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Wrapped M5-branes, consistent truncations and AdS/CMT

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ABSTRACT: At the level of the bosonic fields, we construct consistent Kaluza-Klein reductions of $D = 11$ supergravity on $\Sigma_3 \times S^4$, where $\Sigma_3 = H^3/\Gamma, S^3/\Gamma$ or R^3/Γ where Γ is a discrete group of isometries. The result is the bosonic content of an $N = 2$ $D = 4$ gauged supergravity with a single vector multiplet and two hypermultiplets, whose scalar fields parametrise $SU(1,1)/U(1) \times G_{2(2)}/SO(4)$. When $\Sigma_3 = H^3/\Gamma$ the $D = 4$ theory has an AdS_4 vacuum which uplifts to the known supersymmetric $AdS_4 \times H^3/\Gamma \times S^4$ solution of $D = 11$ supergravity that describes the $N = 2$ $d = 3$ SCFT arising when M5-branes wrap SLAG 3-cycles H^3/Γ in Calabi-Yau three-folds. We use the KK reduction for $\Sigma_3 = H^3/\Gamma$ to construct $D = 11$ black hole solutions that describe these $d = 3$ SCFTs at finite temperature and charge density and show that there is a superconducting instability involving a charged scalar field, and another instability involving neutral fields including both scalar and vector fields. We also use this KK reduction to construct a $D = 11$ Lifshitz solution that is dual to a $d = 3$ field theory with dynamical exponent $z \sim 39$.

KEYWORDS: Holography and condensed matter physics (AdS/CMT), Black Holes in String Theory, AdS-CFT Correspondence, M-Theory

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1 Introduction

The quantum field theories arising on M5-branes are an interesting prediction of string/M-theory. While there are many aspects of these field theories that are still poorly understood, in the limit of a large number of M5-branes they have a good description in terms of $D = 11$ supergravity via the AdS/CFT correspondence. In the simplest setting of coincident planar M5-branes, there is a $d = 6$ maximally supersymmetric CFT which is holographically dual to $AdS_7 \times S^4$.

The AdS/CFT correspondence can also be used to study the supersymmetric field theories arising when M5-branes wrap supersymmetric cycles. Recall that a probe M5-brane can wrap a calibrated-cycle Σ_p in special holonomy manifolds and preserve supersymmetry. On length scales much larger than that of the characteristic size of Σ_p we expect a decoupled lower-dimensional supersymmetric field theory on the unwrapped part of the M5-brane. In special circumstances the AdS/CFT correspondence can again be used to analyse these field theories as first discussed by Maldacena and Nunez [1]. In particular, for certain Σ_p , solutions of $D = 11$ supergravity can be constructed that describe a holographic flow “across-dimensions” from the M5-brane field theory on $\mathbb{R}^{1,5-p} \times \Sigma_p$ (with suitable R -symmetry currents switched on) down to a $d = 6-p$ SCFT field theory on $\mathbb{R}^{1,5-p}$ which is dual to an $AdS_{7-p} \times \Sigma_p \times S^4$ solution of $D = 11$ supergravity (suitably warped and twisted).¹ Such solutions describing M5-branes wrapping holomorphic 2-cycles were considered in [1] and generalised to M5-branes wrapping other calibrated cycles in [2]–[4] (see [5] for a review). In all cases the solutions were first constructed in $D = 7$ gauged supergravity and then uplifted to $D = 11$ on an S^4 .

The example that is of most interest to this paper is when M5-branes wrap special Lagrangian (S Lag) 3-cycles in Calabi-Yau three-folds. In this case, at large distances, one expects a $d = 3$ quantum field theory with $N = 2$ supersymmetry. When $\Sigma = H^3/\Gamma$, where H^3 is hyperbolic three-space and Γ is a freely acting discrete group of isometries (allowing H^3/Γ to be compact), these $d = 3$ field theories are $N = 2$ SCFTs and are dual to solutions of $D = 11$ supergravity of the form $AdS_4 \times H^3/\Gamma \times S^4$, with the S^4 appropriately fibred over H^3/Γ . This was shown in [3] where the dual holographic solutions, including the flow across dimensions, were constructed. Note that when $\Sigma = S^3/\Gamma$, analogous holographic

¹Note that for some Σ_p there are also solutions which do not flow to AdS solutions in the IR as we will shortly recall.

solutions describing the flow across dimensions were also constructed in [3]. However, for this case there is not an analogous $AdS_4 \times S^3 \times S^4$ solution in the IR (instead one finds a singularity) and hence the nature of the $d = 3$ quantum field theory in the far IR for this case is not clear.

In this paper we will be particularly interested in further studying the $d = 3$ SCFTs arising on M5-branes wrapping H^3/Γ , using holographic techniques. A primary motivation is that these theories provide a novel arena for top-down investigations of AdS/CMT. In particular, we will initiate an investigation of the properties of the $d = 3$ $N = 2$ SCFTs at finite temperature and finite charge density (with respect to the abelian R -symmetry), by constructing and analysing appropriate black hole solutions of $D = 11$ supergravity. We show that the high temperature behaviour is described by an (uplifted) AdS-RN type black hole. One interesting question is whether or not the SCFTs exhibit holographic superconductivity [6]–[8] as found in other top-down supergravity settings using consistent Kaluza-Klein (KK) truncations [9]–[13]. We will find two instabilities. The first of these involves charged fields implying that there is a new branch of holographic superconducting black holes with charged hair that spontaneously breaks the R -symmetry, which emerge from the AdS-RN black holes at a branching temperature that we determine. The second instability only involves neutral fields and implies the existence of another branch of charged black holes with neutral hair that do not spontaneously break the R -symmetry. Instabilities involving a single neutral scalar field in the background of an AdS-RN black hole were observed in a bottom up context in [8] and arose because the scalar field has a mass that violates the AdS_2 BF bound but not the AdS_4 BF bound. In our case the situation is more complicated involving two neutral fields and a massive vector field and the instability depends on the detailed couplings including the couplings to the background abelian two-form field strength. This latter feature indicates that there is some similarity with the bottom-up charged dilaton black holes studied in [14]. We will show that the branching temperature for our new charged black holes with neutral scalar and massive vector hair is greater than that of the superconducting black holes.

Our results are therefore suggestive that as one cools the $N = 2$ SCFT at finite charge density the system will undergo a phase transition, moving to a phase described by the new charged black holes with neutral hair. However, there are two important caveats. Firstly, as usual, there could be additional branches of black holes, either inside or outside² the $D = 4$ consistent KK truncation that give rise to a phase transition at even higher temperature. Secondly, the conclusion depends on the order of the two phase transitions since if a phase transition is first order then the critical temperature can be higher than the branching

²As an example, additional instabilities appearing outside of the $D = 4$ KK truncation used to construct holographic superconductors (for the special case of the seven-sphere) in [12, 13] were studied in [15]. It will be interesting to determine the implications of these instabilities, as well as those found in [9], for the phase structure for this case.

temperature. In the present context, it is therefore possible that the superconducting black hole transition is first order and the system moves, discontinuously, from the AdS-RN branch to the superconducting branch at a higher temperature than the critical temperature associated with the charged black holes with neutral hair. We will leave a resolution of this interesting issue to future work.

As with several other top down studies of AdS/CMT we will carry out these investigations using (new) consistent Kaluza-Klein truncations of $D = 11$ supergravity. Recall that such truncations have the key property that the truncated dimensionally reduced theory does not source any of the discarded modes and hence any solution of the reduced theory uplifts to an exact solution of the higher-dimensional theory. There has been significant progress in understanding these truncations over the past few years. For example, it is known that starting with the most general class of supersymmetric AdS_5 solutions of Type IIB or $D = 11$ supergravity there are consistent reductions on the internal manifolds to minimal $N = 2$ $D = 5$ gauged supergravity [16–18]. Similarly, it has been shown for very general classes of supersymmetric AdS_4 solutions (but not yet the most general) that there are analogous reductions to minimal $N = 2$ $D = 4$ gauged supergravity [18]. Building on the work of [19], for the special case of reductions of $D = 11$ supergravity on seven-dimensional Sasaki-Einstein spaces (SE_7), it has been shown that the reductions can be extended to include modes that fill out the bosonic part of an $N = 2$ gauged supergravity coupled to a vector multiplet and a hypermultiplet [10]. Similar results have also been obtained for reductions of Type IIB on SE_5 [20–22] where an interesting enhancement of supersymmetry from $N = 2$ to $N = 4$ was observed (see also [23]). Another development is the addition of the quadratic fermionic sectors to these universal SE truncations [24, 25]. More general truncations for the special case that $SE_5 = T^{1,1}$ have been made in [26–28] and recent results for S^5 and S^7 have been obtained in [29, 30] and [15], respectively.

Here we will construct new consistent KK truncations of $D = 11$ supergravity on $\Sigma_3 \times S^4$ where $\Sigma_3 = H^3, S^3$ or R^3 (or a quotient thereof). We will do this in two steps, generalising the work of [31]. We first use the well known consistent truncation of $D = 11$ supergravity on S^4 to obtain maximal $SO(5)$ gauged supergravity in $D = 7$ [32, 33]. We then reduce this $D = 7$ gauged supergravity on the above Σ_3 to obtain $D = 4$ $N = 2$ gauged supergravities. More precisely, these KK reductions are at the level of the bosonic fields and we find, for each case, the bosonic content of an $N = 2$ gauged supergravity coupled to a single vector multiplet plus two hypermultiplets. The scalars in the vector multiplet parametrise the special Kähler manifold $SU(1,1)/U(1)$, while the scalars in the hypermultiplets parametrise the quaternionic Kähler space $G_{2(2)}/SO(4)$. We find that the gauging of the $N = 2$ supersymmetry is only in the hypermultiplet sector and we find that a $U(1) \times \mathbb{R} \subset G_{2(2)}$ is gauged.

We will also use the new consistent truncations to investigate another interesting issue in AdS/CMT: top down solutions of $D = 11$ supergravity that are dual to field theories

with Lifshitz symmetry. In [34] a class of $d + 1$ -dimensional metrics of the form

$$ds^2 = -r^{2z} dt^2 + r^2 dx^i dx^i + \frac{dr^2}{r^2}, \quad i = 1, \dots, d - 1 \tag{1.1}$$

were proposed to be holographically dual to d -dimensional field theories with anisotropic Lifshitz scaling and dynamical exponent z . It was also shown in [34] that these $\text{Lif}_{d+1}(z)$ solutions arise as solutions of a bottom up $d + 1$ -dimensional phenomenological theory of gravity (see also [35]). Somewhat surprisingly, it has been very difficult to embed these solutions into string/M-theory. However, $\text{Lif}_4(z = 2)$ solutions of type IIB and $\text{Lif}_3(z = 2)$ solutions of $D = 11$ supergravity were recently constructed in [36] and these were significantly extended in [37] where supersymmetric solutions were also presented (which should be stable). Here, using our new consistent truncations, we will construct³ a new $\text{Lif}_4(z) \times H^3/\Gamma \times S^4$ solution of $D = 11$ supergravity with dynamical exponent $z = 39.05\dots$ (determined by solving some algebraic equations numerically).

The plan of the rest of the paper is as follows. In section 2, we first briefly review the consistent KK truncation of $D = 11$ supergravity on S^4 to $D = 7$ $\text{SO}(5)$ gauged supergravity [32, 33] using the presentation of [39]. In section 3 we construct the consistent truncation of $D = 7$ $\text{SO}(5)$ gauged supergravity on H^3, S^3, R^3 (or a quotient by Γ thereof). In section 4 we show that the $D = 4$ truncated theory is the bosonic part of an $N = 2$ gauged supergravity, coupled to a vector multiplet and two hypermultiplets and we elucidate the gauging. The natural degrees of freedom required to exhibit the $N = 2$ supersymmetry in section 4 require some dualisation of the degrees of freedom that naturally appear in the uplifting formulae given in section 3. We emphasise that apart from section 4, we use the variables given in section 3.

In section 5 we recall the supersymmetric AdS_4 solution of the $D = 4$ reduced theory (for the case of H^3/Γ) which uplifts to the $AdS_4 \times H^3/\Gamma \times S^4$ solution dual to the $N = 2$ SCFT on the wrapped M5-branes. Within the $D = 4$ theory we analyse the linearised spectrum of fluctuations and show how they correspond to $OSp(2|4)$ multiplets of operators in the dual SCFT. The reduced $D = 4$ theory has another non-supersymmetric AdS_4 vacuum which uplifts to a non-supersymmetric $AdS_4 \times H^3/\Gamma \times S^4$ solution of $D = 11$ supergravity first found in [31]. For this solution we also analyse the mass spectrum and find that within the $D = 4$ truncation there are no unstable modes. Section 6 briefly considers some additional truncations of the $D = 4$ theory.

