region; this situation is more difficult to analyze than the previous one, however, because disturbances in the fluid change the gravitational field. Only an analysis of the nonradial modes of pulsation of the star will give completely definitive stability criteria. This has been done for Newtonian stellar models (Lebovitz 1966), and the results show an instability if $\rho$ increases outward in any region.

The analogous relativistic analysis has not yet been accomplished because the equations for nonradial pulsation are so intractable (see Thorne and Campolattaro 1967). The case of a fluid in a static *external* relativistic gravitational field (the fluid's motions do not affect the field) has recently been studied by Kovetz (1967), with the result that

* Supported in part by the National Science Foundation [GP-15911, GP-9114] and the Office of Naval Research [Nonr-220(47)].

† NDEA predoctoral research fellow.

[1] Throughout we use geometrized units: $c = G = 1$.

stability requires $\rho$, not $(\rho + p)$, to decrease in the direction of decreasing gravitational field (increasing $|g_{tt}|$). This result encourages us—in the absence of a full analysis of the nonradial modes of pulsation of a relativistic star—to look for other ways to demonstrate the plausibility of $\rho$ rather than $(\rho + p)$ decreasing outward in a relativistic star. The way we have found successful is to use the relativistic Schwarzschild criterion for the onset of convective-type motions. Our proof is fully rigorous if the Schwarzschild criterion is assumed to be fully valid (see below for discussion).

Define

$$\Gamma \equiv \frac{\rho + p}{p}\left(\frac{dp}{d\rho}\right)_{\substack{\text{comoving with the}\\ \text{convecting fluid}}} ; \tag{1}$$

the index $\Gamma$ describes how the fluid's pressure and density change as seen by a comoving observer during its motion. Then the relativistic Schwarzschild criterion is as follows (see Thorne 1966): *If*

$$S(r) \equiv \frac{dp}{dr} - \frac{\Gamma p}{\rho + p}\frac{d\rho}{dr} < 0 \tag{2}$$

*for some* $a < r < b$, *then the fluid is unstable against convection in that layer.*

Although it is easy to proceed directly from the Schwarzschild criterion to the result that $d\rho/dr > 0$ causes instability, it is more instructive to derive an equivalent, but slightly different, form of the criterion, a form that illustrates the different roles played by $\rho$ and $(\rho + p)$. Our derivation, which is similar to one given by Thorne (1966) and logically equivalent to that of Kovetz (1967), is valid only for motions with scale sufficiently small that they do not disturb the thermodynamic or gravitational state of the surrounding fluid. The applicability of the Schwarzschild criterion to motions of a larger scale is discussed below.

One imagines a fluid element at radius $A$ in equilibrium with its surroundings suddenly displaced to radius $B = A + dr$. During its displacement it adjusts its volume so as always to remain in pressure equilibrium with its surroundings. At $B$ it is acted upon by two forces: buoyancy and gravity. If the net result of the two forces is to push it back to $A$, the star is stable; if the forces push the element beyond $B$, the star is unstable. The gravitational force per unit volume acting on the fluid element is [with $|g_{tt}| = \exp(2\Phi)$]

$$F_{\text{grav}} = -(\rho + p)_{\text{fe}}(1 - 2m/r)^{1/2}d\Phi/dr , \tag{3}$$

where $(\rho + p)_{\text{fe}}$ is the inertial mass per unit volume of the fluid element *when it reaches* $B$, and the other quantities are evaluated at $B$. The factor $(1 - 2m/r)^{1/2}$ corrects the radial distance $dr$ to proper distance as measured in a comoving inertial frame. The buoyancy force per unit volume is

$$F_{\text{buoy}} = -(1 - 2m/r)^{1/2}dp/dr , \tag{4}$$

also evaluated at $B$. Now, if it is assumed that the amount of fluid moved is so small that outside the fluid element the initial equilibrium state of the star is undisturbed, the equation relating $dp/dr$ and $d\Phi/dr$ is

$$dp/dr = -(\rho + p)d\Phi/dr . \tag{5}$$

Hence the sum of the forces at $B$ is

$$F_{\text{grav}} + F_{\text{buoy}} = \left(1 - \frac{2m}{r}\right)^{1/2}\frac{d\Phi}{dr}[(\rho + p)_B - (\rho + p)_{\text{fe}}] , \tag{6}$$

where $(\rho + p)_B$ is the inertial mass per unit volume of the surrounding fluid at $B$. If the total force is positive, the fluid is unstable against such displacements. Since $d\Phi/dr > 0$, if

$$(\rho + p)_{\substack{\text{surrounding}\\ \text{medium at } B}} > (\rho + p)_{\substack{\text{fluid element}\\ \text{displaced to } B}} , \tag{7}$$

then the fluid is unstable. This is the form of the Schwarzschild criterion that involves the inertial mass. It requires the inertial mass per unit volume of the fluid element, *when it gets to B*, to be greater than that of the surroundings. This is different from requiring $(\rho + p)$ to be greater at $A$ than at $B$. In fact, as we shall show, it requires $\rho$ at $A$ to be greater than $\rho$ at $B$!

