On the final spin from the coalescence of two black holes

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We provide a compact analytic formula to compute the spin of the black hole produced by the coalescence of two black holes following a quasi-circular inspiral. Without additional fits than those already available for binaries with aligned or antialigned spins, but with a minimal set of assumptions, we derive an expression that can model generic initial spin configurations and mass ratios, thus covering all of the 7-dimensional space of parameters. A comparison with simulations already shows very accurate agreements with all of the numerical data available to date, but we also suggest a number of ways in which our predictions can be further improved.

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The evolution of black hole binary systems is one of the most important problems for general relativity, and more recently for astrophysics, as such systems enter the realm of observation. Recent advances in numerical relativity have made it possible to cover the entire range of the inspiral process, from large separations at which post-Newtonian (PN) calculations provide accurate orbital parameters, through the highly relativistic merger, to ringdown. For many studies of astrophysical interest, such as many-body studies of galactic mergers, or heirarchical models of black-hole formation however, it is impractical to carry out evolutions with the full Einstein, or even post-Newtonian, equations. Fortunately, recent binary black-hole evolutions in full general relativity have shown that certain physical quantities can be estimated to good accuracy if the initial encounter parameters are known. In particular, this paper develops a rather simple and robust formula for determining the spin of the black-hole remnant resulting from the merger of rather generic initial binary configurations.

To appreciate the spirit of our approach it can be convenient to think of the inspiral and merger of two black holes as a mechanism which takes, as input, two black holes of initial masses M_1, M_2 and spin vectors $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and produces, as output, a third black hole of mass $M_{\rm fin}$ and spin $\boldsymbol{S}_{\rm fin}$. In conditions of particular astrophysical interest, the inspiral takes place through quasi-circular orbits since the eccentricity is removed quickly by the gravitational-radiation reaction [1]. Furthermore, at least for nonspinning equal-mass black holes, the final spin does not depend on the value of the eccentricity as long as it is not too large [2]. The determination of $M_{\rm fin}$ and $\boldsymbol{S}_{\rm fin}$ from the knowledge of $M_{1,2}$ and $\boldsymbol{S}_{1,2}$, is of great importance in several fields. In astrophysics, it provides information on the properties of isolated stellar-mass black holes produced at the end of the evolution of a binary system of massive stars. In cosmology, it can be used to model the distribution of masses and spins of the supermassive black holes produced through the merger of galaxies (see ref. [3] for an interesting example). In addition, in gravitational-wave astronomy, the a-priori knowledge of the final spin can help the detection of the ringdown. What makes this a difficult problem is clear: for binaries in quasi-circular orbits the space of initial parameters for the final spin has seven dimensions (*i.e.*, the mass-ratio $q \equiv M_2/M_1$ and the six components of the spin vectors). A number of analytical approaches have been developed over the years to determine the final spin, either exploiting the dynamics of point-particles [4, 5] or the PN approximation [6], or using more sophisticated approaches such as the effective-one-body approximation [7]. Ultimately, however, computing $\boldsymbol{a}_{\rm fin} \equiv \boldsymbol{S}_{\rm fin}/M_{\rm fin}^2$ accurately requires the solution of the full Einstein equations and thus the use of numerical-relativity simulations. Several groups have investigated this problem over the last couple of years [8, 9, 10, 11, 12, 13].

While the recent possibility of measuring accurately the final spin through numerical-relativity calculations represents an enormous progress, the complete coverage of the full parameter space uniquely through simulations is not a viable option. As a consequence, work has been done to derive analytic expressions for the final spin which would model the numerical-relativity data but also exploit as much information as possible either from perturbative studies, or from the symmetries of the system [9, 11, 12, 13, 14, 15]. In this sense, these approaches do not amount to a blind fitting of the numerical-relativity data, but, rather, use the data to construct a physically consistent and mathematically accurate modelling of the final spin. Despite a concentrated effort in this direction, the analytic expressions for the final spin could, at most, cover 3 of the 7 dimensions of the space of parameters [13]. Here, we show that without additional fits and with a minimal set of assumptions it is possible to obtain the extension to the complete space of parameters and reproduce all of the available numerical-relativity data. Although our treatment is intrinsically approximate, we also suggest how it can be improved.