In section 7 we switch gears and study the $N = 2$ SCFT, dual to the supersymmetric $AdS_4 \times H^3/\Gamma \times S^4$ solution, at finite temperature and finite chemical potential. At high temperatures the system is described by an uplifted AdS-RN type black hole with flat spatial horizon (often called a black brane). We show that at zero temperature there are two

³Note that this solution was constructed prior to those in [37] and was announced by one of us (JPG) at the Non-Perturbative Techniques in Field Theory Symposium in Durham, July 2010. Just prior to the submission of this paper, other constructions of Lifshitz solutions were presented in [38].

kinds of instabilities, one of which involves charged fields and is associated with holographic superconductivity and the other just involves neutral fields. By studying the fluctuations about the AdS-RN black hole at finite temperature we then deduce the temperatures at which the two new branches of black holes appear, finding that the non-superconducting charged black holes with neutral hair have a higher branching temperature than the superconducting black holes.

In section 8 we construct the $\text{Lif}_4(z \sim 39) \times H^3/\Gamma \times S^4$ solution of $D = 11$ supergravity. We conclude with some discussion in section 9. The paper contains two appendices: in appendix A we have presented some details of the consistent KK truncation, including the full set of $D = 4$ equations of motion, given in (A.11)–(A.24), that are used throughout this paper. In appendix B we have made some comments concerning an unconventional presentation of a massive vector field that emerges in our truncation. In appendix C we have recorded some details on the coset $G_{2(2)}/\text{SO}(4)$ which we use in elucidating the $N = 2$ supersymmetry of the reduced $D = 4$ theory.

2 Maximal $D = 7$ gauged supergravity and uplift to $D = 11$ supergravity on S^4

The bosonic fields of $D = 7$ gauged supergravity [40] consist of a metric, g_7 , $\text{SO}(5)$ Yang-Mills fields A^{ij} , $i, j = 1, \dots, 5$, five three-forms $S_{(3)}^i$ transforming in the $\mathbf{5}$ of $\text{SO}(5)$ and fourteen scalar fields, given by the symmetric unimodular matrix T_{ij} , which parametrise the coset $\text{SL}(5, \mathbb{R})/\text{SO}(5)$. The seven-form Lagrangian for the bosonic fields is given by

$$\begin{aligned} \mathcal{L}_7 = & R * \mathbb{1} - \frac{1}{4} T_{ij}^{-1} * DT_{jk} \wedge T_{k\ell}^{-1} DT_{\ell i} - \frac{1}{4} T_{ik}^{-1} T_{j\ell}^{-1} * F_{(2)}^{ij} \wedge F_{(2)}^{k\ell} - \frac{1}{2} T_{ij} * S_{(3)}^i \wedge S_{(3)}^j \\ & + \frac{1}{2g} S_{(3)}^i \wedge DS_{(3)}^i - \frac{1}{8g} \epsilon_{ij_1 \dots j_4} S_{(3)}^i \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} + \frac{1}{g} \Omega_{(7)} - V * \mathbb{1}, \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} DT_{ij} & \equiv dT_{ij} + g A_{(1)}^{ik} T_{kj} + g A_{(1)}^{jk} T_{ik} \\ DS_{(3)}^i & \equiv dS_{(3)}^i + g A_{(1)}^{ij} \wedge S_{(3)}^j \\ F_{(2)}^{ij} & \equiv dA_{(1)}^{ij} + g A_{(1)}^{ik} \wedge A_{(1)}^{kj}, \end{aligned} \quad (2.2)$$

the potential V is given by

$$V = \frac{1}{2} g^2 \left(2T_{ij} T_{ij} - (T_{ii})^2 \right), \quad (2.3)$$

and $\Omega_{(7)}$ is a Chern-Simons type of term built from the Yang-Mills fields, which has the property that its variation with respect to $A_{(1)}^{ij}$ gives

$$\delta \Omega_{(7)} = \frac{3}{4} \delta_{i_1 i_2 k \ell}^{j_1 j_2 j_3 j_4} F_{(2)}^{i_1 i_2} \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} \wedge \delta A_{(1)}^{k \ell}. \quad (2.4)$$

An explicit expression can be found in [40].

Any solution to the associated $D = 7$ equations of motion, which are given in appendix A, gives rise to a solution of $D = 11$ supergravity [32, 33]. Using the notation of [39], the $D = 11$ metric and four-form field strength are given by

$$ds_{11}^2 = \Delta^{1/3} ds_7^2 + \frac{1}{g^2} \Delta^{-2/3} T_{ij}^{-1} D\mu^i D\mu^j, \quad (2.5)$$

$$G_{(4)} = \frac{\Delta^{-2}}{g^3 4!} \epsilon_{i_1 \dots i_5} \left[-U \mu^{i_1} D\mu^{i_2} \wedge D\mu^{i_3} \wedge D\mu^{i_4} \wedge D\mu^{i_5} \right. \\ \left. + 4 T^{i_1 m} D T^{i_2 n} \mu^m \mu^n D\mu^{i_3} \wedge D\mu^{i_4} \wedge D\mu^{i_5} + 6g \Delta F_{(2)}^{i_1 i_2} \wedge D\mu^{i_3} \wedge D\mu^{i_4} T^{i_5 j} \mu^j \right] \\ - T_{ij} * S_{(3)}^i \mu^j + \frac{1}{g} S_{(3)}^i \wedge D\mu^i, \quad (2.6)$$

where μ^i , $i = 1, \dots, 5$ are constrained coordinates on S^4 satisfying $\mu^i \mu^i \equiv 1$, and

$$U \equiv 2T_{ij} T_{jk} \mu^i \mu^k - \Delta T_{ii}, \quad \Delta \equiv T_{ij} \mu^i \mu^j, \quad D\mu^i \equiv d\mu^i + g A_{(1)}^{ij} \mu^j. \quad (2.7)$$

For example, the basic AdS_7 vacuum solution of $D = 7$ supergravity, with $A_{(1)}^{ij} = S_{(3)}^i = 0$ and $T_{ij} = \delta_{ij}$ uplifts to the maximally supersymmetric $AdS_7 \times S^4$ solution. Of more interest to this paper is the supersymmetric $AdS_4 \times H^3$ solution found in [3]. This solution uplifts to an $AdS_4 \times H^3 \times S^4$ solution, with a warped product metric and the S^4 non-trivially fibred over the H^3 factor. The solution preserves eight supercharges and the H^3 factor can be replaced with an arbitrary quotient H^3/Γ , possibly compact, and still preserve all supersymmetry. When H^3/Γ is compact these solutions are dual to $N = 2$ superconformal field theories in three spacetime dimensions that arise on the non-compact part of fivebranes wrapping special Lagrangian three-cycles H^3/Γ . We will recall this solution in section 5 below.

It is worth pointing out that the conventions for $D = 7$ gauged supergravity used in [39] and in this paper, slightly differ from those used in [40], which were also used in [3]. In particular $g^{here} = m^{there}$ (and one should be careful since $g^{there} = 2m^{there}$) and also $A^{here} = 2A^{there}$.

3 Consistent KK truncation of $D = 7$ gauged supergravity on S^3 , H^3 or R^3

We now construct the consistent KK ansatz for the reduction of $D = 7$ supergravity on $\Sigma_3 = S^3, H^3$ or R^3 (or a quotient thereof), generalising that of [31]. For the $D = 7$ metric we take

$$ds_7^2 = e^{-6\phi} ds_4^2 + e^{4\phi} ds^2(\Sigma_3) \quad (3.1)$$

where ds_4^2 is an arbitrary metric on the $D = 4$ external spacetime, $ds^2(\Sigma_3)$ is the maximally symmetric metric on S^3 or H^3 or T^3 (or a quotient thereof) normalised so that the Ricci tensor is lg^2 times the metric, for $l = +1, -1$ or 0 respectively, and ϕ is a real “breathing mode” scalar field defined on the $D = 4$ external spacetime.

To construct the ansatz for the remaining fields we introduce an orthonormal frame, \bar{e}^a , for $ds^2(\Sigma_3)$, and let $\bar{\omega}^{ab}$ be the corresponding Levi-Civita spin connection:

$$d\bar{e}^a + \bar{\omega}^a{}_b \wedge \bar{e}^b = 0. \tag{3.2}$$

Then for the $D = 7$ $SO(5)$ vector fields, we consider an $SO(3) \times SO(2)$ split, with $a, b = 1, 2, 3$ and $\alpha, \beta = 4, 5$ and set

$$\begin{aligned} A_{(1)}^{ab} &= \frac{1}{g} \bar{\omega}^{ab} + \beta \epsilon_{abc} \bar{e}^c \\ A_{(1)}^{a\alpha} &= -A_{(1)}^{\alpha a} = \theta^\alpha \bar{e}^a \\ A_{(1)}^{\alpha\beta} &= \epsilon^{\alpha\beta} A_1. \end{aligned} \tag{3.3}$$

This ansatz incorporates a scalar field β , two scalar fields θ^α and a vector field A_1 , all defined on the $D = 4$ external spacetime. We note that we take $\epsilon_{45} = -\epsilon_{54} = 1$. Indices a and α are raised and lowered with δ_{ab} and $\delta_{\alpha\beta}$, respectively.

The split of $SO(5)$ into $SO(3) \times SO(2)$ is also used in the ansatz for the five $D = 7$ three-forms and the fourteen $D = 7$ scalars. Specifically, for the 3-form fields $S_{(3)}^i$ we take:

$$\begin{aligned} S_{(3)}^a &= B_2 \wedge \bar{e}^a + C_1 \wedge \epsilon_{abc} \bar{e}^b \wedge \bar{e}^c \\ S_{(3)}^\alpha &= h_3^\alpha + g \chi^\alpha \text{vol}(\Sigma_3). \end{aligned} \tag{3.4}$$

where $B_2, C_1, \chi^\alpha, h_3^\alpha$ are 2-, 1-, 0-, and 3-forms in $D = 4$, respectively. For the scalars T_{ij} parametrising the coset $SL(5, \mathbb{R})/SO(5)$ we choose:

$$T_{ab} = e^{-4\lambda} \delta_{ab}, \quad T_{a\alpha} = 0, \quad T_{\alpha\beta} = e^{6\lambda} \mathcal{T}_{\alpha\beta}, \tag{3.5}$$

where λ is a scalar and the symmetric, unimodular matrix $\mathcal{T}_{\alpha\beta}$, parametrises the coset $SL(2, \mathbb{R})/SO(2)$ (and thus contains two scalar degrees of freedom), all in $D = 4$.

In summary, the above ansatz incorporates the following $D = 4$ content: 9 scalars $\phi, \lambda, \mathcal{T}_{\alpha\beta}, \beta, \theta_\alpha, \chi_\alpha$, two one-forms A_1, C_1 , one two-form B_2 , two three-forms h_3^α , plus the metric. As we will see in the next section, after some field redefinitions, these arrange themselves into bosonic fields of the following $N = 2$ multiplets: a gravity multiplet (metric plus, a vector), a vector multiplet (vector plus two scalars) and two hypermultiplets (eight scalars).

