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Shock Waves in Plane Symmetric Spacetimes

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We consider Einstein's equations coupled to the Euler equations in plane symmetry, with compact spatial slices and constant mean curvature time. We show that for a wide variety of equations of state and a large class of initial data, classical solutions break down in finite time. The key mathematical result is a new theorem on the breakdown of solutions of systems of balance laws. We also show that an extension of the solution is possible if the spatial derivatives of the energy density and the velocity are bounded, indicating that the breakdown is really due to the formation of shock waves.

Keywords General relativity; Shock waves; Systems of balance laws.

Mathematics Subject Classification 83C55; 35Q75.

1. Introduction

A question of central interest in general relativity is that of the long-time behavior of self-gravitating matter. Mathematically the starting point is to get suitable existence and uniqueness theorems for the Einstein equations coupled to the equations of motion of the matter. Since the resulting system of partial differential equations is difficult to handle it makes sense to begin with solutions of high symmetry. One of the most popular matter models in applications is the perfect fluid described by the relativistic Euler equations. It is to be expected that, as in classical hydrodynamics, a major difficulty in studying the long-time behavior of solutions of these equations is the formation of shocks from smooth initial data. If the solution is to be continued beyond these it is necessary to leave the realm of classical solutions of the equations. It should be noted that until very recently most of the theorems on solutions of the Euler equations involving shocks were a one-dimensional context (plane symmetric

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solutions). This may change following the recent work of Christodoulou [5] on the formation of shocks in special relativity without symmetry assumptions.

One way to avoid the difficulties involved with fluids is to consider instead collisionless matter described by the Vlasov equation. In that case quite a lot of mathematical results are available for the coupled Einstein-matter equations [2, 15]. If, on the other hand, we face the problems associated with the fluid description, as we do in this paper, several natural questions arise. First, are there global existence theorems for weak solutions of the special relativistic Euler equations? Some positive answers have been given in [4, 7, 17]. Second, do these results extend to the case of a self-gravitating fluid? Theorems have been proved under certain assumptions by [3] and [8]. Third, can it be proved that classical solutions of the Einstein–Euler equations break down in finite time? It is the third question which is addressed in the following.

The symmetry assumed in this paper is plane symmetry where the solutions are invariant under the action of the full isometry group of the Euclidean plane. In particular this reduces the full problem to a problem in one time and one space dimension. It is assumed that the position space is compact. This circumvents the need for boundary conditions and is the analogue of periodic boundary conditions in the non-relativistic case. The notion of 'finite time' breakdown is subtle in general relativity. In that theory there is a free choice of time variable and if it has been proved that a solution breaks down after a finite amount of a particular time coordinate it is necessary to think carefully about what this means geometrically. This is what is relevant for physics. In this paper we use a constant mean curvature (CMC) time coordinate. This means that the value of t at some point of spacetime is equal to the mean curvature of the unique compact CMC hypersurface passing through that point.

The main result of this paper (Theorem 5.2) shows that plane symmetric classical solutions of the Einstein–Euler equations exhibit finite-time breakdown for a wide variety of equations of state. The key new analytical result used to prove this is a theorem on the breakdown of solutions of systems of balance laws which is of interest in its own right. This is combined with general estimates for the Einstein equations to give the final result.

In the last section we put the result into context, considering several issues. The Euler equations are compared with other matter models. The assumptions on the equation of state are compared with cases known in the literature, and the relations to existing results on weak solutions are discussed.

2. The Einstein Equations

Let (M, g) be a spacetime, where the manifold is assumed to be $M = I \times \mathbf{T}^3$, I is a real interval and $\mathbf{T}^3 = \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ is the three-torus. We require that the metric g and the matter fields are invariant under the action of the Euclidean group E_2 on the universal cover, and that the spacetime has an E_2-invariant Cauchy surface of constant mean curvature (CMC). As was shown in [12], there is a local in time $3+1$ decomposition of (M, g) where each spatial slice has constant mean curvature. We can introduce spatial coordinates x, y and z on each slice, with ranges $[0, 2\pi]$ and period 2π, and a CMC time coordinate $t = \operatorname{tr} k$ where k is the second fundamental form of the slices.

The metric can be expressed as

$$ds^2 = -\alpha^2 dt^2 + A^2[(dx + \beta dt)^2 + a^2(dy^2 + dz^2)], \tag{1}$$

where α, A and β depend on t and x and a depends on t only. Here α is the lapse and $\beta\delta_1^\mu$ is the shift. The coordinates can be chosen such that $\int_0^{2\pi} \beta\, dx = 0$.

It is convenient to introduce the orthonormal frame

$$\begin{aligned} e_0 &= \alpha^{-1}\partial_t - \alpha^{-1}\beta\partial_x, \\ e_1 &= A^{-1}\partial_x, \\ e_2 &= (aA)^{-1}\partial_y, \\ e_3 &= (aA)^{-1}\partial_z. \end{aligned} \tag{2}$$

We will use indices $a, b, \ldots$ to denote spatial coordinate components and $i, j, \ldots$ to denote spatial frame components.

The frame components of the second fundamental form may be written

$$k_{ij} = -\frac{1}{2}(K - t)\delta_{ij} + \frac{1}{2}(3K - t)\delta_{1i}\delta_{1j}, \tag{3}$$

where K is a function of t and x. We also introduce the notation $\rho = T^{00}$, $j = T^{10}$ and $S^{ik} = T^{ik}$ for frame components of the energy momentum tensor $T^{\mu\nu}$. The Einstein equations can then be written

$$K' + (3K - t)A^{-1}A' = 8\pi A j, \tag{4a}$$

$$(\sqrt{A})'' = -\tfrac{1}{8}A^{5/2}\Big[K^2 + \tfrac{1}{2}(K - t)^2 - t^2 + 16\pi\rho\Big], \tag{4b}$$

$$\alpha'' + A^{-1}A'\alpha' = \alpha A^2\Big[K^2 + \tfrac{1}{2}(K - t)^2 + 4\pi(\rho + \operatorname{tr} S)\Big] - A^2, \tag{4c}$$

$$\partial_t a = a\Big[-\beta' + \tfrac{1}{2}\alpha(3K - t)\Big], \tag{4d}$$

$$\partial_t A = -\alpha K A + (A\beta)', \tag{4e}$$

$$\partial_t K = \beta K' - A^{-2}\alpha'' + A^{-3}A'\alpha' + \alpha[-2A^{-3}A'' + 2A^{-4}(A')^2 + Kt - 4\pi(2S_{11} - \operatorname{tr} S + \rho)]. \tag{4f}$$

Here a prime denotes differentiation by x. Equation (4a) and (4b) are the momentum and Hamiltonian constraints, respectively. The lapse equation (4c) comes from the constant mean curvature condition, while (4d) and (4e) are consequences of the choice of spatial coordinate conditions. The last equation (4f) is the only independent Einstein evolution equation in this case.

Using (4) it is possible to show that many of the fundamental quantities are bounded. This was done in [12] for plane, hyperbolic and spherical symmetry, when the matter satisfies the dominant energy and the non-negative pressures conditions. The non-negative pressures condition was subsequently relaxed to the strong energy condition in [14] (note that plane symmetry is a special case of local $U(1) \times U(1)$ symmetry). Also, a bound for α^{-1} in terms of the previously obtained geometric bounds was found in [13].