Analytic fitting expressions for $\boldsymbol{a}_{\rm fin}$ have so far been built using binaries having spins that are either *aligned* or *antialigned* with the initial orbital angular momentum. This is because in this case both the initial and final spins can be projected in the direction of the orbital angular momentum and it is possible to deal simply with the (pseudo)-scalar quantities a_1, a_2 and $a_{\rm fin}$ ranging between -1 and $+1$. If the black holes have *equal mass* but *unequal* spins that are either *parallel* or *antiparallel*, then the spin of the final black hole has been shown to be accurately described by the simple analytic fit [11]

$$a_{\rm fin}(a_1,a_2)=p_0+p_1(a_1+a_2)+p_2(a_1+a_2)^2\,, \qquad (1)$$

where $p_0 = 0.6883 \pm 0.0003$, $p_1 = 0.1530 \pm 0.0004$, and $p_2 = -0.0088 \pm 0.0005$. When seen as a power series of the initial spins, expression (1) suggests an interesting physical interpretation. Its zeroth-order term, in fact, can be associated with the (dimensionless) orbital angular momentum not radiated in gravitational waves and amounting to $\sim 70\%$ of the final spin at most. The first-order term, on the other hand, can be seen as the contributions from the initial spins and from the spin-orbit coupling, amounting to $\sim 30\%$ at most. Finally, the second-order term, includes the spin-spin coupling, with a contribution to the final spin which is of $\sim 4\%$ at most.

If the black holes have *unequal mass* but spins that are *equal* and *parallel*, the final spin is instead given by the analytic fit [13]

$$a_{\rm fin}(a,\nu)=a+s_4a^2\nu+s_5a\nu^2+t_0a\nu+ 2\sqrt{3}\nu+t_2\nu^2+t_3\nu^3\,, \qquad (2)$$

where ν is the symmetric mass ratio $\nu\equiv M_1M_2/(M_1{+}M_2)^2$, and where the coefficients take the values $s_4 = -0.129 \pm 0.012$, $s_5 = -0.384 \pm 0.261$, $t_0 = -2.686 \pm 0.065$, $t_2 = -3.454 \pm 0.132$, $t_3 = 2.353 \pm 0.548$. Although obtained independently in [11] and [13], expressions (1) and (2) are compatible as can be seen by considering (2) for equal-mass binaries ($\nu = 1/4$) and verifying that the following relations hold within the computed error-bars

$$p_0=\frac{\sqrt{3}}{2}+\frac{t_2}{16}+\frac{t_3}{64}\,,\quad p_1=\frac{1}{2}+\frac{s_5}{32}+\frac{t_0}{8}\,,\quad p_2=\frac{s_4}{16}. \qquad (3)$$

As long as the initial spins are aligned (or antialigned) with the orbital angular momentum, expression (2) can be extended to *unequal-spin, unequal-mass* binaries through the substitution

$$a\ \rightarrow\ \tilde{a}\equiv\frac{a_1+a_2q^2}{1+q^2}\,. \qquad (4)$$

To obtain this result, it is sufficient to consider (1) and (2) as polynomial expressions of the generic quantity

$$\tilde{a}\equiv a_{\rm tot}\frac{(1+q)^2}{1+q^2}\,. \qquad (5)$$

where $a_{\rm tot}\equiv(a_1+a_2q^2)/(1+q)^2$ is the total dimensionless spin for generic aligned binaries. In this way, expressions (1) and (2) are naturally compatible, since $\tilde{a}=(a_1+a_2)/2$ for equal-mass unequal-spin binaries, and $\tilde{a} = a$ for unequal-mass equal-spin binaries. Furthermore, the extreme mass-ratio limit (EMRL) of expression (2) with the substitution (4) yields the expected result: $a_{\rm fin}(a_1,a_2,\nu=0)=a_1$.