We next substitute our ansatz into the $D = 7$ equations of motion. After some arduous calculation we find that they are equivalent to unconstrained equations of motion for the $D = 4$ fields, thus demonstrating the consistency of the ansatz. We have presented a few details in appendix A, and the equations of motion are given in (A.11)–(A.24). We have also verified that these equations of motion can all be derived from the $D = 4$ (four-form) Lagrangian given by

$$2\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{top}}, \tag{3.6}$$

where

$$\begin{aligned} \mathcal{L}_{\text{kin}} = & R^{(4)} \text{vol}_4 + 30d\phi \wedge *d\phi + 30d\lambda \wedge *d\lambda + \frac{1}{4} \text{Tr}(\mathcal{T}^{-1} D\mathcal{T} \wedge *\mathcal{T}^{-1} D\mathcal{T}) \\ & + \frac{3}{2} e^{8\lambda-4\phi} d\beta \wedge *d\beta + \frac{3}{2} e^{-2\lambda-4\phi} D\theta^T \wedge *\mathcal{T}^{-1} D\theta + \frac{1}{2} e^{6\lambda+12\phi} h_3^T \wedge *Th_3 \\ & - \frac{1}{2} e^{-12\lambda+6\phi} F_2 \wedge *F_2 - \frac{3}{2} e^{-4\lambda+2\phi} B_2 \wedge *B_2 + 6e^{-4\lambda-8\phi} C_1 \wedge *C_1, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \mathcal{L}_{\text{pot}} = & g^2 \left\{ 3l e^{-10\phi} - \frac{3}{8} e^{8\lambda-14\phi} (l - 2\beta^2 - 2\theta^T \theta)^2 \right. \\ & + \frac{1}{2} e^{-6\phi} \left[3e^{-8\lambda} + e^{12\lambda} [(\text{Tr}\mathcal{T})^2 - 2\text{Tr}(\mathcal{T}\mathcal{T})] + 6e^{2\lambda} \text{Tr}\mathcal{T} \right] \\ & - \frac{3}{2} e^{-10\phi} \left[e^{10\lambda} (\theta^T \mathcal{T} \theta) - 2\theta^T \theta + e^{-10\lambda} (\theta^T \mathcal{T}^{-1} \theta) \right] \\ & \left. - 6e^{-2\lambda-14\phi} \beta^2 (\theta^T \mathcal{T}^{-1} \theta) - \frac{1}{2} e^{6\lambda-18\phi} (\chi^T \mathcal{T} \chi) \right\} \text{vol}_4 \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} g \mathcal{L}_{\text{top}} = & 6 C_1 \wedge (dB_2 - g\theta^T h_3) - 6 C_1 \wedge d\beta \wedge F_2 - 3g\beta B_2 \wedge B_2 \\ & - 3 B_2 \wedge D\theta^T \wedge \varepsilon D\theta + g\chi^T Dh_3 - 6g\beta h_3^T \varepsilon \theta \wedge d\beta \\ & + 2g \beta^3 F_2 \wedge F_2 + 3\beta D\theta^T \wedge \varepsilon D\theta \wedge F_2 \\ & + \frac{g}{2} (l - 2\beta^2 - 2\theta^T \theta) [-3 B_2 \wedge F_2 + 3 h_3^T \wedge \varepsilon D\theta + 3\beta F_2 \wedge F_2]. \end{aligned} \quad (3.9)$$

In these expressions we have defined

$$F_2 = dA_1 \quad (3.10)$$

and we have introduced the SO(2)-covariant differential D which acts on SO(2) doublets, X^α , as

$$DX^\alpha = dX^\alpha + g\epsilon_{\alpha\beta} A_1 \wedge X^\beta \quad (3.11)$$

and on the coset scalars as

$$D\mathcal{T}^{\alpha\beta} = d\mathcal{T}^{\alpha\beta} + g\epsilon_{\alpha\gamma} A_1 \mathcal{T}^{\gamma\beta} + g\epsilon_{\beta\gamma} A_1 \mathcal{T}^{\alpha\gamma}. \quad (3.12)$$

The $D = 4$ Lagrangian (3.6) is locally SO(2) invariant.

We note that upon setting $\theta = \beta = \chi = 0$ and $\mathcal{T}_{\alpha\beta} = \delta_{\alpha\beta}$ we obtain the consistent truncation studied in [31]. In particular, \mathcal{L} agrees with (2.12), (2.13) of [31]. Additional consistent KK truncations will be considered in section 6.

In the next section we will demonstrate that this $D = 4$ reduced theory comprises the bosonic sector of an $N = 2$ $D = 4$ gauged supergravity coupled to a vector multiplet and two hypermultiplets. The factor of 2 appearing in (3.6) is incorporated to facilitate comparison with some standard conventions used in the $N = 2$ literature. We would like

to emphasise that to do so will require some field redefinitions. In particular, we will see that, essentially, B_2 and C_1 , which we observe do not have the usual kinetic energy terms in (3.6), will be replaced by a vector and a scalar, and the auxiliary three-forms h_3^α will be eliminated. We emphasise that in later sections when we construct new solutions of the $D = 4$ theory we will work in the variables given in (3.6)–(3.9) (i.e. we will analyse the equations of motion (A.11)–(A.24) as these variables are the easiest ones to uplift to $D = 11$. It is straightforward to translate to the standard $N = 2$ variables given in the next section. We have made some clarifying comments on the B_2, C_1 system in a simplified setting in appendix B.

4 Explicit $N = 2$ supersymmetry

4.1 New variables

In order to display the $N = 2$ supersymmetry we will need to change to new variables (see also appendix B). To begin with, we first perform the trivial relabelling

$$B_2 \rightarrow H_2, \quad C_1 \rightarrow G_1, \quad h_3^\alpha \rightarrow F_3^\alpha \quad (4.1)$$

with the capital letters H and F used to indicate that the objects are, or will become, field strengths, or dual field strengths in the new variables.

We continue by defining a new two-form field strength defined by

$$\tilde{H}_2 \equiv e^{-4\lambda+2\phi} * H_2 + 2\beta H_2 - \beta^2 F_2. \quad (4.2)$$

Some manipulation of equations (A.11), (A.12), (A.15), (A.16) allows us to write the following Bianchi identities

$$\begin{aligned} d\tilde{H}_2 &= 0, \\ dG_1 - \frac{1}{2}g\tilde{H}_2 - \frac{1}{4}g(l - 2\theta^T\theta)F_2 - \frac{1}{2}\epsilon_{\alpha\beta}D\theta^\alpha \wedge D\theta^\beta &= 0. \end{aligned} \quad (4.3)$$

These can be integrated to give

$$\tilde{H}_2 = d\tilde{B}_1, \quad (4.4)$$

$$G_1 = Da + \frac{1}{2}\epsilon_{\alpha\beta}\theta^\alpha D\theta^\beta, \quad (4.5)$$

where we have defined

$$Da \equiv da + \frac{1}{4}glA_1 + \frac{1}{2}g\tilde{B}_1. \quad (4.6)$$

Next, notice that equation (A.14) can be written as

$$e^{6\lambda+12\phi} * (TF_3)_\alpha = -D \left[\chi_\alpha + \frac{3}{2}\epsilon_{\alpha\beta}\theta^\beta(l - 2\beta^2 - 2\theta^T\theta) \right] - 6\theta_\alpha G_1 - 6\epsilon_{\alpha\beta}\theta^\beta\theta^\gamma D\theta_\gamma. \quad (4.7)$$

Introducing a scalar

$$\xi_\alpha \equiv -\chi_\alpha - \epsilon_{\alpha\beta}\theta^\beta \left(\frac{3}{2}l - 3\beta^2 - \theta^T\theta \right), \quad (4.8)$$

with one-form field strength

$$F_1^\alpha \equiv D\xi^\alpha - 6\theta^\alpha Da - \epsilon_{\gamma\delta}\theta^\alpha\theta^\gamma D\theta^\delta, \quad (4.9)$$

one can use (4.5) to show that (4.7) is equivalent to

$$F_3 = e^{-6\lambda-12\phi}\mathcal{T}^{-1} * F_1. \quad (4.10)$$

In terms of the new variables, we can identify the degrees of freedom of our theory as a metric, two vectors A_1, \tilde{B}_1 , and ten scalars $\phi, \lambda, \mathcal{T}_{\alpha\beta}, \beta, \theta_\alpha, a, \xi_\alpha$. The $D = 4$ field equations (A.11)–(A.24) can be translated to the new variables by using equations (4.8), (4.9), (4.10) and solving for H_2 from (4.2) as

$$H_2 = \frac{1}{4\beta^2 + e^{-8\lambda+4\phi}} \left[2\beta(\tilde{H}_2 + \beta^2 F_2) - e^{-4\lambda+2\phi} * (\tilde{H}_2 + \beta^2 F_2) \right]. \quad (4.11)$$

In order to write down the Lagrangian, it also proves convenient to define the two dilatons

$$\varphi_0 = -4\lambda + 2\phi, \quad \varphi_1 = 2\sqrt{3}(\lambda + 2\phi), \quad (4.12)$$

and the axion-dilaton

$$\tau = \beta + ie^{\varphi_0}. \quad (4.13)$$

The Lagrangian that gives rise to the equations of motion (A.11), (A.13), (A.16), (A.18)–(A.24) upon variation of $\tilde{B}_1, \xi^\alpha, a, \beta, \theta^\alpha, A_1, \lambda, \mathcal{T}_{\alpha\beta}, \phi$ and the metric, respectively, is

$$\mathcal{L} = \frac{1}{2}R^{(4)}\text{vol}_4 + \mathcal{L}_{\text{VM}} + \mathcal{L}_{\text{HM}} + \mathcal{L}_{\text{pot}}, \quad (4.14)$$

with

$$\begin{aligned} \mathcal{L}_{\text{VM}} = & \frac{3}{4}d\varphi_0 \wedge *d\varphi_0 + \frac{3}{4}e^{-2\varphi_0}d\beta \wedge *d\beta \\ & + \frac{3}{4}\text{Im} [(\tau + \beta)^{-1}] (\tilde{H}_2 + \beta^2 F_2) \wedge *(\tilde{H}_2 + \beta^2 F_2) \\ & + \frac{3}{4}\text{Re} [(\tau + \beta)^{-1}] (\tilde{H}_2 + \beta^2 F_2) \wedge (\tilde{H}_2 + \beta^2 F_2) \\ & - \frac{1}{4}e^{3\varphi_0} F_2 \wedge *F_2 - \frac{3}{2}\beta\tilde{H}_2 \wedge F_2 - \frac{1}{2}\beta^3 F_2 \wedge F_2, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \mathcal{L}_{\text{HM}} = & \frac{1}{4}d\varphi_1 \wedge *d\varphi_1 + \frac{1}{8}\text{Tr}(\mathcal{T}^{-1}D\mathcal{T} \wedge *\mathcal{T}^{-1}D\mathcal{T}) + \frac{3}{4}e^{-\frac{1}{\sqrt{3}}\varphi_1} D\theta^T \wedge *\mathcal{T}^{-1}D\theta \\ & + \frac{1}{4}e^{-\sqrt{3}\varphi_1} F_1^T \wedge *\mathcal{T}^{-1}F_1 + 3e^{-\frac{2}{\sqrt{3}}\varphi_1} G_1 \wedge *G_1, \end{aligned} \quad (4.16)$$

where G_1, F_1 are given in (4.5), (4.9) and

$$\begin{aligned} \mathcal{L}_{\text{pot}} = \frac{1}{2}g^2 \left\{ 3le^{-10\phi} - \frac{3}{8}e^{8\lambda-14\phi}(l - 2\beta^2 - 2\theta^T\theta)^2 \right. \\ \left. + \frac{1}{2}e^{-6\phi} \left[3e^{-8\lambda} + e^{12\lambda}[(\text{Tr}\mathcal{T})^2 - 2\text{Tr}(\mathcal{T}\mathcal{T})] + 6e^{2\lambda}\text{Tr}\mathcal{T} \right] \right. \\ \left. - \frac{3}{2}e^{-10\phi} \left[e^{10\lambda}(\theta^T\mathcal{T}\theta) - 2\theta^T\theta + e^{-10\lambda}(\theta^T\mathcal{T}^{-1}\theta) \right] \right. \\ \left. - 6e^{-2\lambda-14\phi}\beta^2(\theta^T\mathcal{T}^{-1}\theta) - \frac{1}{2}e^{6\lambda-18\phi}(\chi^T\mathcal{T}\chi) \right\} \text{vol}_4, \end{aligned} \quad (4.17)$$

where (4.8) should be used to write χ^α in terms of $\xi^\alpha, \theta^\alpha$ and β .

The $D = 4$ Lagrangian (4.14) has now a local $U(1) \times \mathbb{R}$ symmetry. Note, in particular, from (4.6) that a non-compact, local \mathbb{R} shift of the scalar field a can be cancelled by a gauge transformation of the vector $lA_1 + 2\tilde{B}_1$. Furthermore, the scalar potential (4.17) does not depend on a . Note that the Lagrangian (3.6), with its local $SO(2) \cong U(1)$ symmetry, corresponds, in part, to an \mathbb{R} -gauge-fixed version of (4.14).

4.2 $N = 2$ supersymmetry

We will now show that our $D = 4$ reduced theory corresponds to the bosonic part of $D = 4$ $N = 2$ supergravity coupled to a vector multiplet and two hypermultiplets, with an Abelian $U(1) \times \mathbb{R}$ gauging in the hypermultiplet sector. The scalar manifold is the symmetric space

$$\frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{G_{2(2)}}{\text{SO}(4)}, \quad (4.18)$$

where the first factor is the special Kähler manifold parametrised by the two real scalars in the vector multiplet, and the second factor is the quaternionic-Kähler manifold parametrised by the eight real scalars in the hypermultiplets.