Theorem 2.1 (See [12–14]). *Let a solution of the Einstein equations with plane symmetry be given and suppose that when coordinates are chosen which cast the metric into the form* (1) *with constant mean curvature time slices the time coordinate takes all values in the finite interval* $[t_0, t_1)$ *with* $t_1 < 0$. *Suppose further that the dominant and strong energy conditions hold. Then the following quantities are bounded on the interval* $[t_0, t_1)$:

$$\alpha, \alpha^{-1}, \alpha', A, A^{-1}, A', \partial_t A, a, a^{-1}, \partial_t a, K, \beta, \beta'. \tag{5}$$

The bounds involve only

$$t_0, t_1, \|a\|_0, \|a^{-1}\|_0, \|A\|_0, \|A^{-1}\|_0, \tag{6}$$

where $\|\cdot\|_0$ *is the supremum norm on the initial surface* $t = t_0$.

3. The Euler Equations

In this section we will rewrite the matter equations for a perfect fluid in plane symmetry as a system of balance laws.

The basic matter variables are the pressure p, the energy density μ and the unit 4-velocity U^μ of the fluid. Because of the plane symmetry the frame components of U can be written as

$$U^0 = (1-u^2)^{-1/2}, \quad U^1 = u(1-u^2)^{-1/2} \quad \text{and} \quad U^2 = U^3 = 0, \tag{7}$$

where $u \in (-1, 1)$. The energy momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\mu + p)U^\mu U^\nu + p g^{\mu\nu}, \tag{8}$$

which implies

$$\rho = T^{00} = \frac{(\mu + p)}{1-u^2} - p, \tag{9a}$$

$$j = T^{01} = \frac{(\mu + p)u}{1-u^2}, \tag{9b}$$

$$S^{11} = T^{11} = \frac{(\mu + p)u^2}{1-u^2} + p, \tag{9c}$$

$$S^{22} = S^{33} = T^{22} = T^{33} = p. \tag{9d}$$

It will be convenient to introduce two new variables

$$w = \left(\frac{\mathrm{d}p}{\mathrm{d}\mu}\right)^{1/2}, \tag{10a}$$

$$\varphi = \int_{\mu_-}^{\mu} \left(\frac{\mathrm{d}p}{\mathrm{d}m}\right)^{1/2} \frac{\mathrm{d}m}{m + p}, \tag{10b}$$

where m is a dummy integration variable and the constant μ_- is arbitrary.

The matter equations are given by the vanishing of the divergence $\nabla_\nu T^{\nu\sigma}$ of the energy momentum tensor. The spatial frame components $\nabla_\nu T^{\nu 2}$ and $\nabla_\nu T^{\nu 3}$ vanish identically. Expressing the remaining two components in terms of φ, w and u gives

$$w[2ue_0(u) + (1+u^2)e_1(u)] + (1-u^2)[(1+u^2w^2)e_0(\varphi) + u(1+w^2)e_1(\varphi)]$$
$$-w(1-u^2)[Ku^2 + t - 2uA^{-1}e_1(A) - 2u\alpha^{-1}e_1(\alpha)] = 0, \tag{11a}$$
$$w[(1+u^2)e_0(u) + 2ue_1(u)] + (1-u^2)[u(1+w^2)e_0(\varphi) + (u^2+w^2)e_1(\varphi)]$$
$$-w(1-u^2)[(K+t)u - 2u^2A^{-1}e_1(A) - (1+u^2)\alpha^{-1}e_1(\alpha)] = 0. \tag{11b}$$

Adding and subtracting (11a) and (11b) we get

$$E_0 = w[e_0(u) + e_1(u)] + (1-u)[(1+uw^2)e_0(\varphi) + (u+w^2)e_1(\varphi)]$$
$$- w(1-u)[Ku + t - 2uA^{-1}e_1(A) - (1+u)\alpha^{-1}e_1(\alpha)] = 0, \tag{12a}$$
$$E_1 = w[e_0(u) - e_1(u)] - (1+u)[(1-uw^2)e_0(\varphi) + (u-w^2)e_1(\varphi)]$$
$$-w(1+u)[Ku - t + 2uA^{-1}e_1(A) - (1-u)\alpha^{-1}e_1(\alpha)] = 0. \tag{12b}$$

The linear combinations $(1+u)(1\pm w)E_0 + (1-u)(1\mp w)E_1$ give

$$D_+u + (1-u^2)D_+\varphi = (1-u^2)\left[\frac{Ku + tw - 2uwA^{-1}e_1(A)}{1+uw} - \alpha^{-1}e_1(\alpha)\right], \tag{13a}$$

$$D_-u - (1-u^2)D_-\varphi = (1-u^2)\left[\frac{Ku - tw + 2uwA^{-1}e_1(A)}{1-uw} - \alpha^{-1}e_1(\alpha)\right], \tag{13b}$$

with differentiation operators

$$D_+ = e_0 + \frac{u+w}{1+uw}e_1 \quad \text{and} \quad D_- = e_0 + \frac{u-w}{1-uw}e_1. \tag{14}$$

Next, we introduce the variables

$$r = \varphi + \frac{1}{2}\ln\frac{1+u}{1-u} \quad \text{and} \quad s = \varphi - \frac{1}{2}\ln\frac{1+u}{1-u}. \tag{15}$$

These are analogues of the Riemann invariants found by Taub [18]. In contrast with the special relativistic case, r and s are not invariant. In particular, it follows from (13) that instead of a system of conservation laws, r and s satisfy a system of balance laws:

$$D_+r = \frac{tw + Ku - 2uwA^{-1}e_1(A)}{1+uw} - \alpha^{-1}e_1(\alpha), \tag{16a}$$

$$D_-s = \frac{tw - Ku - 2uwA^{-1}e_1(A)}{1-uw} + \alpha^{-1}e_1(\alpha). \tag{16b}$$

Here u and w should be regarded as functions of r and s given by (15) and (10).

We will now make some assumptions on the equation of state to ensure that the maps between the different matter variables introduced in this section are well behaved.

Lemma 3.1. *Suppose that the equation of state is given as a smooth function p of μ such that*

1. *the strong energy condition* ($\mu + p \geq 0$ *and* $\mu + 3p \geq 0$) *holds,*
2. *the dominant energy condition* ($-\mu \leq p \leq \mu$) *holds,*
3. $0 < \frac{\mathrm{d}p}{\mathrm{d}\mu} < 1$ *and*
4. $\frac{\mathrm{d}^2 p}{\mathrm{d}\mu^2} \geq 0$.

Then the maps

$$(r, s) \mapsto (u, \varphi), \quad \mathbf{R}^2 \to (-1, 1) \times \mathbf{R}, \tag{17}$$

$$\mu \mapsto \varphi, \quad (0, \infty) \to (\varphi_-, \infty), \quad -\infty \leq \varphi_- < 0 \tag{18}$$

are smooth bijections,

$$(\mu, u) \mapsto (\rho, j, S) \tag{19}$$

is smooth and 1-1 *on* $(0, \infty) \times (-1, 1)$,

$$\mu \mapsto w, \quad (0, \infty) \to (0, 1) \tag{20}$$

is smooth and increasing, and

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} \geq 0. \tag{21}$$

Proof. From (15),

$$\varphi = \frac{r + s}{2} \quad \text{and} \quad u = \tanh \frac{r - s}{2}, \tag{22}$$

which establishes (17) without any restrictions on the equation of state. Applying conditions 1 and 3 to (10b) shows that φ is a smooth strictly increasing function of μ and thus 1-1. Moreover, from conditions 4 and 2 it follows that

$$\lim_{\mu \to \infty} \varphi = \infty. \tag{23}$$

Thus φ is a smooth bijection from $(0, \infty)$ to (φ_-, ∞), where $\varphi_- = \lim_{\mu \to 0^+} \varphi \geq -\infty$.