As already commented above, the predictions of expressions (2) and (4) cover 3 of the 7 dimensions of the space of parameters for binaries in quasi-circular orbits; we next show how to to cover the remaining 4 dimensions and derive an analytic expression for the dimensionless spin *vector* $\boldsymbol{a}_{\rm fin}$ of the black hole produced by the coalescence of two generic black holes in terms of the mass ratio q and of the initial dimensionless spin vectors $\boldsymbol{a}_{1,2}$. To make the problem tractable analytically, 4 assumptions are needed. While some of these are very natural, others can be relaxed if additional accuracy in the estimate of $\boldsymbol{a}_{\rm fin}$ is necessary. It should be noted, however, that removing any of these assumptions inevitably complicates the picture, introducing additional dimensions, such as the initial separation in the binary or the radiated mass, in the space of parameters.

As a result, in the simplest and yet accurate description the required assumptions are as follows:

(i) The mass radiated to gravitational waves $M_{\rm rad}$ can be neglected i.e., $M_{\rm fin} = M \equiv M_1 + M_2$. We note that $M_{\rm rad}/M = 1 - M_{\rm fin}/M \approx 5 - 7\times10^{-2}$ for most of the binaries evolved numerically. The same assumption was applied in the analyses of [11, 13], as well as in [5]. Relaxing this assumption would introduce a dependence on $M_{\rm fin}$ which can only be measured through a numerical simulation.

(ii) At a sufficiently large but finite initial separation the final spin vector $\boldsymbol{S}_{\rm fin}$ can be well approximated as the sum of the two initial spin vectors and of a third vector $\tilde{\boldsymbol{\ell}}$

$$\boldsymbol{S}_{\rm fin}=\boldsymbol{S}_1+\boldsymbol{S}_2+\tilde{\boldsymbol{\ell}}\,, \qquad (6)$$

Differently from refs. [4] and [5], where a definition similar to (6) was also introduced, here we will constrain $\tilde{\boldsymbol{\ell}}$ by exploiting the results of numerical-relativity calculations rather than by relating it to the orbital angular momentum of a test particle at the innermost stable circular orbit (ISCO). When viewed as expressing the conservation of the total angular momentum, eq. (6) also defines the vector $\tilde{\boldsymbol{\ell}}$ as the difference between the orbital angular momentum when the binary is widely separated $\boldsymbol{L}$, and the angular momentum radiated until the merger $\boldsymbol{J}_{\rm rad}$, *i.e.*, $\tilde{\boldsymbol{\ell}}=\boldsymbol{L}-\boldsymbol{J}_{\rm rad}$.

(iii) The vector $\tilde{\boldsymbol{\ell}}$ is parallel to $\boldsymbol{L}$. This assumption is correct when $\boldsymbol{S}_1=-\boldsymbol{S}_2$ and $q=1$ [this can be seen from the PN equations at 2.5 order], or by equatorial symmetry when the spins are aligned with $\boldsymbol{L}$ or when $\boldsymbol{S}_1=\boldsymbol{S}_2=0$ (also these cases can be seen from the PN equations). For more general configurations one expects that $\tilde{\boldsymbol{\ell}}$ will also have a component orthogonal to $\boldsymbol{L}$ as a result, for instance, of spin-orbit or spin-spin couplings, which will produce in general a precession of $\tilde{\boldsymbol{\ell}}$. In practice, the component of $\tilde{\boldsymbol{\ell}}$ orthogonal to $\boldsymbol{L}$ will correspond to the angular momentum $\boldsymbol{J}^{\perp}_{\rm rad}$ radiated in a plane orthogonal to $\boldsymbol{L}$, with a resulting error in the estimate of $|\tilde{\boldsymbol{\ell}}|$ which is $\sim |\boldsymbol{J}^{\perp}_{\rm rad}|^2/|\tilde{\boldsymbol{\ell}}|^2 \sim |\boldsymbol{J}^{\perp}_{\rm rad}|^2/(2\sqrt{3}M_1M_2)^2$[25]. Although these errors are small in all the configurations that we have analysed, they may be larger in general configurations. Measuring $\boldsymbol{J}^{\perp}_{\rm rad}$ via numerical-relativity simulations, or estimating it via high-order PN equations, is an obvious way to improve our approach. A similar assumption was also made in ref. [5].