We will show this by casting the Lagrangian (4.14)–(4.17) into the canonical $N = 2$ form (see, for example, [42, 43])

$$\mathcal{L} = \frac{1}{2}R^{(4)}\text{vol}_4 + \mathcal{L}_{\text{VM}} + \mathcal{L}_{\text{HM}} - V\text{vol}_4, \quad (4.19)$$

where

$$\mathcal{L}_{\text{VM}} = g_{\tau\bar{\tau}}d\tau \wedge *d\bar{\tau} + \frac{1}{2}\text{Im}\mathcal{N}_{IJ}F^I \wedge *F^J + \frac{1}{2}\text{Re}\mathcal{N}_{IJ}F^I \wedge F^J \quad (4.20)$$

is the piece corresponding to the (ungauged) vector multiplet,

$$\mathcal{L}_{\text{HM}} = h_{uv}Dq^u \wedge *Dq^v \quad (4.21)$$

the piece corresponding to the gauged hypermultiplets, and V is the scalar potential. We now elaborate on each of these terms.

In (4.20), τ is a complex coordinate and $g_{\tau\bar{\tau}}$ a Kähler metric, with Kähler potential K_V , on the special Kähler manifold $SU(1,1)/U(1)$; $F_I = dA^I$, $I = 0, 1$ are the abelian two-form field strengths of the graviphoton and vector in the vector multiplet; and \mathcal{N}_{IJ} is a τ -dependent matrix, specified by supersymmetry, governing the couplings of the scalars to the gauge kinetic terms and the Chern-Simons terms. Specifically, if X^I are homogeneous coordinates on the special Kähler manifold and a prepotential \mathcal{F} exists, in terms of which the Kähler potential can be written as

$$K_V = -\log(i\bar{X}^I \mathcal{F}_I - iX^I \bar{\mathcal{F}}_I) , \tag{4.22}$$

then \mathcal{N}_{IJ} is given by

$$\mathcal{N}_{IJ} \equiv \bar{\mathcal{F}}_{IJ} + 2i \frac{(\text{Im}\mathcal{F}_{IK})(\text{Im}\mathcal{F}_{JL})X^K X^L}{(\text{Im}\mathcal{F}_{AB})X^A X^B} , \tag{4.23}$$

where $\mathcal{F}_I = \partial_I \mathcal{F}$ and $\mathcal{F}_{IJ} = \partial_I \partial_J \mathcal{F}$ are the derivatives of the prepotential with respect to X^I .

In (4.21), q^u , $u = 1, \dots, 8$, are coordinates and h_{uv} the homogeneous metric on the quaternionic-Kähler manifold $G_{2(2)}/SO(4)$, normalised so that its Ricci tensor is $-2(2 + n_H) = -8$ times the metric, where $n_H = 2$ is the number of hypermultiplets. The covariant derivatives of q^u are defined in terms of two specific Killing vectors k_I^u , $I = 0, 1$, of $G_{2(2)}/SO(4)$ as $Dq^u = dq^u - gk_I^u A^I$.

Finally, when a gauging is turned on in the hypermultiplet sector only, as in the present case, the $N = 2$ scalar potential V in (4.19) is given by

$$V = e^{K_V} X^I \bar{X}^J 4h_{uv} k_I^u k_J^v - \left(\frac{1}{2} \text{Im}\mathcal{N}^{-1IJ} + 4e^{K_V} X^I \bar{X}^J \right) P_I^x P_J^x , \tag{4.24}$$

where the only symbols that remain to be defined are the momentum maps P_I^x . First recall that the quaternionic-Kähler manifold has $Sp(1) \times Sp(n_H)$ holonomy and the $Sp(1)$ factor is associated to the existence of a triplet of complex structures. Let ω^x be the $Sp(1)$ part of the spin connection and K^x the corresponding curvature (see appendix C for more details). Then for each of the two Killing vectors k_I^u , $I = 0, 1$, of $G_{2(2)}/SO(4)$ along which the gauging is turned on, the scalars P_I^x , $x = 1, 2, 3$, are a triplet of potentials for the $Sp(1)$ part of the curvature K^x of $G_{2(2)}/SO(4)$ along k_I^u satisfying:⁴

$$2\iota_{k_I} K^x = DP_I^x \equiv (dP_I^x + \epsilon^{xyz} \omega^y P_I^z) . \tag{4.25}$$

We now show that our $D = 4$ Lagrangian (4.14)–(4.17) is of this form.

⁴Note that in [10] there should be a factor of 2 appearing on the left hand side of equation (C.7) and a factor of 1/2 on the right hand side of (C.13).

4.2.1 Vector multiplet

Let us first deal with \mathcal{L}_{VM} . Introducing the homogeneous coordinates $X^I = (1, \tau^2)$, $I = 0, 1$, with τ defined in (4.13), we can write down the holomorphic prepotential

$$\mathcal{F} = \sqrt{X^0(X^1)^3}. \quad (4.26)$$

From (4.22), this gives the Kähler potential

$$K_V = -\log i(\tau - \bar{\tau})^3 + \log 2, \quad (4.27)$$

from which we obtain the Kähler metric

$$g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} K_V = -\frac{3}{(\tau - \bar{\tau})^2} \quad (4.28)$$

on $\text{SU}(1,1)/\text{U}(1)$.

Next, equation (4.23) allows us to compute

$$\mathcal{N}_{IJ} = \frac{1}{2(\tau + \beta)} \begin{pmatrix} -\tau^3 \bar{\tau} & 3\beta\tau \\ 3\beta\tau & 3 \end{pmatrix}. \quad (4.29)$$

Finally, defining the Abelian gauge fields $A^I = (A_1, -\tilde{B}_1)$, $I = 0, 1$, with field strengths $F^I = dA_1^I = (F_2, -\tilde{H}_2)$ (see (4.4)), it is straightforward to now show that the Lagrangian (4.15) can indeed be cast in the canonical form (4.20).

Observe that exactly the same vector multiplet structure arises in the $D = 4$, $N = 2$ theory obtained from consistent truncation of $D = 11$ supergravity on an arbitrary Sasaki-Einstein seven-fold [10]. Indeed, we find agreement with the analysis of section 2.3 of [10] with the identifications $\varphi_0 = 2U + V$, $\beta = h$ and identical H_2 and F_2 . The quantities corresponding to X^I , \mathcal{F} , K_V and \mathcal{N}_{IJ} were denoted with tildes in [10].

4.2.2 Gauged hypermultiplets

We next show that the Lagrangian \mathcal{L}_{HM} corresponds to a gauged non-linear sigma model with $G_{2(2)}/\text{SO}(4)$ target space. Some details about this coset space are given in appendix C. We find it useful here to use the $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ Iwasawa parametrisation for $\mathcal{T}_{\alpha\beta}$,

$$\mathcal{T} = \begin{pmatrix} e^{\varphi_2} & \zeta e^{\varphi_2} \\ \zeta e^{\varphi_2} & e^{-\varphi_2} + \zeta^2 e^{\varphi_2} \end{pmatrix}, \quad (4.30)$$

which allows us to write the Lagrangian (4.16) as

$$\begin{aligned} \mathcal{L}_{\text{HM}} = & \frac{1}{4} d\varphi_1 \wedge *d\varphi_1 + \frac{1}{4} D\varphi_2 \wedge *D\varphi_2 + \frac{1}{4} e^{2\varphi_2} D\zeta \wedge *D\zeta + \frac{3}{4} e^{-\frac{1}{\sqrt{3}}\varphi_1 - \varphi_2} D\theta^1 \wedge *D\theta^1 \\ & + \frac{3}{4} e^{-\frac{1}{\sqrt{3}}\varphi_1 + \varphi_2} (D\theta^2 - \zeta D\theta^1) \wedge *(D\theta^2 - \zeta D\theta^1) + 3e^{-\frac{2}{\sqrt{3}}\varphi_1} G_1 \wedge *G_1 \\ & + \frac{1}{4} e^{-\sqrt{3}\varphi_1 - \varphi_2} F_1^1 \wedge *F_1^1 + \frac{1}{4} e^{-\sqrt{3}\varphi_1 + \varphi_2} (F_1^2 - \zeta F_1^1) \wedge *(F_1^2 - \zeta F_1^1). \end{aligned} \quad (4.31)$$

Recall that G_1, F_1^α , $\alpha = 1, 2$, are given in (4.5), (4.9), that the covariant derivatives $D\theta^\alpha$, $D\xi^\alpha$ are defined in (3.11), and that Da is defined in (4.6). We have also defined

$$\begin{aligned} D\varphi_2 &= d\varphi_2 + 2gA_1\zeta, \\ D\zeta &= d\zeta + gA_1(e^{-2\varphi_2} - \zeta^2 - 1). \end{aligned} \quad (4.32)$$

In this form, it is now apparent that the Lagrangian (4.31) is equivalent to (4.21) with h_{uv} being the metric on $G_{2(2)}/\text{SO}(4)$ given in (C.19), and q^u , $u = 1, \dots, 8$, the scalars given in (C.11).

From the definition of the covariant derivatives, we can read off

$$\begin{aligned} k_0 &= -2\zeta\partial_{\varphi_2} - (e^{-2\varphi_2} - \zeta^2 - 1)\partial_\zeta - \theta_2\partial_{\theta_1} + \theta_1\partial_{\theta_2} - \frac{1}{4}l\partial_a - \xi_2\partial_{\xi_1} + \xi_1\partial_{\xi_2}, \\ k_1 &= \frac{1}{2}\partial_a, \end{aligned} \quad (4.33)$$

as the Killing vectors k_I , $I = 0, 1$, of $G_{2(2)}/\text{SO}(4)$ along which the gauging is turned on. It can be checked that the vectors (4.33) do indeed leave the metric (C.19) invariant. We have a $U(1) \times \mathbb{R}$ gauging, as noted at the end of section 4.1. To see this explicitly, we introduce the $\mathfrak{u}(1) \oplus \mathbb{R} \subset \mathfrak{g}_2$ -algebra valued gauge field

$$X_1 = A_1\mathbf{K}_1 + \frac{\sqrt{3}}{2}(lA_1 + 2\tilde{B}_1)\mathbf{F}_4 \quad (4.34)$$

and use the embedding tensor approach (see [44] for a review) to write the covariant derivative of the hyperscalars matrix \mathcal{M} defined in (C.15) as

$$D\mathcal{M} = d\mathcal{M} + g(X_1\mathcal{M} + \mathcal{M}X_1^\sharp) \quad (4.35)$$

(\sharp being the generalised transpose defined in appendix C) and the Lagrangian (4.31) as

$$\mathcal{L}_{\text{HM}} = \frac{1}{16}\text{Tr}(\mathcal{M}^{-1}D\mathcal{M} \wedge *\mathcal{M}^{-1}D\mathcal{M}). \quad (4.36)$$

Note that from (4.34) we deduce that the Killing vector $k_0 + (l/2)k_1$ is associated to the compact generator $\mathbf{K}_1 = \mathbf{E}_1 - \mathbf{F}_1$, while $-(1/\sqrt{3})k_1$ is related to the negative root non-compact generator \mathbf{F}_4 .

4.2.3 Scalar potential

Given the $\text{Sp}(1)$ spin connection ω^x on $G_{2(2)}/\text{SO}(4)$, and its curvature K^x , presented in equations (C.26) and (C.29), we can work out the momentum maps P_I^x corresponding to the Killing vectors in (4.33) via the definition (4.25). For k_0 we find

$$\begin{aligned} P_0^1 &= \frac{1}{2}e^{\frac{1}{2}\vec{\alpha}_5 \cdot \vec{\varphi}} \left(-\xi_2 + \theta_1 \left(\frac{3}{2}l - \theta^T\theta \right) \right) + \frac{3}{2}e^{\frac{1}{2}\vec{\alpha}_3 \cdot \vec{\varphi}} (\theta_1 + \zeta\theta_2), \\ P_0^2 &= \frac{1}{2}e^{\frac{1}{2}\vec{\alpha}_6 \cdot \vec{\varphi}} \left(\xi_1 + \zeta\xi_2 + (\theta_2 - \zeta\theta_1) \left(\frac{3}{2}l - \theta^T\theta \right) \right) + \frac{3}{2}e^{\frac{1}{2}\vec{\alpha}_2 \cdot \vec{\varphi}}\theta_2, \\ P_0^3 &= \frac{1}{2}e^{\frac{1}{2}\vec{\alpha}_1 \cdot \vec{\varphi}} (e^{-2\varphi_2} + \zeta^2 + 1) - \frac{3}{4}e^{\frac{1}{2}\vec{\alpha}_4 \cdot \vec{\varphi}} (l - 2\theta^T\theta), \end{aligned} \quad (4.37)$$

where $\vec{\alpha}_i$, $i = 1, \dots, 6$, are the G_2 roots given in (C.1), $\vec{\varphi} \equiv (\varphi_1, \varphi_2)$, and a dot denotes Euclidean scalar product. For k_1 we have

$$P_1^1 = -\frac{3}{2}e^{\frac{1}{2}\vec{\alpha}_5 \cdot \vec{\varphi}}\theta_1, \quad P_1^2 = -\frac{3}{2}e^{\frac{1}{2}\vec{\alpha}_6 \cdot \vec{\varphi}}(\theta_2 - \zeta\theta_1), \quad P_1^3 = -\frac{3}{2}e^{\frac{1}{2}\vec{\alpha}_4 \cdot \vec{\varphi}}. \quad (4.38)$$

Equipped with all these definitions, we can verify, after some calculation, that the scalar potential of our $D = 4$ reduced theory in (4.17) agrees with the canonical $N = 2$ expression (4.24): $\mathcal{L}_{\text{pot}} = -g^2 V \text{vol}_4$.