That ρ, j and S are smooth functions of μ and u is evident from (9). The map is 1-1 since p is 1-1 because of condition 3. Finally, (20) follows directly from conditions 3 and 4, and

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} = \frac{\mathrm{d}w}{\mathrm{d}\mu} \Big/ \frac{\mathrm{d}\varphi}{\mathrm{d}\mu} = \frac{1}{2}(\mu + p)\left(\frac{\mathrm{d}p}{\mathrm{d}\mu}\right)^{-1} \frac{\mathrm{d}^2 p}{\mathrm{d}\mu^2} \geq 0 \tag{24}$$

because of conditions 1, 3 and 4. □

4. Balance Laws

In this section we will obtain a blowup result for the system of balance laws

$$D_+r = e_0(r) + \kappa(r,s)e_1(r) = f(t,x,r,s), \tag{25a}$$

$$D_-s = e_0(s) + \lambda(r,s)e_1(s) = g(t,x,r,s), \tag{25b}$$

where $\{e_0, e_1\}$ is a pseudo-orthonormal frame with respect to a Lorentzian metric on $[t_0, t_1) \times \mathbf{S}^1$ for some real interval $[t_0, t_1)$. The operators $D_+ = e_0 + \kappa e_1$ and $D_- = e_0 + \lambda e_1$ are called characteristic derivatives and we denote the corresponding integral curves (or characteristics) by ϱ_x and σ_x. The suffix shows where the curves intersect the surface $t = t_0$, i.e., $\varrho_x(0) = (t_0, x)$ and $\sigma_x(0) = (t_0, x)$. We also denote the commutation coefficients by c_{ij}^k so that $[e_i, e_j] = c_{ij}^k e_k$.

Let $\mathscr{U}_\delta$ be an open subset of $\mathbf{R}^2$, to be further specified below. We assume that at least one of κ and λ is genuinely nonlinear on $\mathscr{U}_\delta$, i.e., $\partial\kappa/\partial r \neq 0$ or $\partial\lambda/\partial s \neq 0$. Without loss of generality we may assume that $\partial\kappa/\partial r > 0$. We also assume that $\kappa - \lambda \neq 0$ on $\mathscr{U}_\delta$ so that the system is strictly hyperbolic.

Let the initial values of r and s be $r_0(x) = r(t_0, x)$ and $s_0(x) = s(t_0, x)$. We will show that for certain choices of r_0 and s_0, r and s are bounded while $\partial_x r$ or $\partial_x s$ blow up in finite time, indicating the presence of a shock wave. There are similar results for systems of conservation laws (see, e.g., [6]), but we cannot hope for the same generality here. For more specific results, a better control of the source terms f and g is needed. For example, if the sources are superlinear in r and s we can expect blowup of r and/or s themselves [11]. On the other hand, global existence of smooth solutions have been shown under certain conditions [20]. We will leave these considerations aside and aim for a more general but less sharp result.

4.1. *Bounds on r and s*

First we need to specify the set $\mathscr{U}_\delta$. The initial data (r_0, s_0) is a smooth map $\mathbf{S} \to \mathbf{R}^2$. The image is a plane curve with convex hull $\mathscr{U}_0$. For any $\delta > 0$ we define

$$\mathscr{U}_\delta = \{P \in \mathbf{R}^2;\ d(P, \mathscr{U}_0) < \delta\} \quad \text{and} \quad \Omega_\delta = [t_0, t_1) \times \mathbf{S}^1 \times \mathscr{U}_\delta, \tag{26}$$

where d is the Euclidean distance in $\mathbf{R}^2$. Note that $\mathscr{U}_\delta$ is convex as well, as follows easily from the triangle inequality.

To be able to perform integrations along lines of constant r in a well defined way we need to introduce the following construction. Since $\mathbf{S}^1$ is compact, there are points (r_-, s_-) and (r_+, s_+) on the initial data curve in $\mathbf{R}^2$ where r_0 attains its minimum and maximum, respectively. These are also global extremal points of r in $\mathscr{U}_0$ since $\mathscr{U}_0$ is convex. By construction, $(r_- - \delta, s_-)$ and $(r_+ + \delta, s_+)$ are extremal points of r in the closure of $\mathscr{U}_\delta$, and we let $s = \gamma(r)$ be the straight line between those two points. We can now state the following lemma. Here $\|\cdot\|$ is the supremum norm on $\mathbf{S}^1$.

Lemma 4.1. *The line from (r, s) to $(r, \gamma(r))$ is contained in $\mathscr{U}_\delta$ for all $(r, s) \in \mathscr{U}_\delta$. Moreover, $|\gamma'| \leq \delta^{-1}\|s_0\|$ and $|s - \gamma(r)| < 2\|s_0\| + \delta$ on $\mathscr{U}_\delta$.*

Proof. Since $r_- - \delta < r_+ + \delta$, the line $s = \gamma(r)$ intersects the balls with radius δ around (r_-, s_-) and (r_+, s_+). It follows from the convexity of $\mathscr{U}_\delta$ that $s = \gamma(r)$ is contained in $\mathcal{U}_\delta$ except for its endpoints $(r_- - \delta, s_-)$ and $(r_+ + \delta, s_+)$. The first statement then follows directly from the convexity of $\mathcal{U}_\delta$.

The second statement is just the rough estimate

$$|\gamma'| = \frac{|s_+ - s_-|}{r_+ - r_- + 2\delta} \leq \frac{\|s_0\|}{\delta}, \tag{27}$$

and the third follows from the fact that $|\gamma(r)| \leq \|s_0\|$ and $|s| < \|s_0\| + \delta$ on $\mathcal{U}_\delta$. $\square$

Let

$$F = \sup_{\Omega_\delta}\{|f|, |g|\} \quad \text{and} \quad E = \inf_{\Omega_\delta}\{e_0^0 + \kappa e_1^0, e_0^0 + \lambda e_1^0\}. \tag{28}$$

The following lemma provides a crude estimate of r and s.