(iv) When the initial spin vectors are equal and opposite ($\boldsymbol{S}_1=-\boldsymbol{S}_2$) and the masses are equal ($q=1$), the spin of the final black hole is the same as for the nonspinning binaries. Stated differently, equal-mass binaries with equal and

opposite-spins behave as nonspinning binaries, at least when it comes down to the properties of the final black hole. While this result cannot be derived from first principles, it reflects the expectation that if the spins are the same and opposite, their contributions to the final spin cancel for equal-mass binaries. Besides being physically reasonable, this expectation is met by all of the simulations performed to date, both for spins aligned with $\boldsymbol{L}$ [11, 13] and orthogonal to $\boldsymbol{L}$ [10]. In addition, this expectation is met by the leading-order contributions to the spin-orbit and spin-spin point-particle Hamiltonians and spin-induced radiation flux [7, 16]. A similar assumption is also made, although not explicitly, in ref. [5] which, for $\boldsymbol{S}_{\rm tot} = 0$, predicts $\iota = 0$ and $|\boldsymbol{a}_{\rm fin}| = L_{\rm orb}(\iota = 0, |\boldsymbol{a}_{\rm fin}|)/M =$const. [*cf.* eqs. (12)–(13) in ref. [5]].

Using these assumptions we can now derive the analytic expression for the final spin. We start by expressing the vector relation (6) as

$$\boldsymbol{a}_{\rm fin} = \frac{1}{(1+q)^2}\left(\boldsymbol{a}_1 + \boldsymbol{a}_2 q^2 + \boldsymbol{\ell} q\right)\,, \tag{7}$$

where $\boldsymbol{a}_{\rm fin} = \boldsymbol{S}_{\rm fin}/M^2$ [*cf.* assumption *(i)*], $\boldsymbol{\ell} \equiv \tilde{\boldsymbol{\ell}}/(M_1 M_2)$, $\boldsymbol{a}_{1,2} \equiv \boldsymbol{S}_{1,2}/M^2_{1,2}$, and its norm is then given by

$$|\boldsymbol{a}_{\rm fin}| = \frac{1}{(1+q)^2}\Big[|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 q^4 + 2|\boldsymbol{a}_2||\boldsymbol{a}_1|q^2\cos\alpha + 2\left(|\boldsymbol{a}_1|\cos\beta + |\boldsymbol{a}_2|q^2\cos\gamma\right)|\boldsymbol{\ell}|q + |\boldsymbol{\ell}|^2 q^2\Big]^{1/2}, \tag{8}$$

where the three (cosine) angles α, β and γ are defined by

$$\cos\alpha \equiv \hat{\boldsymbol{a}}_1\cdot\hat{\boldsymbol{a}}_2\,, \quad \cos\beta \equiv \hat{\boldsymbol{a}}_1\cdot\hat{\boldsymbol{\ell}}\,, \quad \cos\gamma \equiv \hat{\boldsymbol{a}}_2\cdot\hat{\boldsymbol{\ell}}\,. \tag{9}$$

Because $\boldsymbol{a}_{1,2} \parallel \boldsymbol{S}_{1,2}$ and $\boldsymbol{\ell} \parallel \boldsymbol{L}$ [*cf.* assumption *(iii)*], the angles α, β and γ are also those between the initial spin vectors and the initial orbital angular momentum, so that it is possible to replace $\hat{\boldsymbol{a}}_{1,2}$ with $\hat{\boldsymbol{S}}_{1,2}$ and $\hat{\boldsymbol{\ell}}$ with $\hat{\boldsymbol{L}}$ in (9). Note that α, β and γ are well-defined if the initial separation of the two black holes is sufficiently large [*cf.* assumption *(ii)*] and that the error introduced by assumption *(iii)* in the measure of $\cos\alpha, \cos\beta$ and $\cos\gamma$ is also of the order of $|\boldsymbol{J}^{\perp}_{\rm rad}|/|\tilde{\boldsymbol{\ell}}|$.