5 AdS_4 vacua and mass spectrum

In this section we will discuss the AdS_4 vacua of the $D = 4$ reduced theory described in the preceding sections. We find two known AdS_4 vacua both of which have $l = -1$ i.e. $\Sigma_3 = H^3$ (or H^3/Γ). One of these $AdS_4 \times H^3$ solutions [3] is supersymmetric and after uplifting on S^4 to $D = 11$ is interpreted as being dual to the SCFT arising on M5-branes wrapping SLag 3-cycle H^3 . The second $AdS_4 \times H^3$ solution is not supersymmetric and was found in [31]. Here we shall recall these solutions and, within the consistent truncation, determine the spectrum of operators in the dual CFTs.

As mentioned previously, in this and the remaining sections of the paper, we will use the field variables of section 3, for which the equations of motion are given in (A.11)–(A.24).

5.1 Supersymmetric AdS_4

The supersymmetric AdS_4 solution is obtained by setting $l = -1$,

$$e^{-20\phi} = 2, \quad e^{10\lambda} = 2, \quad (5.1)$$

with all other fields trivial, and the AdS_4 radius squared L^2 is given by

$$L^2 = \frac{\sqrt{2}}{g^2}. \quad (5.2)$$

We now consider the masses of the fields in this vacuum. We find that the ϕ, λ fields mix to give masses

$$M^2 L^2 = 3 \pm \sqrt{17}, \quad (5.3)$$

corresponding to operators in the dual SCFT with scaling dimensions

$$\Delta = \frac{1}{2} + \frac{1}{2}\sqrt{17}, \quad \Delta = \frac{5}{2} + \frac{1}{2}\sqrt{17}. \quad (5.4)$$

The β field doesn't mix and has mass given by

$$M^2 L^2 = 2 \quad (5.5)$$

and hence scaling dimension

$$\Delta = \frac{3}{2} + \frac{1}{2}\sqrt{17}.$$

We next consider the fields B_2 , C_1 and A_1 . Using (A.12) to solve for B_2 and then substituting into (A.11) (at linearised order), and then combining with (A.20) we obtain coupled equations for two vector fields C_1 and A_1 . After diagonalisation these give masses

$$M^2 L^2 = 0, 4 \tag{5.6}$$

and hence

$$\Delta = 2, \quad \Delta = \frac{3}{2} + \frac{1}{2}\sqrt{17}.$$

Note that the massive mode is given by C_1 while the massless mode is given by the combination $A_1 + (3/g)C_1$ and corresponds to the abelian R -symmetry current of the dual $N = 2$ SCFT.

We now turn to the charged scalar fields. We first consider the fields χ , θ and h_3 . After using (A.14) to solve for h_3 and then substituting into (A.13) we obtain coupled equations for χ , θ . After taking suitable linear combinations of χ and $\epsilon\theta$ we can diagonalise the mass matrix leading to masses

$$M^2 L^2 = 10, 2 \tag{5.8}$$

and hence scaling dimensions

$$\Delta = 5, \quad \Delta = \frac{3}{2} + \frac{1}{2}\sqrt{17}.$$

From (3.11) we observe that these scalars have $SO(2)$ charge g . We next analyse the two scalar degrees of freedom in \mathcal{T} . To do so it will be useful to now choose the explicit parametrisation of $SL(2, \mathbb{R})/SO(2)$ given by⁵

$$\begin{aligned} \mathcal{T}[\rho, \sigma] &= R[\sigma]^{-1} \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix} R[\sigma], \\ R[\sigma] &= \begin{pmatrix} \cos(\frac{\sigma}{2}) & \sin(\frac{\sigma}{2}) \\ -\sin(\frac{\sigma}{2}) & \cos(\frac{\sigma}{2}) \end{pmatrix}, \end{aligned} \tag{5.10}$$

where σ is a periodic coordinate with period 2π and $\rho > 0$ since $\mathcal{T}[-\rho, \sigma] = \mathcal{T}[\rho, \sigma + \pi]$. Using this we find that the corresponding kinetic term in the Lagrangian (3.7) can be written as

$$\frac{1}{4} \text{Tr}(\mathcal{T}^{-1} D\mathcal{T} \wedge * \mathcal{T}^{-1} D\mathcal{T}) = \frac{1}{2} [d\rho \wedge *d\rho + \sinh^2 \rho (d\sigma - 2gA_1) \wedge * (d\sigma - 2gA_1)]. \tag{5.11}$$

⁵Changing between the parametrisations (4.30) and (5.10) of \mathcal{T} is equivalent to the change of coordinates $e^{\varphi_2} = \cosh \rho + \cos \sigma \sinh \rho$ and $e^{\varphi_2} \zeta = \sin \sigma \sinh \rho$ on $SL(2, \mathbb{R})/SO(2)$.

After expanding about the supersymmetric AdS_4 vacuum we find a complex scalar field with mass given by

$$M^2 L^2 = 4 \tag{5.12}$$

and hence scaling dimension

$$\Delta = 4. \tag{5.13}$$

Using (3.11) we also observe that the R -charge of the complex scalar in \mathcal{T} is $2g$ i.e. twice that of the complex scalar degree of freedom in θ and χ . Note that the above scalar operators in the dual SCFT, except one coming from the ϕ, λ sector, and the massive vector are all irrelevant ($\Delta > 3$).

By considering the conformal dimensions of the fields and their R -charges we can now arrange these into $O\text{Sp}(2|4)$ multiplets (see e.g. [45]). The graviton ($\Delta = 2$) and the massless vector ($\Delta = 3$) are both neutral and form a massless graviton multiplet (see table 8 of [45]). The complex scalar in χ, θ with $\Delta = 5$ and R -charge one (in units of g) combined with the complex scalar in \mathcal{T} with $\Delta = 4$ and R -charge two form a hypermultiplet (see table 7 of [45]). The remaining fields, three neutral scalars with $\Delta = E_0, E_0 + 1, E_0 + 2$, one complex scalar with $\Delta = E_0 + 1$ and unit charge and the massive neutral vector with $\Delta = E_0 + 1$, where $E_0 = (1 + \sqrt{17})/2$ form a long vector multiplet (table 3 of [45]). Note that since the spectrum contains irrational scaling dimensions the abelian R -symmetry group of the SCFT is a non-compact \mathbb{R} .

5.2 Non-susy vacuum

We now consider the non-supersymmetric AdS_4 solution first found in [31]. This is obtained by setting $l = -1$,

$$e^{-20\phi} = \frac{486}{625}, \quad e^{10\lambda} = 10 \tag{5.14}$$

and the AdS_4 radius squared L^2 is given by

$$L^2 = \frac{5\sqrt{2}}{3\sqrt{3}} \frac{1}{g^2}. \tag{5.15}$$

We next discuss the mass spectrum about this vacuum. The ϕ, λ fields mix and give masses

$$M^2 L^2 = \frac{23}{5} \pm \frac{1}{5} \sqrt{409} \tag{5.16}$$

corresponding to scaling dimensions

$$\Delta = \frac{3}{2} + \frac{1}{10} [685 \pm 20\sqrt{409}]^{1/2}. \tag{5.17}$$

The β field has mass given by

$$M^2 L^2 = \frac{6}{5} \tag{5.18}$$

corresponding to

$$\Delta = \frac{3}{2} + \frac{1}{10}\sqrt{345}. \tag{5.19}$$

For the B_2 , C_1 and A_1 fields, by again solving for B_2 and then considering the linearised equations for A_1 and C_1 we are led to two vectors with masses

$$M^2 L^2 = 0, 28/5 \tag{5.20}$$

and hence

$$\Delta = 2, \quad \Delta = \frac{3}{2} + \frac{3}{10}\sqrt{65}. \tag{5.21}$$

The combination $A_1 + (27/7g)C_1$ is the massless mode and C_1 is the massive mode.

Finally we consider the charged fields. After eliminating h_3 we again find that $\epsilon\theta, \chi$ mix to give a complex field with mass

$$M^2 L^2 = \frac{134}{5} \pm \frac{4}{5}\sqrt{241} \tag{5.22}$$

and hence

$$\Delta = \frac{3}{2} + \frac{1}{10}\sqrt{2905 \pm 80\sqrt{241}}. \tag{5.23}$$

Using the parametrisation of \mathcal{T} given in (5.10) at linearised order we find a complex scalar field with mass

$$M^2 L^2 = 68 \tag{5.24}$$

and hence

$$\Delta = \frac{3}{2} + \frac{1}{2}\sqrt{281}. \tag{5.25}$$

Note that all of the above scalar fields and also the massive vector are dual to irrelevant operators in the dual CFT.

5.3 Additional AdS_4 vacua?

In searching for additional AdS_4 vacua, we must impose that $C_1 = B_2 = A_1 = h_3 = 0$ and that all scalar fields are constant. Then (A.13) immediately implies that $\chi = 0$.

We now show there are no additional AdS_4 solutions when $l = 0$ or $l = -1$. From (A.18) we deduce that $\beta = 0$. Next, (A.19) can be written

$$\left[e^{-4\phi+8\lambda}(-l + 2\theta^T\theta) + \mathcal{T}^{-1}(e^{5\lambda}\mathcal{T} - e^{-5\lambda})^2 \right] \theta = 0 \tag{5.26}$$

which implies (for $l = 0, -1$) that $\theta = 0$. We then just have the $\phi, \lambda, \mathcal{T}$ system. Next using the parametrisation of \mathcal{T} given in (5.10), we deduce from (A.22) that, without loss of

generality, we can take \mathcal{T} to be diagonal. From (A.22) we immediately deduce that either $\rho = 0$ or $\cosh \rho = (3/2)e^{-10\lambda}$. In the former case we easily conclude that there is just the supersymmetric and non-supersymmetric AdS_4 solutions discussed above [31]. In the latter case we find that (A.21) and (A.23) imply that

$$\begin{aligned} x^2 l^2 + 16 - 8y &= 0, \\ 40xl - 7x^2 l^2 + 48 + 16y &= 0, \end{aligned} \tag{5.27}$$

respectively, where we have defined $x \equiv e^{8\lambda-4\phi}$, $y \equiv e^{20\lambda}$. It is now simple to see that there are no solutions when $l = 0$. When $l = -1$ these equations have a positive solution, but it gives rise to a complex value of ρ and hence there are no additional AdS_4 solutions when $l = -1$ either (in the ϕ, λ, ρ sector this was already stated in [31]).

The above analysis also implies that there are no additional solutions when $l = +1$ and we impose $\beta = \theta = 0$. If we take $\theta = 0$ and $\beta \neq 0$ we see from (A.18) that $\beta^2 = 1/2$ and following similar arguments we again find no additional solutions. This just leaves open the possibility of AdS_4 solutions with $l = +1$ and $\theta \neq 0$, which we will not address here.

6 Additional consistent truncations

In this section we shall discuss some additional truncations of the consistent KK truncation that we presented in section 3. We make no attempt to be comprehensive.

6.1 Minimal gauged supergravity — Einstein-Maxwell theory

The supersymmetric AdS_4 vacuum discussed in section 5.1 is a specific example of the general class of supersymmetric AdS_4 solutions of $D = 11$ supergravity, dual to $N = 2$ SCFTs in $d = 3$, that were classified using G -structure techniques in [41]. For any solution in this general class, it has already been shown that there is a consistent truncation to minimal gauged supergravity, with bosonic fields consisting of a metric and a gauge-field [18]. Thus we should be able to further truncate the ansatz in (3.1)–(3.5) to obtain the bosonic content of minimal gauged supergravity. This is simple to do.