Lemma 4.2. *If F is finite and $E > 0$, any smooth solution (r, s) of (25) with initial data (r_0, s_0) remains within $\mathcal{U}_\delta$ for $t < \min\{t_0 + 2^{-1/2}\delta E F^{-1}, t_1\}$.*

Proof. From (25) we have that $|r - r_0| \leq F\xi$ along a characteristic ϱ_x as long as $t < t_1$, where ξ is the parameter along ϱ_x, chosen such that $D_+ = \partial/\partial\xi$ and $\varrho_x(t_0) = (t_0, x)$. By the definition of D_+, $\mathrm{d}t/\mathrm{d}\xi = e_0{}^0 + \kappa e_1{}^0 \geq E$, so $\xi \leq E^{-1}(t - t_0)$ and $|r - r_0| \leq FE^{-1}(t - t_0)$. We can obtain similar estimates for s along the characteristics σ_x. Thus the distance between (r, s) and (r_0, s_0) is at most $\sqrt{2}\, FE^{-1}(t - t_0)$, and the conclusion follows. $\square$

As we saw in Lemma 3.1, $\mu \mapsto \varphi : (0, \infty) \to (\varphi_-, \infty)$ is smooth and 1-1. It is quite possible that φ_- is finite, i.e., that $\mu = 0$ for finite r and s. This is indeed the case for a relativistic polytropic perfect fluid, for example. This complication can be circumvented by a further restriction on δ.

Lemma 4.3. *Let $\varphi_0 = \frac{1}{2}\inf_{x \in \mathbf{S}}\{r_0(x) + s_0(x)\}$. If*

$$0 < \delta < \varphi_0 - \varphi_- \tag{29}$$

then $\varphi > \varphi_-$ on $\mathcal{U}_\delta$.

Proof. If $\mu = 0$ at some point on the initial surface, then $\varphi = (r_0 + s_0)/2 = \varphi_-$ there. Thus the right hand side of (29) vanishes and the statement is void. On the other hand, if $\mu > 0$ for all $x \in \mathbf{S}^1$ at $t = 0$, the right hand side of (29) is a positive number because $\mathbf{S}^1$ is compact. The conclusion follows from the fact that if $(r, s) \in \mathcal{U}_\delta$ then

$$\varphi - \varphi_- = \frac{1}{2}(r + s) - \varphi_- > \varphi_0 - \varphi_- - \delta > 0. \tag{30}$$

$\square$

4.2. *The Blowup Equation*

For a homogeneous system of two conservation laws, it is possible to show that shock waves will form for suitable initial data by introducing a new unknown which is a rescaling of the spatial derivative of r. The scale factor can be chosen such that the new unknown satisfies a differential equation which does not involve derivatives of s. We follow a similar route, but as we will see below it is a bit harder to decouple the equations when source terms are present.

We are interested in the spatial derivative $e_1(r)$. First note that

$$[e_1, e_0] = c^0_{10}e_0 + c^1_{10}e_1 = c^0_{10}D_+ + (c^1_{10} - \kappa c^0_{10})e_1 \tag{31}$$

from the definition of D_+. It follows that

$$D_+e_1 = (e_1 - c^0_{10})D_+ - (\kappa_r e_1(r) - \kappa_s e_1(s) + c^1_{10} - \kappa c^0_{10})e_1. \tag{32}$$

We adopt the convention of denoting partial derivatives by subscripts, and we also write $f_{e_1} = e^0_1 f_t + e^1_1 f_x$ for the partial frame derivative of f with respect to e_1, i.e., $e_1(f)$ with r and s are regarded as constants.

Applying the operator D_+e_1 to r and using (25a) gives

$$\begin{aligned} D_+e_1(r) = &-\kappa_r e_1(r)^2 - \kappa_s e_1(s)e_1(r) - (c^1_{10} - \kappa c^0_{10})e_1(r) \\ &+ f_{e_1} + f_r e_1(r) + f_s e_1(s) - c^0_{10} f. \end{aligned} \tag{33}$$

In particular, there is a term quadratic in $e_1(r)$ which might cause blowup of $e_1(r)$. The problem is that there is a term involving both $e_1(r)$ and $e_1(s)$. As in the homogeneous case, the mixed term can be eliminated by introducing a function $h(r, s)$ by

$$h(r, s) = \int_{\gamma(r)}^{s} (\kappa - \lambda)^{-1} \frac{\partial \kappa}{\partial s}\, \mathrm{d}s, \tag{34}$$

where γ is defined in Lemma 4.1. Now let $R = -\mathrm{e}^h e_1(r)$. Differentiating R and simplifying gives

$$D_+R = a_2R^2 + a_1R + a_0 - \mathrm{e}^h f_s e_1(s), \tag{35}$$

where

$$a_2 = \mathrm{e}^{-h}\kappa_r, \tag{36a}$$

$$a_1 = f_r + f h_r + (\kappa - \lambda)^{-1} g \kappa_s + c^1_{01} - \kappa c^0_{01}, \tag{36b}$$

$$a_0 = -\mathrm{e}^h (f c^0_{01} + f_{e_1}). \tag{36c}$$

We would like to use (35) to show that R has to blow up for appropriately chosen initial data. But there is still a problematic term involving $e_1(s)$ which cannot be estimated using bounds on r and s alone. This term is not present in the homogeneous case since then f_s vanish. It is of course possible to define a quantity similar to R based on $e_1(s)$ instead of $e_1(r)$. The result is a system of two coupled equations, but then we cannot use ODE techniques directly.

It is, however, possible to remove the $e_1(s)$ term by the following procedure. By construction,

$$D_+s = D_-s + (\kappa - \lambda)e_1(s) = g + (\kappa - \lambda)e_1(s), \tag{37}$$

where we have used (25b). Since $\kappa - \lambda \neq 0$, we can introduce a function

$$\phi(t, x, r, s) = -\int_{\gamma(r)}^{s} \mathrm{e}^{h}(\kappa - \lambda)^{-1} f_s \, \mathrm{d}s. \tag{38}$$

By the chain rule, (37) and (25a),

$$D_+\phi = \phi_{e_0} + \kappa\phi_{e_1} + \phi_s(\kappa - \lambda)e_1(s) + \phi_s g + \phi_r f, \tag{39}$$

where ϕ_{e_0} and ϕ_{e_1} are the frame partial derivatives of ϕ, regarding r and s as constants. Applying (38) to (39) gives

$$D_+\phi = \phi_{e_0} + \kappa\phi_{e_1} - \mathrm{e}^{h} f_s e_1(s) + \phi_s g + \phi_r f. \tag{40}$$

Now (35) may be written

$$D_+(R - \phi) = A_2(R - \phi)^2 + A_1(R - \phi) + A_0, \tag{41}$$

where the coefficients are

$$A_2 = \mathrm{e}^{-h}\kappa_r, \tag{42a}$$

$$A_1 = f_r + f h_r + (\kappa - \lambda)^{-1} g\kappa_s + c_{01}^1 - \kappa c_{01}^0 + 2\,\mathrm{e}^{-h}\kappa_r\phi, \tag{42b}$$

$$\begin{aligned} A_0 = {} & \mathrm{e}^{-h}\kappa_r\phi^2 + (f_r + f h_r + (\kappa - \lambda)^{-1} g\kappa_s + c_{01}^1 - \kappa c_{01}^0)\phi \\ & -\mathrm{e}^{h}(f c_{01}^0 + f_{e_1} - f_s(\kappa - \lambda)^{-1} g) - \phi_r f - \phi_{e_0} - \kappa\phi_{e_1}. \end{aligned} \tag{42c}$$

Note that (41) is a first order ODE in $R - \phi$ whose coefficients can be estimated in terms of r and s.

4.3. *Blowup of R*

We start with a simple lemma about blow-up of solutions to an ODE with a quadratic nonlinearity (see, e.g., [1, p. 72] or [9, Lemma 1.3.2]).