The angle $\theta_{\rm fin}$ between the final spin vector and the initial orbital angular momentum can be easily calculated from $|\boldsymbol{a}_{\rm fin}|$. Because of assumption *(iii)*, the component of the final spin in the direction of $\boldsymbol{L}$ is [*cf.* eq. (7)]

$$a^{\parallel}_{\rm fin} \equiv \boldsymbol{a}_{\rm fin}\cdot\hat{\boldsymbol{\ell}} = \frac{|\boldsymbol{a}_1|\cos\beta + |\boldsymbol{a}_2|q^2\cos\gamma + |\boldsymbol{\ell}|q}{(1+q)^2}\,, \tag{10}$$

so that $\cos\theta_{\rm fin} = a^{\parallel}_{\rm fin}/|\boldsymbol{a}_{\rm fin}|$, and the component orthogonal to the initial orbital angular momentum is $a^{\perp}_{\rm fin} = |\boldsymbol{a}_{\rm fin}|\sin\theta_{\rm fin}$.

In essence, therefore, our approach consists of considering the dimensionless spin vector of the final black hole as the sum of the two initial spins and of a third vector parallel to the initial orbital angular momentum when the binaries are widely separated. Implicit in the assumptions made, and in the logic of mapping an initial-state of the binary into a final one, is the expectation that the length of this vector is an intrinsic "property" of the binary, depending on the initial spin vectors and mass ratio, but not on the initial separation. This is indeed a consequence of assumption *(ii)*: because the vector $\tilde{\boldsymbol{\ell}}$ measures the orbital angular momentum that cannot be radiated, it can be thought of as the angular momentum of the binary at the "effective" ISCO and, as such, it cannot be dependent on the initial separation.

A very important consequence of our assumptions is that $\boldsymbol{a}_{\rm fin}$ for a black-hole binary is already fully determined by the set of coefficients s_4, s_5, t_0, t_2, t_3 computed to derive expression (2). The latter, in fact, is simply the final spin for a special set of values for the cosine angles; since the fitting coefficients are constant, they must hold also for generic binaries.

In view of this, all that is needed is to measure $|\boldsymbol{\ell}|$ in terms of the fitting coefficients computed in refs. [11, 13]. This can be done by matching expression (10) with (2) [with the condition (4)] for parallel and aligned spins ($\alpha = \beta = \gamma = 0$), for parallel and antialigned spins ($\alpha = 0, \beta = \gamma = \pi$), and for antiparallel spins which are aligned or antialigned ($\alpha = \beta = \pi$, $\gamma = 0$ or $\alpha = \gamma = \pi$, $\beta = 0$). This matching is not unique, but the degeneracy can be broken by exploiting assumption *(iv)* and by requiring that $|\boldsymbol{\ell}|$ depends linearly on $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. We therefore obtain

$$|\boldsymbol{\ell}| = \frac{s_4}{(1+q^2)^2}\left(|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 q^4 + 2|\boldsymbol{a}_1||\boldsymbol{a}_2|q^2\cos\alpha\right) + \left(\frac{s_5\nu + t_0 + 2}{1+q^2}\right)\left(|\boldsymbol{a}_1|\cos\beta + |\boldsymbol{a}_2|q^2\cos\gamma\right) + 2\sqrt{3} + t_2\nu + t_3\nu^2\,. \tag{11}$$

We now consider some limits of expressions (8) and (11). First of all, when $q \to 0$, (8) and (11) yield the correct EMRL, *i.e.*, $|\boldsymbol{a}_{\rm fin}| = |\boldsymbol{a}_1|$. Secondly, for equal-mass binaries having spins that are equal and antiparallel, (8) and (11) reduce to

$$|\boldsymbol{a}_{\rm fin}| = \frac{|\boldsymbol{\ell}|}{4} = \frac{\sqrt{3}}{2} + \frac{t_2}{16} + \frac{t_3}{64} = p_0 \simeq 0.687\,. \tag{12}$$