We set

$$l = -1, \quad e^{-20\phi} = 2, \quad e^{10\lambda} = 2, \tag{6.1}$$

as in the supersymmetric AdS_4 vacuum and also

$$B_2 = -\frac{1}{\sqrt{2}} * F_2. \tag{6.2}$$

Finally, we set $C_1, \chi, \theta, \mathcal{T}, h_3$ and β to their trivial values. We then find that all equations of motion (A.11)–(A.24) boil down to

$$\begin{aligned} R_{\mu\nu} &= -\frac{3g^2}{\sqrt{2}} g_{\mu\nu} + \frac{1}{\sqrt{2}} \left(F_{\mu\rho} F_{\nu}{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \\ d * F &= 0. \end{aligned} \tag{6.3}$$

These equations of motion come from the bosonic Lagrangian of minimal gauged supergravity, which is just Einstein-Maxwell theory with a negative cosmological constant:

$$\mathcal{L}_{\text{kin}} = \left[R^{(4)} + \frac{6}{L^2} \right] \text{vol}_4 - \frac{1}{\sqrt{2}} F_2 \wedge *F_2, \quad (6.4)$$

where (c.f. (5.2))

$$L^2 = \frac{\sqrt{2}}{g^2}. \quad (6.5)$$

For example, one solution is the standard electrically charged AdS Reissner-Nördstrom black hole with flat spatial sections (also called a black brane) given by

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + \frac{r^2}{L^2} (dx_1^2 + dx_2^2), \\ F_2 &= \frac{\mu_e r_+}{r^2} dt \wedge dr, \end{aligned} \quad (6.6)$$

with

$$f = \frac{r^2}{L^2} - \left(\frac{r_+^2}{L^2} + \frac{\mu_e^2}{2\sqrt{2}} \right) \frac{r_+}{r} + \frac{\mu_e^2}{2\sqrt{2}} \frac{r_+^2}{r^2}. \quad (6.7)$$

This solution, after uplifting to $D = 11$, describes the SCFT on M5-branes wrapped on SLAG 3-cycles, H^3/Γ , when held at finite temperature and finite chemical potential. The stability of these black holes will be discussed in section 7.

6.2 Charged fields to zero

It is also consistent with the full equations of motion (A.11)–(A.24) to set

$$\chi = h_3 = \theta = 0. \quad (6.8)$$

It is also consistent to then, in addition, set

$$\mathcal{T} = \delta. \quad (6.9)$$

This latter truncation sets all of the fields carrying non-zero $\text{SO}(2)$ charge to zero.

7 M5-branes wrapping SLAG 3-cycles at finite T, μ

We now use the results obtained so far to initiate a study of the $N = 2$ $d = 3$ SCFTs, dual to the supersymmetric $AdS_4 \times H^3/\Gamma \times S^4$ solutions, when held at finite temperature T and chemical potential μ with respect to the global R -symmetry. As we have already mentioned these SCFTs arise on M5-branes wrapped on SLAG 3-cycles H^3/Γ in Calabi-Yau three-folds. The conformal invariance implies that the system will just depend on the dimensionless parameter T/μ .

At high temperatures the system is described by the (uplifted) electrically charged AdS-RN black hole that was given in (6.6)–(6.7). In this section we will investigate the possibility that the AdS-RN black hole has unstable linearised modes below given “branching

temperatures”. At a branching temperature the corresponding linearised mode becomes a zero mode and indicates that a new branch of black hole solutions is appearing. We will see that there are two new types of black hole branches, one with charged hair, corresponding to holographic superconductivity, and the other without. We will see that the branching temperature of the superconducting black holes is lower than that of the other branch. In order to determine which is thermodynamically preferred one will need to go beyond the linearised analysis that we perform here to determine the order of the phase transition. Specifically, if the transition to the superconducting black holes is first order, the “critical temperature” at which the system moves, discontinuously, from the AdS-RN black holes to the superconducting branch is higher than that of the superconducting black hole branching temperature⁶ and could be higher than the critical temperature for the neutral black holes. Furthermore, there could be additional black hole branches either within or outside the $D = 4$ truncation, which could be associated with even higher critical temperatures.

The simplest way to look for new branches of $D = 4$ black hole solutions is to study the zero temperature, near horizon $AdS_2 \times \mathbb{R}^2$ limit of the AdS-RN black hole and look for modes that violate the AdS_2 BF bound [8, 9, 49]. Using this approach in section 7.1, we find that there are indeed charged modes that violate the AdS_2 BF bound indicating holographic superconductivity. However, we also find some neutral modes that violate the AdS_2 BF bound. This indicates that there are two new branches of black hole solutions emerging. By a more careful analysis in section 7.2, we will show that non-superconducting black holes have a higher branching temperature.

7.1 Instabilities of the $AdS - RN$ black hole at $T = 0$

The near horizon limit of the $T = 0$ AdS RN black hole (6.6)–(6.7) gives the $AdS_2 \times \mathbb{R}^2$ solution

$$\begin{aligned} ds_4^2 &= L_{(2)}^2 ds^2(AdS_2) + dx_1^2 + dx_2^2, \\ F_2 &= q \text{Vol}(AdS_2), \\ B_2 &= \frac{q}{\sqrt{2}L_{(2)}^2} dx_1 \wedge dx_2, \end{aligned} \tag{7.1}$$

where

$$L_{(2)}^2 = \frac{1}{3\sqrt{2}g^2}, \quad q = \frac{1}{\sqrt{3}g}. \tag{7.2}$$

We now consider some linearised fluctuations about this solution. We will consider various perturbations that are independent of the coordinates on \mathbb{R}^2 and look for perturbations whose AdS_2 mass violates the BF bound. For a unit radius AdS_2 space this condition is

$$M^2 < -\frac{1}{4}. \tag{7.3}$$

⁶Some general discussion of related issues appear in [46] and some bottom up examples of holographic superconductors with first order transitions are described in [47, 48].

We will not consider any perturbations of the metric, but we have checked that, at linearised order, the perturbations that we consider do not source any metric perturbations: specifically we have checked that the right hand side of (A.24) vanishes at leading order.

We first consider fluctuations in the h_3, χ, θ sector. After eliminating h_3 we find that χ and $\epsilon\theta$ mix, exactly as in the AdS_4 vacua studied in section 5.1. At linearised order we find

$$\begin{aligned}
 -D *_2 D\chi + \frac{3}{2}D *_2 D(\epsilon\theta) - 2\sqrt{2}g^2\chi \text{Vol}(AdS_2) &= 0, \\
 D *_2 D(\epsilon\theta) + g^2 \left[\sqrt{2}\chi + \frac{5}{\sqrt{2}}(\epsilon\theta) \right] \text{Vol}(AdS_2) &= 0.
 \end{aligned}
 \tag{7.4}$$

This gives AdS_2 masses equal to $5/3$ and $1/3$. Hence there is no instability in this sector.

We next consider the scalars parametrising the $SL(2, \mathbb{R})/SO(2)$ coset, \mathcal{T} . Using the parametrisation (5.10) we consider small fluctuations of ρ . After checking that it is consistent with the equations of motion, we set $\sigma = 0$ and then find that at linearised order ρ decouples and satisfies

$$\square_2 \rho + \frac{2}{3}\rho = 0,
 \tag{7.5}$$

where \square_2 is the Laplacian on a unit radius AdS_2 . This gives $M^2 = -2/3$ which violates the AdS_2 BF bound. This shows that the system is unstable to condensing a charged mode. As we will discuss in the next subsection we can use the results of [9] to argue that there will be a new branch of black holes that will spontaneously break the abelian R-symmetry and hence exhibit holographic superconductivity.

We now consider fluctuations of the neutral scalars that mix with some fluctuations of the gauge fields. Specifically, for the scalars we consider

$$\begin{aligned}
 \beta &= \delta\beta, \\
 \lambda &= \frac{1}{10} \ln 2 + \delta\lambda, \\
 \phi &= -\frac{1}{20} \ln 2 + \delta\phi
 \end{aligned}
 \tag{7.6}$$

and for the gauge-fields

$$\begin{aligned}
 F_2 &= (q + \delta F) \text{Vol}(AdS_2), \\
 B_2 &= \left(\frac{qR^2}{\sqrt{2}} + \delta B \right) dx_1 \wedge dx_2 + \delta B' \text{Vol}(AdS_2),
 \end{aligned}
 \tag{7.7}$$

with, from (A.11),

$$C_1 = -\frac{1}{2g} *_2 d(\delta B).
 \tag{7.8}$$

In order to solve the equations of motion (A.14), (A.20) at leading order, we quickly see that these fluctuations are not independent. We find that we should set

$$\begin{aligned}
 \delta B' &= 2q\delta\beta, \\
 \delta F &= -3\sqrt{2}R_{(2)}^2 \delta B - 6q(\delta\phi - 2\delta\lambda).
 \end{aligned}
 \tag{7.9}$$

Substituting into the rest of the equations we find that the mode $\delta\beta$ decouples from the others and satisfies

$$\square_2 \delta\beta - \frac{4}{3} \delta\beta = 0 \tag{7.10}$$

and thus does not give rise to any instability. The remaining modes remain coupled and satisfy

$$\begin{aligned} \square_2(q\delta B) - \frac{2}{3}(q\delta B) - \frac{4}{3}(\delta\phi - 2\delta\lambda) &= 0, \\ \square_2\delta\lambda - \frac{2}{3}\delta\lambda + \frac{2}{3}\delta\phi + \frac{2}{5}(q\delta B) &= 0, \\ \square_2\delta\phi + \frac{2}{3}\delta\lambda - \frac{4}{3}\delta\phi - \frac{1}{5}(q\delta B) &= 0, \end{aligned} \tag{7.11}$$

yielding the mass spectrum on AdS_2

$$M^2 = \frac{1}{3} \left(3 - \sqrt{17} \right); \quad \frac{1}{3} \left(3 + \sqrt{17} \right); \quad \frac{2}{3}. \tag{7.12}$$

Thus the $F_2, B_2, C_1, \phi, \lambda$ sector also contains an unstable mode. Recall from section 5.1 that in the supersymmetric AdS_4 vacuum the two neutral scalars ϕ, λ are dual to one relevant operator (with $\Delta \approx 2.56$) and one irrelevant operator, and that B_2, C_1 fields describe a neutral massive vector which is dual to an irrelevant operator. The detailed interactions between these fields and also with F_2 give rise to the violation of the AdS_2 BF bound (c.f. the simpler mechanism involving a single neutral scalar field in a bottom up setting discussed in [8]).

We have thus found two unstable modes. One is charged and comes from the \mathcal{T} sector while the other is neutral and comes from the $F_2, B_2, C_1, \phi, \lambda$ sector. Each of these unstable modes will give rise to a new branch of charged black holes that will appear at some branching temperature, the former with charged hair and the latter with neutral hair. To determine the branching temperatures we need to consider the linearised fluctuations about the electrically charged AdS-RN black hole at non-zero temperature.

7.2 Instabilities of the AdS-RN black hole at $T \neq 0$

We now study the perturbative stability of the finite temperature electrically charged AdS RN black hole of section 6.1, focussing on the modes associated with those violating the AdS_2 BF bound in the zero temperature limit that we identified in the last subsection. We begin by writing the AdS RN black hole metric and vector potential (6.6)–(6.7) as

$$\begin{aligned} ds_4^2 &= -f dt^2 + \frac{dr^2}{f} + \frac{r^2}{L^2} (dx_1^2 + dx_2^2), \\ F_2 &= \frac{F_0}{r^2} Vol_2, \\ B_2 &= \frac{F_0}{\sqrt{2}L^2} Vol_{\mathbb{R}^2}, \end{aligned} \tag{7.13}$$

where $\text{Vol}_2 = dt \wedge dr$, $\text{Vol}_{\mathbb{R}^2} = dx^1 \wedge dx^2$ and

$$\begin{aligned} f &= \frac{r^2}{L^2} - \left(\frac{r_+^2}{L^2} + \frac{\mu^2}{2\sqrt{2}} \right) \frac{r_+}{r} + \frac{\mu^2}{2\sqrt{2}} \frac{r_+^2}{r^2}, \\ F_0 &= -\mu r_+, \quad L^2 = \frac{\sqrt{2}}{g^2}. \end{aligned} \quad (7.14)$$

We will consider modes which are functions of t, r only. For the \mathcal{T} sector, in the parametrisation (5.10) we again consider a perturbation $\delta\rho$ with $\sigma = 0$. Using that $F_2 = dA_1$ with $A_1 = A dt$, $A = \mu \left(1 - \frac{r_\pm}{r}\right)$ we find that $\delta\rho$ must satisfy

$$\square_4 \delta\rho + \frac{4g^2 A^2}{f} \delta\rho - 2\sqrt{2}g^2 \delta\rho = 0, \quad (7.15)$$

where \square_4 is the $D = 4$ Laplacian for the metric in (7.13).