Lemma 4.4. *Let $v(t)$ be a solution on $[t_0, t_1)$ of*

$$\frac{\mathrm{d}v}{\mathrm{d}t} = A_2(t)v^2 + A_1(t)v + A_0(t), \quad v(t_0) = v_0, \tag{43}$$

where A_2, A_1 and A_0 are continuous and bounded on $[t_0, t_1)$ with $A_2 \geq 0$. Put

$$K_1(t) = \int_{t_0}^{t} A_1 \, \mathrm{d}t, \quad K_0(t) = \int_{t_0}^{t} |A_0|\mathrm{e}^{-K_1} \, \mathrm{d}t, \quad K_2(t) = \int_{t_0}^{t} A_2\mathrm{e}^{K_1} \, \mathrm{d}t. \tag{44}$$

If $v_0 > K_0(t_1)$ then

$$K_2(t_1) < (v_0 - K_0(t_1))^{-1} \tag{45}$$

and we have the following lower bound on v in $[t_0, t_1)$:

$$\mathrm{e}^{-K_1(t)}v(t) \geq K_0(t_1) - K_0(t) + [(v_0 - K_0(t_1))^{-1} - K_2(t)]^{-1}. \tag{46}$$

Proof. The linear term can be dealt with by putting $V = v\,\mathrm{e}^{-K_1}$, which transforms (43) into

$$\frac{\mathrm{d}V}{\mathrm{d}t} = A_2\,\mathrm{e}^{K_1}V^2 + A_0\,\mathrm{e}^{-K_1}, \quad V(t_0) = v_0, \tag{47}$$

Let W be the solution of

$$\frac{\mathrm{d}W}{\mathrm{d}t} = A_2\,\mathrm{e}^{K_1}(W - K_0(t_1) + K_0(t))^2 - |A_0|\,\mathrm{e}^{-K_1}, \quad W(t_0) = v_0, \tag{48}$$

which can be found explicitly as

$$W(t) = K_0(t_1) - K_0(t) + [(v_0 - K_0(t_1))^{-1} - K_2(t)]^{-1}. \tag{49}$$

We want to show (46), i.e., $V \geq W$. Now

$$\frac{\mathrm{d}W}{\mathrm{d}t} \leq A_2\mathrm{e}^{K_1}W^2 + A_0\,\mathrm{e}^{-K_1}, \tag{50}$$

so subtracting (47) from (50) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(W - V) \leq A_2\mathrm{e}^{K_1}(W + V)(W - V), \tag{51}$$

and a Gronwall estimate implies that $W - V \leq 0$ as long as $W + V \geq 0$. Whenever $W - V \leq 0$, $W + V \geq 2W$, so $V \geq W$ as long as $W \geq 0$. By definition, $W \geq 0$ when $K_2(t) < (v_0 - K_0(t_1))^{-1}$. But $W \to \infty$ as $K_2(t) \to (v_0 - K_0(t_1))^{-1}$, so since V is bounded on $[t_0, t_1)$ we must have $K_2(t_1) < (v_0 - K_0(t_1))^{-1}$. □

Note that in [1] and [9], the factors involving K_1 are moved outside the integrals. We avoid this since we want to keep the simple form of (46).

Lemma 4.5. *Let $\mathcal{U}$ be an open set in $\mathbf{R}^2$ and put $\Omega = [t_0, t_1) \times \mathbf{S}^1 \times \mathcal{U}$. Assume that*

1. *(r, s) remains in $\mathcal{U}$ for $t \in [t_0, t_1)$,*
2. *$e_0^0 + \kappa e_1^0$ is continuous and positive on Ω,*
3. *A_2, A_1 and A_0, given by (42), are continuous with $A_2 > 0$ and*

$$C_2 = \inf_{\mathcal{U}} A_2, \quad C_1 = \sup_{\Omega} |A_1| \quad \textit{and} \quad C_0 = \sup_{\Omega} |A_0| \tag{52}$$

   are all finite with $C_2 > 0$, and

4. *ϕ is bounded on Ω.*

If

$$R(t_0, x) \geq \phi(t_0, x, r_0(x), s_0(x)) + C_0\chi(t_1) + (C_2\chi(t_1))^{-1} + C_1/C_2 \tag{53}$$

for some $x \in \mathbf{S}^1$, where

$$\chi(t_1) = \begin{cases} (\mathrm{e}^{C_1\xi(t_1)} - 1)/C_1 & \textit{if } C_1 > 0, \\ \xi(t_1) & \textit{if } C_1 = 0 \end{cases} \tag{54}$$

and $\xi(t_1)$ *is the parameter value along the characteristic* ϱ_x *corresponding to the time* t_1, *then* R *cannot be bounded on* $[t_0, t_1] \times \mathbf{S}^1$.

Proof. Assumption 1 ensures that the other assumptions apply along the characteristic ϱ_x for $t \in [t_0, t_1]$. Along ϱ_x, the parameter ξ satisfies $\mathrm{d}t/\mathrm{d}\xi = e_0^0 + \kappa e_1^0$, so it follows from assumption 2 that ξ is a continuously differentiable and increasing function of t along ϱ_x.

Because of assumption 3 and assuming that $R - \phi$ is bounded, we can apply Lemma 4.4 to (41) along ϱ_x, giving

$$R(t_0, x) - \phi(t_0, x, r_0(x), s_0(x)) < K_0(\xi(t_1)) + K_2(\xi(t_1))^{-1}. \tag{55}$$

If $C_1 > 0$, we have

$$|K_1| \leq C_1\xi(t), \quad K_0 \leq \frac{C_0}{C_1}(\mathrm{e}^{C_1\xi(t_1)} - 1) \quad \text{and} \quad K_2 \geq \frac{C_2}{C_1}(1 - \mathrm{e}^{-C_1\xi(t_1)}), \tag{56}$$

so

$$R(t_0, x) - \phi(t_0, x, r_0(x), s_0(x)) < C_0\chi(t_1) + (C_2\chi(t_1))^{-1} + C_1/C_2, \tag{57}$$

where $\chi(t_1) = (\mathrm{e}^{C_1\xi(t_1)} - 1)/C_1$. When $C_1 = 0$ the same inequality holds but with $\chi(t_1) = \xi(t_1)$. This contradicts the assumptions of the lemma and so in fact $R - \phi$ is unbounded. It follows that R must blow up since ϕ does not. □

Theorem 4.6. *Let* $\mathcal{U}_\delta$ *and* Ω_δ *be as in* (26). *Assume that*

I. λ, λ_r, λ_s, κ_r *and* κ_s *are continuous and bounded on* $\mathcal{U}_\delta$, *with* κ_r *and* $\kappa - \lambda$ *bounded away from* 0,
II. f, f_r, f_s, f_{se_0}, f_{e_1} *and* g *are continuous and bounded on* Ω_δ,
III. $e_0^0 + \kappa e_1^0$ *and* $e_0^0 + \lambda e_1^0$ *are continuous, bounded, positive and bounded away from* 0 *on* Ω_δ, *and*
IV. c_{01}^0 *and* c_{01}^1 *are continuous and bounded on* $[t_0, t_1) \times \mathbf{S}$.