This result allows us now to qualify more precisely a comment made before: because for equal-mass black holes which are either nonspinning or have equal and opposite spins, the vector $|\boldsymbol{\ell}|$ does not depend on the initial spins, expression (12) states that $|\boldsymbol{\ell}|M^2_{\rm fin}/4 = |\boldsymbol{\ell}|M^2/4 = |\boldsymbol{\ell}|M_1M_2$ is, for such systems, the orbital angular momentum at the effective ISCO. We can take this a step further and conjecture that $|\boldsymbol{\ell}|M_1M_2 = |\tilde{\boldsymbol{\ell}}|$ is the series expansion of the dimensionless orbital angular momentum at the ISCO also for *unequal-mass* binaries which are either nonspinning or with equal and opposite spins. The zeroth-order term of this series (namely, the term $2\sqrt{3}M_1M_2$) is exactly the one predicted from the EMRL. We note that although numerical simulations do not reveal the presence of an ISCO, the concept of an effective ISCO can nevertheless be useful for the construction of gravitational-wave templates [22, 23].

Finally, we consider the case of equal, parallel and aligned/antialigned spins ($|\boldsymbol{a}_2| = |\boldsymbol{a}_1|, \alpha = 0, \beta = \gamma = 0,\, \pi$), for which expressions (10) and (11) become

$$a_{\rm fin} \;=\; |\boldsymbol{a}_1|\cos\beta\left[1 + \nu(s_4|\boldsymbol{a}_1|\cos\beta + t_0 + s_5\nu)\right] + \nu(2\sqrt{3} + t_2\nu + t_3\nu^2)\,, \tag{13}$$