Clearly, the pressures on both sides of equation (7) are equal. Then we write

$$\rho_{\text{fe}} = \rho_A + dp\left(\frac{d\rho}{dp}\right)_{\substack{\text{comoving with the}\\ \text{convecting fluid}}} \tag{8}$$

and

$$\rho_B = \rho_A + dp\left(\frac{d\rho}{dp}\right)_{\substack{\text{star in}\\ \text{equilibrium}}} \tag{9}$$

Noting that $dp < 0$ for $dr > 0$, we see that, if

$$\left(\frac{d\rho}{dp}\right)_{\substack{\text{star in}\\ \text{equilibrium}}} < \left(\frac{d\rho}{dp}\right)_{\substack{\text{comoving with the}\\ \text{convective fluid}}}, \tag{10}$$

then the fluid is unstable. This is equivalent to the relativistic Schwarzschild criterion (eq. [2], with eq. [1] defining $\Gamma$).

We argue below that, for all fluids of which a star might be composed, the right-hand side of the inequality (10) (or equivalently, the index $\Gamma$) must be positive or zero. Therefore, if $(d\rho/dp)_{\text{star}}$ is negative, the inequality (10) is satisfied and the star is unstable. Of course, since $dp/dr < 0$ in all stars, $(d\rho/dp)_{\text{star}} < 0$ means $d\rho/dr > 0$. *So if $d\rho/dr > 0$, the star is unstable against convective-type motions.*

This result is based on the assumption that $\Gamma \geq 0$. One can easily convince oneself that $\Gamma \geq 0$ for any physically reasonable fluid motions. A few examples follow.

1. *Adiabatic motions of a compressible fluid.*—These are usually expected when the motions occur on the hydrodynamic time scale. If $\Gamma < 0$, an increase in the density of a fluid element would decrease its internal pressure, and hence decrease its resistance to being crushed by the rest of the star. This is itself unstable. Such a fluid element would be compressed rapidly and adiabatically by the surrounding fluid until $\Gamma$ became positive and its internal pressure began to build up.

2. *Nearly incompressible fluids.*—These have $\Gamma \sim +\infty$.

3. *Isothermal motions.*—If the fluid were such a good conductor of heat that the motion proceeded isothermally rather than adiabatically, the fluid would change its state along an isotherm in a $(p, V)$-diagram. A negative $\Gamma$ corresponds to a positive slope for the isotherm. This is never realized physically (except for such unstable systems as superheated or supercooled fluids) because the fluid prefers to change its phase along a curve $p =$ constant, which corresponds to $\Gamma = 0$ (see Fermi 1936).

The conclusion is that the density of total mass-energy must decrease monotonically outward from the center of a relativistic star.

In order to put our proof on a completely firm foundation, we would have to prove the relativistic Schwarzschild criterion rigorously. This has never been done. However, we should emphasize that the criterion is certainly not in any serious doubt. Kovetz (1967) has proved it for the case of a static external gravitational field; and Thorne (1965, 1969) has given a number of strong arguments for it in the general case. Thorne's arguments include: (i) two *completely rigorous* proofs from different viewpoints that $S(r) = 0$ corresponds to neutral stability for an infinity of different small-scale and large-scale nonradial modes of motion—i.e., that $S(r) = 0$ is the dividing line between stability and instability for an infinity of modes; and (ii) a proof equivalent to the one given in this Note that, for motions of scale so small as not to disturb the state of the surrounding fluid and field, $S(r) < 0$ means instability, while $S(r) > 0$ means stability. Together these arguments and those of this Note imply that infinitely many modes of nonradial pulsation—with low as well as high values of $l$, modes that substantially alter the gravitational field as well as modes that do not—will be unstable if $\rho$ increases outward in any region of a spherically symmetric relativistic star.

I am grateful to Dr. Robert Geroch for his correspondence and to Professor Kip S. Thorne for many helpful comments. This Note is an expanded version of remarks delivered to the Los Angeles meeting of the American Physical Society, December 1969.

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