Suppose also that

$$e_1(r)(t_0, x) \geq \mathrm{e}^{-h(r_0, s_0)}(\phi(t_0, x, r_0, s_0) + C_0\chi(\hat{t}_1) + (C_2\chi(\hat{t}_1))^{-1} + C_1/C_2) \tag{58}$$

for some $x \in \mathbf{S}$, *where* $\hat{t}_1 = \min\{t_0 + 2^{-1/2}\delta E F^{-1}, t_1\}$, F *and* E *are given by* (28), h *is given by* (34), ϕ *by* (38), C_i *by* (52) *and* χ *by* (54). *Then there is no smooth solution of* (25) *on* $[t_0, t_1]$ *with initial data* (r_0, s_0).

Moreover, the right hand side of (58) *can be estimated in terms of* $\|r_0\|$, $\|s_0\|$, t_0, t_1, x, δ *and the bounds in assumptions* I–IV.

Proof. First of all, that $(r, s) \in \mathcal{U}_\delta$ for $t \in [t_0, \hat{t}_1)$ follows from Lemma 4.2, which applies because of assumptions II and III. Thus condition 1 of Lemma 4.5 holds. Condition 2 follows directly from assumption III.

To simplify the presentation we will denote the supremum norms over $\mathbf{S}$, $\mathcal{U}_\delta$ and Ω_δ with the same symbol $\|\cdot\|$. From the definition (34) and Lemma 4.1, h is continuous on $\mathcal{U}_\delta$ with

$$\|h\| \leq \|(\kappa - \lambda)^{-1}\|\|\kappa_s\|(2\|s_0\| + \delta). \tag{59}$$

It follows that h is bounded because of assumption I. Differentiating (34) by r and performing a partial integration to get rid of the mixed second partial derivative κ_{rs} gives an estimate

$$\|h_r\| \leq \|(\kappa-\lambda)^{-1}\|(2\|\kappa_r\| + \|\kappa_s\|) + \|(\kappa-\lambda)^{-1}\|^2(\|\kappa_r\|\|\lambda_s\| + \|\kappa_s\|\|\lambda_r\|)(2\|s_0\| + \delta), \tag{60}$$

so it follows from assumption I that h_r is bounded as well.

From the definition (38) and assumptions I and II, ϕ is continuous on Ω_δ with

$$\|\phi\| \leq \|(\kappa-\lambda)^{-1}\|e^{\|h\|}\|f_s\|(2\|s_0\| + \delta). \tag{61}$$

Differentiating ϕ with respect to r and performing a partial integration to avoid the term involving f_{rs} gives an estimate

$$\|\phi_r\| \leq e^{\|h\|}\|(\kappa-\lambda)^{-1}\|[2\|f_r\| + \|f_s\| + (\|f_r\|\|H_s\| + \|f_s\|\|H_r\|)(2\|s_0\| + \delta)], \tag{62}$$

where $H = h - \ln(\kappa - \lambda)$. Differentiating ϕ with respect to e_0 gives the estimate

$$\|\phi_{e_0}\| \leq e^{\|h\|}\|(\kappa-\lambda)^{-1}\|\|f_{se_0}\|(2\|s_0\| + \delta), \tag{63}$$

while for ϕ_{e_1} we perform a partial integration before estimating, giving

$$\|\phi_{e_1}\| \leq e^{\|h\|}\|(\kappa-\lambda)^{-1}\|\|f_{e_1}\|(2 + \|H_s\|(2\|s_0\| + \delta)). \tag{64}$$

By assumptions I and II, ϕ, ϕ_r, ϕ_{e_0} and ϕ_{e_1} are bounded. We conclude that all terms in (42) are continuous and bounded.

Now $\inf_{\mathcal{U}_\delta} A_2 > 0$ because of assumption I, so all conditions of Lemma 4.5 are met. Since $R = e^h e_1(r)$, $e_1(r)$ cannot be bounded on the characteristic ϱ_x if (58) holds.

Finally, the only quantity in the right hand side of (58) whose dependence on the estimates has not been established by the arguments above is $\chi(\hat{t}_1)$. From the definition (28) of E and F, $\hat{t}_1$ can be estimated both from below and above by t_0, t_1, x, δ, $\|r_0\|$, $\|s_0\|$ and the bounds in assumptions I–IV. That the same holds for $\chi(\hat{t}_1)$ follows from the definition (54) of χ together with the estimate of A_1 above. □

The condition on f_{se_0} can of course be replaced by a condition on f_{e_0} by performing a partial integration as we did with h_r, f_{rs} and f_{se_1}. The reason for not doing so will become evident below. Note also that there is no explicit dependence of f_{e_0} in (42).

Let us define the blowup time as

$$t_* = \sup\{t > t_0;\ (r, s) \text{ is smooth on } [t_0, t]\}. \tag{65}$$

Then we also have the following.

Corollary 4.7. *Under the conditions of Theorem* 4.6, *the blowup time* t_* *satisfies* $\chi(t_*) \leq \chi_*$ *where* χ_* *is the smaller root of*

$$e_1(r)(t_0, x) = e^{-h(r_0,s_0)}(\phi(t_0, x, r_0, s_0) + C_0\chi + (C_2\chi)^{-1} + C_1/C_2). \tag{66}$$

5. Blowup in Spacetime

We will now combine the results of the preceding sections to show finite life span of solutions to the Einstein–Euler equations. We first need to control higher derivatives of some geometric quantities, given bounds on the matter variables.

Lemma 5.1. *If the conditions of Lemma* 3.1 *are satisfied and* r *and* s *are bounded with* $r + s > 2\varphi_-$, *the following quantities are bounded*:

$$K', A'', \alpha'', \partial_t K, \beta'' \quad \text{and} \quad \partial_t A'. \tag{67}$$

The bounds involve only those in (6) *and the a priori bounds on* r *and* s.

Proof. From Lemma 3.1, ρ, j and S can be bounded in terms of r and s for a given equation of state. Using these bounds together with (5) in (4a), (4b), (4c) and (4f) immediately gives bounds on K', A'', α'' and $\partial_t K$. Differentiating (4d) by x gives β'' expressed in α, α', K, K' and t, so it is bounded too. Finally, differentiating (4e) by x gives a bound on $\partial_t A'$ in terms of the previously obtained bounds. □

The previous results can now be combined to show a blow up result for the Einstein–Euler equations in plane symmetry.

Theorem 5.2. *Let* $t_0 < t_1 < 0$ *and suppose that the conditions of Lemma* 3.1 *hold. Then there are smooth initial data on the surface* $t = t_0$ *such that the corresponding smooth solution of the Einstein–Euler equations* (4) *and* (11) *does not extend beyond* $t = t_1$.

Proof. Theorem 2.1 shows that all the geometric quantities in (5) are bounded in terms of the quantities in (6). Let $\mathcal{U}_\delta$ and Ω_δ be as in (26), with δ as in (29). Then $\mathcal{U}_\delta$ and Ω_δ depend on r_0, s_0, t_0, t_1 and δ, and δ is limited by $\varphi_0 - \varphi_-$, which in turn depends only on the equation of state and the minimum of $r_0 + s_0$. By construction, r and s can be bounded on Ω_δ in terms of δ and the maximum values of r_0 and s_0. It then follows from Lemma 5.1 that the geometric quantities (67) also are bounded, in terms of (6) and

$$\delta, \|r_0\|, \|s_0\|. \tag{68}$$

We need to establish the bounds in Theorem 4.6. First Theorem 2.1 and Lemma 5.1 give that

$$e_0^0 + \kappa e_1^0 = e_0^0 + \lambda e_1^0 = \alpha^{-1} \tag{69}$$

are continuous, positive and bounded away from 0 and

$$c_{01}^0 = \alpha^{-1} e_1(\alpha) \quad \text{and} \quad c_{01}^1 = \alpha^{-1} A e_1(\beta) - A^{-1} e_0(A) \tag{70}$$

are continuous and bounded in terms of (6).