We next consider the F_2, B_2, ϕ, λ sector. For the scalars we consider

$$\begin{aligned} \lambda &= \frac{1}{10} \ln 2 + \delta\lambda, \\ \phi &= -\frac{1}{20} \ln 2 + \delta\phi. \end{aligned} \quad (7.16)$$

For the two forms we will take

$$\begin{aligned} F_2 &= \left[\frac{F_0}{r^2} + \delta F \right] \text{Vol}_2, \\ B_2 &= \left[\frac{F_0}{\sqrt{2}L^2} + \delta B \right] \text{Vol}_{\mathbb{R}^2}, \end{aligned} \quad (7.17)$$

with

$$\begin{aligned} C_1 &= -\frac{1}{2g} *_4 d(\delta B), \\ \delta F &= -\frac{3\sqrt{2}L^2}{r^2} \delta B - \frac{6F_0}{r^2} (\delta\phi - 2\delta\lambda). \end{aligned} \quad (7.18)$$

We then find that the equation (A.12) reads

$$L^2 r^2 d \left(\frac{1}{r^2} *_2 d(\delta B) \right) + \frac{4}{\sqrt{2}} g^2 L^2 \delta B + 4F_0 g^2 (\delta\phi - 2\delta\lambda) = 0. \quad (7.19)$$

After defining

$$\delta B = r^2 \frac{F_0}{L^2 r_+^2} \delta b, \quad (7.20)$$

we find that this equation and all remaining equations reduce to the coupled system

$$\begin{aligned} -\square_4 \delta b + 2 \left(-\partial_r \left(\frac{f}{r} \right) + \sqrt{2}g^2 \right) \delta b + \frac{4g^2 r_+^2}{r^2} (\delta\phi - 2\delta\lambda) &= 0, \\ -\square_4 \delta\lambda - g^2 \frac{2\sqrt{2}}{5} (\delta\lambda + 2\delta\phi) - \frac{2F_0^2}{5} \left(\frac{1}{r^2 r_+^2} \delta b + \frac{1}{r^4 \sqrt{2}} (\delta\phi - 2\delta\lambda) \right) &= 0, \\ -\square_4 \delta\phi + g^2 \frac{\sqrt{2}}{5} (-4\delta\lambda + 17\delta\phi) + \frac{F_0^2}{5} \left(\frac{1}{r^2 r_+^2} \delta b + \frac{1}{r^4 \sqrt{2}} (\delta\phi - 2\delta\lambda) \right) &= 0. \end{aligned} \quad (7.21)$$

One can check that upon setting $r = r_+$, one recovers the AdS_2 equations given in (7.11) (after rescaling δb).

By numerically solving (7.15) and (7.21) we can determine the temperatures at which the new branches of black hole solutions appear. Since we are looking for zero modes we consider perturbations that are independent of time (i.e. $e^{-i\omega t}$ with $\omega = 0$). The modes we are interested in are independent of the spatial coordinates and so the modes just depend on r . We then expand out near the horizon and integrate out to infinity, looking for the temperature at which the non-normalisable asymptotic behaviour vanishes. In fact for (7.15) this analysis has already been performed by Deneff and Hartnoll in [9]. In their notation we have⁷ $\gamma_{DH}q_{DH} = 4$ and, as we showed in (5.13), $\Delta = 4$. We have solved the numerical problem and we find the critical temperature $\gamma_{DH}T_c/\mu \approx .001$, which agrees with figure 1 of [9]. We also numerically solved (7.21) and find $\gamma_{DH}T_c/\mu \approx .0045$.

Thus, in conclusion, we have shown that as we lower the temperature of the AdS RN black hole, two new branches of black holes appear. The first branch that appears is a new class of charged black holes carrying neutral scalar and massive vector hair. At a lower temperature a second branch of charged black holes appear carrying charged scalar hair which spontaneously break the R -symmetry and hence exhibit holographic superconductivity. It would be interesting to determine if the phase transitions are first or second order. It would also be interesting to construct the fully back reacted thermodynamically preferred black hole solutions and study their behaviour at lower temperatures.

8 Lifshitz solutions

In this section we investigate the possibility that the equations of motion of the $D = 4$ reduced theory, given in (A.11)–(A.24), admits $Lif_4(z)$ solutions. After uplifting to $D = 11$ such solutions would be dual to $d = 3$ field theories with Lifshitz scaling and dynamical exponent z . After reducing the problem to solving a set of algebraic equations we find (using Mathematica) one solution with $z = 39.05\dots$

For simplicity we restrict our analysis to the truncation where $\chi = h_3 = \theta = 0$, discussed in section 6.2, and consider the following ansatz. For the metric we take

$$ds_4^2 = -\frac{r^{2z}}{L^{2z}} dt^2 + \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} (dx_1^2 + dx_2^2), \quad (8.1)$$

which is the standard $Lif_4(z)$ metric in $D = 4$. We also take

$$\begin{aligned} A_1 &= q \frac{r^z}{L^z} dt, \\ C_1 &= -c \frac{r^z}{L^{z+1}} dt, \\ B_2 &= b \frac{r^2}{L^3} dx_1 \wedge dx_2, \end{aligned} \quad (8.2)$$

⁷In more detail, we should set $M_{DH}^2 = 2$, $g_{DH}^2 = 1/\sqrt{2}$, $L_{DH}^2 = \sqrt{2}/g^2$, $\gamma_{DH}^2 = 4/g^2$, $q_{DH} = 2g$.

where q, c, b are constants, and all remaining scalar fields are taken to be constant. Note that the scaling symmetry (A.25) can be used to set $L = 1$ if desired. We observe that this ansatz is consistent with the $\text{Lif}_4(z)$ scaling symmetry

$$t \rightarrow s^z t, \quad x_i \rightarrow s x_i, \quad r \rightarrow s^{-1} r, \quad (8.3)$$

where z is the (constant) dynamical exponent. One can check that it is consistent with the $D = 4$ equations of motion to now further set $\beta = 0$ and, in the parametrisation of \mathcal{T} given in (5.10), $\sigma = 0$, and we shall do so for additional simplicity. The ansatz is thus specified by eight constants: $q, b, c, z, \rho, \lambda, \phi$ and gL .

Substituting this ansatz into (A.11) and (A.12) we get

$$c = \frac{e^{4\lambda+8\phi}}{gL} b, \quad (8.4)$$

$$2 \left[(gL)^2 - 2ze^{8\lambda+6\phi} \right] b = (gL)^2 l q z e^{4\lambda-2\phi}. \quad (8.5)$$

From the equation of motion for the gauge field (A.20) we obtain

$$2qze^{-12\lambda+6\phi} - 4(gL)^2 q \sinh^2 \rho + 3lb = 0. \quad (8.6)$$

From the λ, ϕ and ρ equations of motion, (A.21)–(A.23), and using (8.4), we obtain

$$(gL)^2 \left[l^2 e^{20\lambda} + 4e^{4\lambda+8\phi} \left(1 + 2e^{20\lambda} \sinh^2 \rho - e^{10\lambda} \cosh \rho \right) \right. \\ \left. + 2q^2 z^2 e^{20\phi} - e^{8\lambda+16\phi} \left[2 - \frac{8}{(gL)^2} e^{8\lambda+6\phi} \right] b^2 \right] = 0, \quad (8.7)$$

$$(gL)^2 \left[10le^{12\lambda+4\phi} - \frac{7}{4} l^2 e^{20\lambda} + e^{4\lambda+8\phi} \left(3 - 4e^{20\lambda} \sinh^2 \rho + 12e^{10\lambda} \cosh \rho \right) \right. \\ \left. - q^2 z^2 e^{20\phi} + e^{8\lambda+16\phi} \left[1 + \frac{16}{(gL)^2} e^{8\lambda+6\phi} \right] b^2 \right] = 0, \quad (8.8)$$

$$\sinh \rho \left[-3 + 2e^{10\lambda} \cosh \rho - 2q^2 e^{-2\lambda+6\phi} \cosh \rho \right] = 0. \quad (8.9)$$

We now turn to Einstein's equations (A.24). Observe that for our metric ansatz the non-zero components for the Ricci tensor are given by

$$R_{tt} = \frac{z(z+2)r^{2z}}{L^{2+2z}}, \quad R_{rr} = -\frac{z^2+2}{r^2}, \quad R_{ij} = -\frac{(z+2)r^2}{L^4} \delta_{ij}, \quad (8.10)$$

where $i, j = 1, 2$. We then find that the (tt) , (rr) and (ii) components of (A.24) give:

$$\begin{aligned} z(z+2) &= A + B - C, \\ -(z^2+2) &= -B + C, \\ -(z+2) &= B + C, \end{aligned} \quad (8.11)$$

where, again using (8.4),

$$\begin{aligned}
 A &= 2(gL)^2 q^2 \sinh^2 \rho + \frac{6}{(gL)^2} e^{4\lambda+8\phi} b^2, \\
 B &= \frac{q^2 z^2}{4} e^{-12\lambda+6\phi} + \frac{3}{4} e^{-4\lambda+2\phi} b^2, \\
 C &= -\frac{(gL)^2}{2} \left[3l e^{-10\phi} - \frac{3}{8} e^{8\lambda-14\phi} l^2 + \frac{1}{2} e^{-6\phi-8\lambda} \left(3 - 4e^{20\lambda} \sinh^2 \rho + 12e^{10\lambda} \cosh \rho \right) \right].
 \end{aligned}
 \tag{8.12}$$

The three equations (8.11) can be rewritten as

$$\begin{aligned}
 A + 2B + 2C + 6 &= 0, \\
 A^2 + 2A - 8B &= 0
 \end{aligned}
 \tag{8.13}$$

and

$$z = \frac{4B}{A}.
 \tag{8.14}$$

We now observe that (8.14) is actually already implied by the previous equations (8.4), (8.5) and (8.6).

To summarise, c can be obtained from (8.4). If we assume that $(gL)^2 \neq 2ze^{8\lambda+6\phi}$, as we shall do, then b can be obtained from (8.5). This leaves six algebraic equations to be solved, (8.6)–(8.9) and (8.13), (8.13), for six remaining constants ϕ , λ , ρ , q , z and (gL) . Using Mathematica we found one solution with $l = -1$ and

$$\begin{aligned}
 z &= 39.059617\dots \\
 gL &= 19.592485\dots \\
 \lambda &= 0.068678\dots \\
 \phi &= 0.043883\dots \\
 \rho &= 0.299400\dots \\
 q &= -0.907857\dots
 \end{aligned}
 \tag{8.15}$$

This solution uplifts to a solution of $D = 11$ supergravity that is a product of a $\text{Lif}_4(z \sim 39)$ factor with an $H^3 \times S^4$ factor, with the latter fibred over the former (due to the fact that $A_1 = (qr^z/L^z)dt$). It would be interesting to explore this solution further. It would also be interesting to know if the $D = 4$ equations of motion admit further $\text{Lif}_4(z)$ solutions.

9 Discussion

We have presented a new consistent KK reduction of $D = 11$ supergravity on $\Sigma_3 \times S^4$, where $\Sigma_3 = H^3/\Gamma, S^3/\Gamma, R^3/\Gamma$, to obtain $N = 2$ gauged supergravities in $D = 4$. For the case of H^3/Γ , the $D = 4$ theory admits a supersymmetric AdS_4 vacuum which uplifts to a $D = 11$ solution dual to the $d = 3$ $N = 2$ SCFT arising on M5-branes wrapping SLAG 3-cycles H^3/Γ .

We showed that the $D = 4$ theory also admits another non-supersymmetric AdS_4 solution as well as a $Lif_4(z)$ solution with $z \sim 39$. It would be interesting to determine whether or not there are additional Lifshitz solutions. It would also be interesting to investigate whether or not there are $D = 4$ domain wall type solutions interpolating between these solutions that would describe dual RG flows between the different critical points.

We also studied the $N = 2$ SCFT arising on the wrapped M5-branes at finite temperature and chemical potential by studying black holes. The high temperature limit is described by an uplifted $D = 4$ AdS-RN type black hole. We also showed that these black holes have two instabilities corresponding to the existence of two new branches of black hole solutions. One branch, the new charged black holes with neutral hair, preserve the abelian R -symmetry and arise from an instability involving two neutral scalars and a massive vector field. The other branch of black holes spontaneously break the R -symmetry and thus comprise a new class of holographic superconducting black holes. We showed that the branching temperature of the charged black holes with neutral hair is higher than that of the superconducting black holes. Therefore the charged black holes with neutral hair will be thermodynamically preferred unless the superconducting black hole transition is first order with a critical temperature sufficiently higher than its branching temperature. We leave this interesting issue, and the construction of the fully back reacted black hole solutions to future work. One particularly interesting issue is to determine the zero temperature ground state of the system. There are two natural candidates for such a ground state solution: the new $Lif_4(z \sim 39)$ solution and the non-supersymmetric AdS_4 solution.