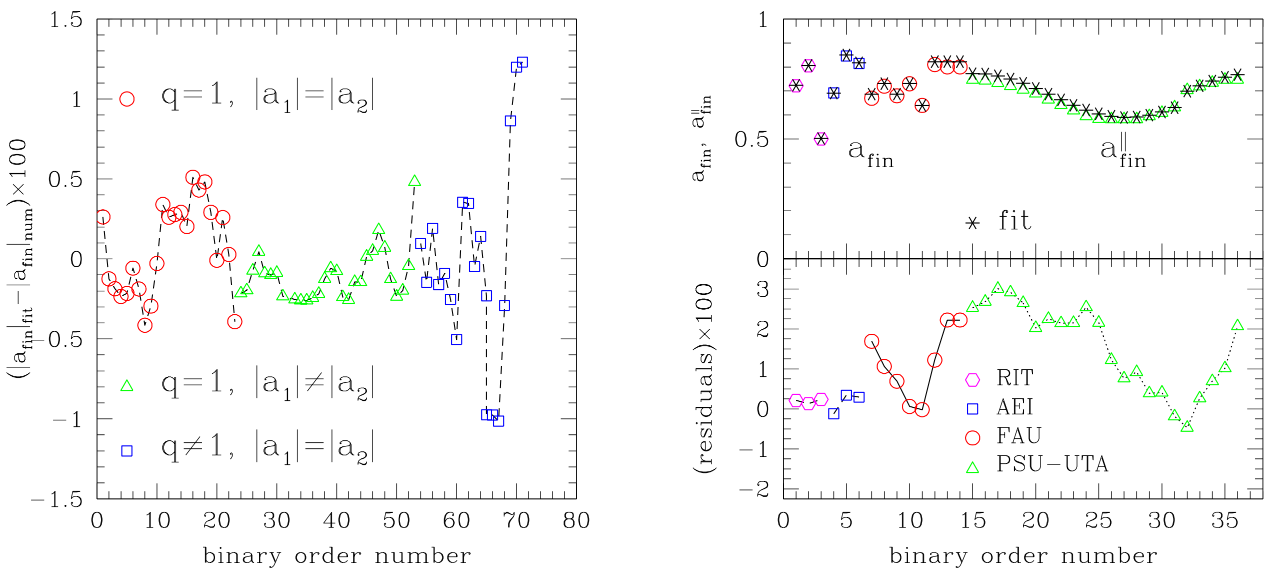

Figure 1: *Left panel:* Rescaled residual for aligned binaries. The circles refer to equal-mass, equal-spin binaries presented in refs. [11, 12, 13, 17, 18, 19], triangles to equal-mass, unequal-spin binaries presented in ref. [11, 18], and squares to unequal-mass, equal-spin binaries presented in refs. [13, 17, 18, 19]. Here and in the right panel the "binary order number" is just a dummy index labelling the different configurations. *Right panel:* The top part reports with asterisks the final spin computed for misaligned binaries. Hexagons refer to data from ref. [8] (labelled "RIT"), squares to the data Table I (labelled "AEI"), circles to data from ref. [20] (labelled "FAU"), and triangles to data from ref. [21] (labelled "PSU-UTA"). Note that these latter data points refer to the aligned component $a^{\parallel}_{\rm fin}$ since this is the only component available from ref. [21]. The bottom part of this panel shows instead the rescaled residuals for these misaligned binaries.

a^x_1	a^y_1	a^z_1	a^x_2	a^y_2	a^z_2	ν	$\vert\boldsymbol{a}_{\rm fin}\vert$	$\theta_{\rm fin}(^\circ)$
0.151	0.000	-0.563	0.000	0.000	0.583	0.250	0.692	2.29
0.151	0.000	0.564	0.000	0.151	0.564	0.250	0.846	3.97
0.413	0.000	0.413	0.000	0.413	0.413	0.250	0.815	7.86

Table I: Initial parameters of the new misaligned AEI binaries.

where $\cos\beta = \pm 1$ for aligned/antialigned spins. As expected, expression (13) coincides with (2) when $|\boldsymbol{a}_1|\cos\beta = a$ and with (1) [through the coefficients (3)] when $q = 1$ and $2|\boldsymbol{a}_1|\cos\beta = a_1 + a_2$. Similarly, (10) and (11) reduce to (2) for equal, antiparallel and aligned/antialigned spins ($|\boldsymbol{a}_2| = |\boldsymbol{a}_1|$, $\alpha = 0$, $\beta = 0, \gamma = \pi$, or $\beta = \pi, \gamma = 0$).

The only way to assess the validity of expressions (8) and (11) is to compare their predictions with the numerical-relativity data. This is done in Figs. 1 and 2, which collect all of the published data, together with the three additional binaries computed with the `CCATIE` code [9] and reported in Table I. In these plots, the "binary order number" is just a dummy index labelling the different configurations. The left panel of Fig. 1, in particular, shows the rescaled residual, *i.e.*, $(|\boldsymbol{a}_{\rm fin}|_{\rm fit} - |\boldsymbol{a}_{\rm fin}|_{\rm num.}) \times 100$, for aligned binaries. The plot shows the numerical-relativity data with circles referring to equal-mass, equal-spin binaries from refs. [11, 12, 13, 17, 18, 19], triangles to equal-mass, unequal-spin binaries from refs. [11, 18], and squares to unequal-mass, equal-spin binaries from refs. [13, 17, 18, 19]. Although the data is from simulations with different truncation errors, the residuals are all very small and with a scatter of $\sim 1\%$.

A more stringent test is shown in the right panel of Fig. 1, which refers to misaligned binaries. In the top part, hexagons indicate the numerical values for $|\boldsymbol{a}_{\rm fin}|$ from ref. [8], squares the ones in Table I, circles those from ref. [20] and triangles those from ref. [21]; note that these latter data points refer to the aligned component $a^{\parallel}_{\rm fin}$ since this is the only component available from ref. [21]. The agreement is again very good, with errors of a couple of percent (see bottom part of the same panel), even if the binaries are generic and for some the initial and final spins differ by almost 180° [8].