From the definition of u and the construction of $\mathcal{U}_\delta$, $\ln(1-u^2)$, $(1-u^2)^{-1}$ and $(1 \pm uw)^{-1}$ can be bounded in terms of (68). Also,

$$\frac{1}{2}\inf_{S^1}\{r_0 + s_0\} - \delta < \varphi < \frac{1}{2}\sup_{S^1}\{r_0 + s_0\} + \delta, \tag{71}$$

so for a given equation of state, Lemma 3.1 gives that w^{-1}, $(1-w^2)^{-1}$ and $\frac{\mathrm{d}w}{\mathrm{d}\varphi}$ are bounded in terms of (68) and

$$(\varphi_0 - \varphi_- - \delta)^{-1} = \left(\frac{1}{2}\inf_{\mathrm{S}^1}\{r_0 + s_0\} - \varphi_- - \delta\right)^{-1}. \tag{72}$$

The balance law coefficients can be read off directly from (16). They are

$$\kappa(r,s) = \frac{u+w}{1+uw}, \tag{73a}$$

$$\lambda(r,s) = \frac{u-w}{1-uw}, \tag{73b}$$

$$f(t,x,r,s) = \frac{tw + Ku - 2uwA^{-1}e_1(A)}{1+uw} - \alpha^{-1}e_1(\alpha), \tag{73c}$$

$$g(t,x,r,s) = \frac{tw - Ku - 2uwA^{-1}e_1(A)}{1-uw} + \alpha^{-1}e_1(\alpha). \tag{73d}$$

From the previously obtained bounds it follows that κ and λ are continuous on $\mathcal{U}_\delta$ and can be bounded in terms of (68). Differentiating and using that $w_r = w_s = \frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}\varphi}$ and $u_r = -u_s = \frac{1}{2}(1-u^2)$ we get

$$\kappa_r = \frac{1-u^2}{2(1+uw)^2}\left(1 - w^2 + \frac{\mathrm{d}w}{\mathrm{d}\varphi}\right), \tag{74}$$

with similar expressions for κ_s, λ_r and λ_s. These quantities are continuous on $\mathcal{U}_\delta$ and can be bounded in terms of (68) and (72). Moreover, κ_r and

$$\kappa - \lambda = \frac{2w(1-u^2)}{1-u^2w^2} \tag{75}$$

are positive, and κ_r^{-1} and $(\kappa-\lambda)^{-1}$ are bounded in terms of (68) and (72).

From (73), f and g are continuous on Ω_δ and bounded in terms of (6) and (68). Since $w_r = w_s = \frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}\varphi}$ and $u_r = -u_s = \frac{1}{2}(1-u^2)$, the same holds for their r and s derivatives.

Finally the quantities appearing in f_{se_0} and f_{e_1} not present in f are $\partial_t K$, K', $\partial_t A'$, A'' and α''. It follows from Lemma 5.1 that f_{se_0} and f_{e_1} are continuous and bounded in terms of (6), (68) and (72). Note that f_{se_0} does not depend on $\partial_t\alpha'$ since the term in (73c) containing α is independent of s. This is the reason for not replacing f_{se_0} by f_{e_0} in Theorem 4.6.

Now all the conditions of Theorem 4.6 have been established and we can conclude that if the initial data satisfies (58) for some x, there is no smooth solution on $[t_0, t_1]$. By definition, $e_1(r) = A^{-1}\partial_x r$. Since all quantities in (58) except $\partial_x r$ can be estimated in terms of the geometric initial data (6) and the matter initial data (68) and (72), choosing initial data with sufficiently large gradient $\partial_x r$ will prevent a smooth extension up to $t = t_1$. □

The question, how big the class of solutions of the constraints is which satisfy the hypotheses of the theorem will not be treated in great generality here. It will, however, be shown that there are some data which do so for which the time of existence is arbitrarily short. Consider data for which u is identically zero. Then a particular solution of equation (4a) is given by $K = t/3$. Substituting this into the remaining constraint (4b) gives $(\sqrt{A})'' = -\frac{1}{8}A^{5/2}(-\frac{2}{3} + 16\pi\mu)$. The equation to be solved is a special case of one treated in [10] where the existence of a solution A was shown by the method of sub- and supersolutions. This method also gives uniform bounds for the solution which in particular shows that the derivative μ' at a point can be made arbitrarily large within a family of solutions while maintaining a fixed L^∞ bound for A. At the same time fixed L^∞ bounds for μ and μ^{-1} can be maintained. This suffices to show that $e_1(r)$ becomes arbitrarily large within the family and provides the desired data for which the time of existence is arbitrarily short.

6. An Extension Result

In the previous section we established that if the initial data has sufficiently large gradients, there is no smooth solution of the Einstein–Euler equations beyond a certain time. It seems quite plausible that the obstruction to extending the solution is the blowup of first derivatives of the matter variables, although it must be pointed out that we have not shown that. To investigate this further, we will show that if the first derivatives of the matter variables are bounded then the spacetime can be extended. The argument will be similar to that in [12]. First we need bounds on r and s, which can be established under a mild extra condition on the equation of state.

Theorem 6.1. *Assume that in addition to the conditions in Lemma* 3.1, $\mathrm{d}p/\mathrm{d}\mu$ *is bounded away from* 1. *Then the following quantities are bounded*:

$$r, s, \rho, j, S. \tag{76}$$

Proof. The extra condition and Lemma 3.1 implies that the factors in the denominators of (73) are positive and bounded away from 0. Thus the right hand sides of (25 can be bounded in terms of the geometric quantities (5). Integrating (25) shows that r and s are bounded along the characteristics. We may pass from these estimates to estimates in time because of (69). Finally, ρ, j and S must also be bounded because of Lemma 3.1. □

Lemma 6.2. *Under the conditions of Lemma* 3.1, *if all spatial derivatives of order up to n of the geometric quantities* (5) *and the matter quantities* (76) *are bounded, the spatial derivatives of order up to $n+1$ of the geometric quantities* (5) *are bounded.*

Proof. Note first that some of the spatial derivatives are already included in (5). As was shown in the proof of Lemma 5.1, we can solve (4) for the remaining quantities α'', A'', $\partial_t A'$, K' and β''. Differentiating the resulting equations n times gives the desired bounds. □

Lemma 6.3. *Suppose that the conditions of Lemma* 6.2 *hold,* $\mathrm{d}p/\mathrm{d}\mu$ *is bounded away from* 1 *and $\partial_x r$ and $\partial_x s$ are bounded. Then the spatial derivatives of the matter quantities* (76) *of order up to $n+1$ are bounded.*

Proof. Note first that because of the definition (2) of e_1, using e_1^n instead of ∂_x^n introduces spatial derivatives of A^{-1} of order at most $n-1$, which are bounded by assumption.

Put $r_n = e_1^n(r)$ and $s_n = e_1^n(s)$. Applying (32) to r_{n+1} gives

$$D_- r_{n+1} = (e_1 - c_{10}^0)D_+ r_n - (\kappa_r e_1(r) - \kappa_s e_1(s) + c_{10}^1 - \kappa c_{10}^0) r_{n+1}. \tag{77}$$

If we can show that (77) is a linear equation in $D_+ r_{n+1}$ with the coefficients given by smooth functions of spatial derivatives of r and s of order up to n and of the geometric quantities (5) of order up to $n+1$, we would be done because of Lemma 6.2.

Putting $n = 1$ and using the expression (33) for $D_+ r_1$ shows that $D_+ r_2$ is linear in r_2, and the coefficients have the right dependence because of (73) and (70). For $n > 2$ the coefficients of $D_+ r_n$ are differentiated by x at most once in (77) so the result follows by induction. The case with s is completely analogous. □

Lemma 6.4. *Under the conditions of Lemma* 6.3, *if all derivatives of the geometric quantities* (5) *and the matter quantities* (76) *of the form* $\partial_t^k \partial_x^n$ *with* $n \geq 0$ *and* $0 \leq k \leq m$ *are bounded then the* $\partial_t^{m+1} \partial_x^n$ *derivatives of the matter quantities* (76) *are bounded.*

Proof. As was the case with e_1 and ∂_x, we may use e_0 instead of ∂_t because of (2). The balance laws (25) can be used to bound an extra time derivative by introducing an extra space derivative, which is bounded by Lemma 6.3. □

Lemma 6.5. *Under the conditions of Lemma* 6.3, *if*

1. *all derivatives of the geometric quantities* (5) *and the matter quantities* (76) *of the form* $\partial_t^k \partial_x^n$ *with* $n \geq 0$ *and* $0 \leq k \leq m$ *are bounded, and*
2. *all derivatives of the matter quantities* (76) *of the form* $\partial_t^{m+1} \partial_x^n$ *are bounded,*

then the derivatives of the geometric quantities (5) *of the form* $\partial_t^{m+1} \partial_x^n$ *are bounded.*

Proof. See [12]. □

Putting together Theorem 6.1 and Lemma 6.2–6.5 we see that under the hypotheses of Lemma 6.3, all derivatives of the geometric and matter quantities are bounded. When all the derivatives are bounded on a given interval, the solution can be smoothly extended to the closure of that interval. This results in a new initial data set and applying the local existence and uniqueness result from [12] gives an extension of the solution. We thus have the following theorem.

Theorem 6.6. *Let a solution of the Einstein equations with plane symmetry be given. Suppose that when coordinates are chosen which cast the metric into the form* (1) *with constant mean curvature time slices, the time coordinate takes all values in the finite interval* $[t_0, t_1)$ *with* $t_1 < 0$. *Assume that the equation of state satisfies the conditions in Lemma* 3.1, *and that* $\mathrm{d}p/\mathrm{d}\mu$ *is bounded away from* 1. *If the first spatial derivatives of the energy density* μ *and the velocity parameter* u *are bounded on* $[t_0, t_1)$, *the solution can be extended beyond* $t = t_1$.

7. Discussion

In this paper it has been shown that plane symmetric classical solutions of the Einstein–Euler system can break down in finite time under some general assumptions on the equation of state. Time here is measured with respect to a CMC foliation. That the formation of singularities is due to the fluid and not just to a breakdown of the CMC foliation is shown by the following fact. If instead of the Euler equations we describe the matter content of spacetime by a collisionless gas satisfying the Vlasov equations and maintain all other assumptions the solutions exist globally in the future [12]. In the case of the Euler equations for a fluid without pressure (dust) a non-existence result was proved in [13]. The proofs of all these results have common elements. It is shown that the spacetime geometry has certain good properties right up to the singularity. Using this it is shown that when there are no singularities in flat spacetime there are also none after coupling to the Einstein equations (Vlasov) while in the cases there are singularities (Euler with pressure, dust) the mechanisms of blow-up in the flat space case (formation of shocks, blow-up of the density) provide a reliable guide as to what quantities are bounded or not and how to prove it in the case of coupling to the Einstein equations.

A family of data along which the time of existence becomes arbitrarily small can be used to show that the results of [10] on solutions of the Einstein-dust system generalize to the case of fluids with non-zero pressure. The conclusion is that there are data for the Einstein–Euler equations with an equation of state of the kind considered in the present paper on a CMC hypersurface such that the corresponding maximal Cauchy development cannot be foliated by compact CMC hypersurfaces. To see that the proofs of [10] can be adapted note first that under the assumption $u = 0$ the constraints for the fluid with pressure are identical to those for dust so that many of the arguments can be taken over directly. Cauchy stability for the Einstein–Euler system with $0 < \frac{dp}{d\mu} < 1$, which is necessary for the argument, follows from the fact that it can be written in symmetric hyperbolic form so that standard results apply. See for instance [16], in particular p. 140. The final element which is required is to know that the argument for breakdown of the solution can be localized in space. Since the contradiction to existence on a certain time interval is obtained by an analysis of the evolution of certain quantities along a single characteristic it is only necessary to show that in a short time interval the characteristic can only move a short (coordinate) distance in space. This follows immediately from the available a priori bounds for the geometry.

The assumptions on the equation of state in Theorem 5.2 are compatible with cases of physical relevance. Consider, for instance, the relativistic polytrope with

$$\mu = m + Knm^{1+\frac{1}{n}}, \quad p = Km^{1+\frac{1}{n}} \tag{78}$$

where $n > 1$ and $K > 0$ are constants and m the rest mass density. In this case all the assumptions are satisfied. They are also satisfied for an equation of state leading to a linear relation $p = K\mu$ with $K > 0$, as considered in [3] and [17]. This is true in spite of the fact that if a full thermodynamic treatment of this case is attempted the internal energy is negative at low densities and so its physical interpretation is questionable.

In the case of Theorem 5.2 the density remains bounded on the maximal interval of existence of a classical solution. This cannot be expected to hold for dust, although it is not actually proved in [13] that the density is unbounded. Thus some restrictions on the equation of state are to be expected. It is probable that blow-up of the density occurs when the equation of state is such that the pressure is a bounded function of the density (cf. [19]). In Theorem 5.2 this case is excluded by condition 4 of Lemma 3.1.

Ideally, there would be a useful interaction between the results of [3] and those of the present paper. It would then be possible to conclude that the classical solutions with finite-time breakdown can be continued as weak solutions and on the other hand that the weak solutions which have been constructed are not always classical. Unfortunately the time coordinates used in the two cases are different so that a direct comparison is not possible.

Finally, some possible extensions of the results of this paper will be mentioned. In the case of a special relativistic fluid the analogue of Theorem 5.2 holds with the same proofs. In that case the time is the ordinary Minkowski time. Also, Theorem 6.6 shows that under the extra condition that $\mathrm{d}p/\mathrm{d}\mu$ is bounded away from 1, a solution for which the first derivatives of m and u remain bounded remains smooth, i.e. it can be extended smoothly to a longer time interval. The results obtained here therefore support the picture that the breakdown of the classical solutions is due to the formation of shock waves.

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