Finally, Fig. 2 reports the angle between the final spin vector and the initial orbital angular momentum $\theta_{\rm fin}$ using the same data (and convention for the symbols) as in the right panel of Fig. 1. Measuring the final angle accurately is not trivial, particularly due to the fact that the numerical evolutions start at a finite separation which does not account for earlier evolution of the orbital angular momentum vector. The values reported in [8] (and the relative error-bars) are shown with hexagons, while the squares refer to the binaries in Table I, and have been computed using a new approach for the calculation of the Ricci scalar on the apparent horizon [24]. Shown with asterisks and circles are instead the values predicted for the numerical data (as taken from refs. [8, 20, 21] and from Table I) by our analytic fit (asterisks) and by the point-particle approach suggested in ref. [5] (circles).

Clearly, when a comparison with numerical data is possible, the estimates of our fit are in reasonable agreement with the data and yield residuals in the final angle (*i.e.*, $(\theta_{\rm fin})_{\rm fit} - (\theta_{\rm fin})_{\rm num.}$) which are generally smaller than those obtained with the point-particle approach of ref. [5]. However, for two of the three binaries from ref. [8] the estimates are slightly outside the error-bars. Note that the reported angles are relative to the orbital plane at a small initial binary-

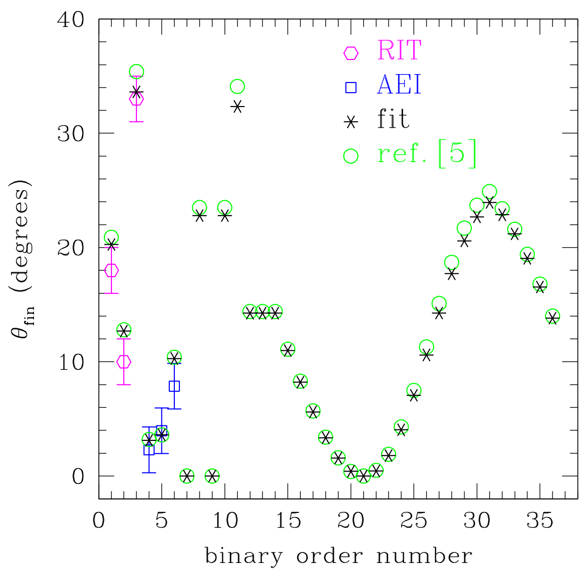

Figure 2: Using the same data (and convention for the symbols) as in the right panel of Fig. 1, we here report the angle between the final spin vector and the initial orbital angular momentum $\theta_{\rm fin}$. Shown instead with asterisks and circles are the values predicted for the numerical data (as taken from refs. [8, 20, 21] and from Table I) by our analytic fit (asterisks) and by the point-particle approach suggested in ref. [5] (circles).

separation, and thus are likely to be underestimates as they do not take into account the evolution from asymptotic distances; work is in progress to clarify this. When the comparison with the numerical data is not possible because $\theta_{\rm fin}$ is not reported (as for the data in ref. [21]), our approach and the one in ref. [5] yield very similar estimates.

In summary: we have considered the spin vector of the black hole produced by a black-hole binary merger as the sum of the two initial spins and of a third vector, parallel to the initial orbital angular momentum, whose norm depends only on the initial spin vectors and mass ratio, and measures the orbital angular momentum not radiated. Without additional fits than those already available to model aligned/antialigned binaries, we have measured the unknown vector and derived a formula that accounts therefore for all of the 7 parameters describing a black-hole binary inspiralling in quasi-circular orbits. The equations (8) and (11), encapsulate the near-zone physics to provide a convenient, but also robust and accurate over a wide range of parameters, determination of the merger product of rather generic black-hole binaries.

Testing the formula against all of the available numerical data has revealed differences between the predicted and the simulated values of a few percent at most. Our approach is intrinsically approximate and it has been validated on a small set of configurations, but it can be improved, for instance: by reducing the χ^2 of the fitting coefficients as new simulations are carried out; by using fitting functions that are of higher-order than those in expressions (1) and (2); by estimating $\boldsymbol{J}^{\perp}_{\rm rad}$ through PN expressions or by measuring it via numerical simulations.

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