The supersymmetric $AdS_4 \times H^3 \times S^4$ solution of $D = 11$ supergravity [3] is a specific example of a general class of supersymmetric $AdS_4 \times \mathcal{N}_7$ solutions with magnetic four-form flux, all describing M5-branes wrapping SLAG 3-cycles, that were classified using G -structures in section 9.5 of [41]. As we discussed in section 6 there is a consistent KK reduction on any of these \mathcal{N}_7 to minimal gauged supergravity in $D = 4$ [18]. Given we have shown in this paper that for the specific example when $\mathcal{N}_7 = H^3 \times S^4$ (with suitable twisting and four-form flux) there is a much bigger consistent KK reduction, it would be interesting to know whether there is a similarly enlarged KK truncation for other \mathcal{N}_7 .

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A Consistent KK truncation formulae

A.1 $D = 7$ gauged supergravity equations of motion

We begin by recording the equations of motion for $D = 7$ gauged supergravity arising from (2.1):

$$DS_{(3)}^i = gT_{ij} * S_{(3)}^j + \frac{1}{8} \epsilon_{ij_1 \dots j_4} F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4}, \quad (\text{A.1})$$

$$\begin{aligned} D\left(T_{ik}^{-1} T_{j\ell}^{-1} * F_{(2)}^{ij}\right) &= -2gT_{i[k}^{-1} * DT_{\ell]i} - \frac{1}{2g} \epsilon_{i_1 i_2 i_3 k \ell} F_2^{i_1 i_2} \wedge DS_{(3)}^{i_3} \\ &\quad + \frac{3}{2g} \delta_{i_1 i_2 k \ell}^{j_1 j_2 j_3 j_4} F_{(2)}^{i_1 i_2} \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} - S_{(3)}^k \wedge S_{(3)}^\ell = 0, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} D\left(T_{ik}^{-1} * D(T_{kj})\right) &= 2g^2(2T_{ik}T_{kj} - T_{kk}T_{ij})\epsilon_{(7)} + T_{im}^{-1} T_{kl}^{-1} * F_{(2)}^{m\ell} \wedge F_{(2)}^{kj} + T_{jk} * S_{(3)}^k \wedge S_{(3)}^i \\ &\quad - \frac{1}{5} \delta_{ij} \left[2g^2 \left(2T_{ik}T_{ik} - (T_{ii})^2 \right) \epsilon_{(7)} + T_{nm}^{-1} T_{kl}^{-1} * F_{(2)}^{m\ell} \wedge F_{(2)}^{kn} + T_{kl} * S_{(3)}^k \wedge S_{(3)}^\ell \right], \end{aligned} \quad (\text{A.3})$$

$$R_{\mu\nu} = \frac{1}{4} T_{ij}^{-1} D_\mu T_{jk} T_{kl}^{-1} D_\nu T_{li} + \frac{1}{4} T_{ik}^{-1} T_{jl}^{-1} F_{\mu\rho}^{ij} F_{\nu}^{kl\rho} + \frac{1}{4} T_{ij} S_{\mu\rho_1\rho_2}^i S_{\nu}^{j\rho_1\rho_2} + \frac{1}{10} g_{\mu\nu} X, \quad (\text{A.4})$$

where

$$X \equiv -\frac{1}{4} T_{ik}^{-1} T_{jl}^{-1} F_{\rho_1\rho_2}^{ij} F^{kl\rho_1\rho_2} - \frac{1}{3} T_{ij} S_{\rho_1\rho_2\rho_3}^i S^{j\rho_1\rho_2\rho_3} + 2V. \quad (\text{A.5})$$

A typo in [39] has been fixed in (A.4).

A.2 Consistency

We now substitute the KK ansatz (3.1), (3.3), (3.4) and (3.5) into the equations of motion (A.1)–(A.4) of $D = 7$ maximal gauged supergravity. To carry out the computation it is helpful to note that the ansatz implies that

$$\begin{aligned} DT^{ab} &= -4e^{-4\lambda} d\lambda \delta_{ab}, \\ DT^{a\alpha} &= g \left[e^{6\lambda} (\mathcal{T}\theta)_\alpha - e^{-4\lambda} \theta_\alpha \right] \bar{e}^a, \\ DT^{\alpha\beta} &= e^{6\lambda} [6d\lambda \mathcal{T}_{\alpha\beta} + D\mathcal{T}_{\alpha\beta}]. \end{aligned} \quad (\text{A.6})$$

Furthermore

$$\begin{aligned} F_{(2)}^{ab} &= \frac{1}{g} \bar{\mathcal{R}}_{ab} - g(\theta^T \theta + \beta^2) \bar{e}^a \wedge \bar{e}^b - \epsilon_{abc} \bar{e}^c \wedge d\beta \\ &= \frac{g}{2} [l - 2(\theta^T \theta + \beta^2)] \bar{e}^a \wedge \bar{e}^b - \epsilon_{abc} \bar{e}^c \wedge d\beta, \\ F_{(2)}^{a\alpha} &= D\theta_\alpha \wedge \bar{e}^a - g\beta \theta_\alpha \epsilon_{abc} \bar{e}^b \wedge \bar{e}^c, \\ F_{(2)}^{\alpha\beta} &= \epsilon_{\alpha\beta} F_2, \end{aligned} \quad (\text{A.7})$$

where $\bar{\mathcal{R}}_{ab}$ is the Riemann tensor of $ds^2(\Sigma_3)$ in (3.1), we have defined $F_2 = dA_1$, and

$$\begin{aligned} DS_{(3)}^a &= -\bar{e}^a \wedge (dB_2 - g\theta_\alpha h_\alpha) + \epsilon_{abc} \bar{e}^b \wedge \bar{e}^c \wedge (dC_1 - g\beta B_2), \\ DS_{(3)}^\alpha &= -g \text{vol}(\Sigma_3) \wedge (D\chi_\alpha + 6\theta_\alpha C_1) + Dh_\alpha. \end{aligned} \quad (\text{A.8})$$

Using the obvious orthonormal frame in $D = 7$ we can calculate the components of the $D = 7$ Ricci-tensor and find:

$$\begin{aligned} R_{mn} &= e^{6\phi} \left[R_{mn}^{(4)} + 3\nabla^2 \phi \eta_{mn} - 30\nabla_m \phi \nabla_n \phi \right], \\ R_{am} &= 0, \\ R_{ab} &= e^{6\phi} \left[-2\nabla^2 \phi + lg^2 e^{-10\phi} \right] \delta_{ab}, \end{aligned} \quad (\text{A.9})$$

$m = 0, 1, 2, 3$. We also find

$$\begin{aligned} -e^{-6\phi} X &= 3e^{8\lambda-4\phi} (\nabla\beta)^2 + 3e^{-2\lambda-4\phi} (D_m \theta^T \mathcal{T}^{-1} D^m \theta) + \frac{1}{2} e^{-12\lambda+6\phi} F_{mn} F^{mn} \\ &+ \frac{3g^2}{4} [l - 2(\theta^T \theta + \beta^2)]^2 e^{8\lambda-14\phi} + 12g^2 \beta^2 e^{-2\lambda-14\phi} (\theta^T \mathcal{T}^{-1} \theta) \\ &+ 3e^{-4\lambda+2\phi} B_{mn} B^{mn} + 24e^{-4\lambda-8\phi} C_m C^m + \frac{1}{3} e^{6\lambda+12\phi} (h_{p_1 p_2 p_3}^T \mathcal{T} h^{p_1 p_2 p_3}) \\ &+ 2g^2 e^{6\lambda-18\phi} (\chi^T \mathcal{T} \chi) \\ &+ g^2 e^{-6\phi} \left[3e^{-8\lambda} + e^{12\lambda} [(\text{Tr} \mathcal{T})^2 - 2\text{Tr}(\mathcal{T} \mathcal{T})] + 6e^{2\lambda} \text{Tr} \mathcal{T} \right]. \end{aligned} \quad (\text{A.10})$$

Proceeding, we now find that eq. (A.1) gives:

$$dB_2 - g(\theta^T h_3) + 2e^{-4\lambda-8\phi} g * C_1 - d\beta \wedge F_2 = 0, \quad (\text{A.11})$$

$$dC_1 - g\beta B_2 - \frac{1}{2} e^{-4\lambda+2\phi} g * B_2 - \frac{g}{4} [l - 2(\theta^T \theta + \beta^2)] F_2 - \frac{1}{2} \epsilon_{\alpha\beta} D\theta^\alpha \wedge D\theta^\beta = 0 \quad (\text{A.12})$$

and

$$Dh_3^\alpha - e^{6\lambda-18\phi} g^2 (\mathcal{T} \chi)_\alpha \text{vol}_4 = 0, \quad (\text{A.13})$$

$$D\chi_\alpha + 6\theta_\alpha C_1 + e^{6\lambda+12\phi} * (\mathcal{T} h_3)_\alpha - 6\epsilon_{\alpha\beta} \theta^\beta \beta d\beta + \frac{3}{2} [l - 2(\theta^T \theta + \beta^2)] \epsilon_{\alpha\beta} D\theta^\beta = 0, \quad (\text{A.14})$$

where $*$ and vol_4 are the Hodge dual and volume form corresponding to the four-dimensional metric ds_4^2 in (3.1). Observe that (with $g \neq 0$) these equations imply that

$$d(e^{-4\lambda+2\phi} * B_2) + 2g\beta \theta^T h_3 - 4ge^{-4\lambda-8\phi} \beta * C_1 + 2B_2 \wedge d\beta = 0, \quad (\text{A.15})$$

$$d(e^{-4\lambda-8\phi} * C_1) - \frac{1}{2} D\theta^T \wedge h_3 - \frac{1}{2} g^2 e^{6\lambda-18\phi} (\theta^T \mathcal{T} \chi) \text{vol}_4 = 0, \quad (\text{A.16})$$

$$D(e^{6\lambda+12\phi} \mathcal{T} * h_3) + 3ge^{-4\lambda+2\phi} \theta * B_2 - 6C_1 \wedge D\theta + 6g\beta \theta B_2 + g\epsilon \chi F_2 = 0. \quad (\text{A.17})$$

Next we find that eq. (A.2) gives:

$$d(e^{-4\phi+8\lambda} * d\beta) + g^2 \beta \left[4e^{-14\phi-2\lambda} (\theta^T \mathcal{T}^{-1} \theta) - e^{-14\phi+8\lambda} (l - 2\beta^2 - 2\theta^T \theta) - 2e^{-18\phi+6\lambda} (\theta^T \epsilon \mathcal{T} \chi) \right] \text{vol}_4 + e^{-4\lambda+2\phi} F_2 \wedge *B_2 + B_2 \wedge B_2 = 0, \quad (\text{A.18})$$

$$D(e^{-4\phi-2\lambda} \mathcal{T}^{-1} * D\theta) + g^2 \left[4e^{-14\phi-2\lambda} \beta^2 \mathcal{T}^{-1} \theta - e^{-14\phi+8\lambda} (l - 2\beta^2 - 2\theta^T \theta) \theta + e^{-10\phi} (e^{10\lambda} \mathcal{T} \theta - 2\theta + e^{-10\lambda} \mathcal{T}^{-1} \theta) + \frac{1}{2} e^{-18\phi+6\lambda} (l - 2\beta^2 - 2\theta^T \theta) \epsilon \mathcal{T} \chi \right] \text{vol}_4 + 4e^{-4\lambda-8\phi} * C_1 \wedge \epsilon D\theta + 2C_1 \wedge h_3 = 0, \quad (\text{A.19})$$

$$d(e^{6\phi-12\lambda} * F_2) + 3ge^{-4\phi-2\lambda} (\theta^T \epsilon \mathcal{T}^{-1} * D\theta) + g \text{Tr}(\epsilon \mathcal{T}^{-1} * DT) - g(\chi^T \epsilon h_3) - 3ge^{-4\lambda-8\phi} (l - 2\beta^2 - 2\theta^T \theta) * C_1 + 3e^{-4\lambda+2\phi} d\beta \wedge *B_2 = 0 \quad (\text{A.20})$$

and we have used (A.11)–(A.13) to simplify the expressions.

We next turn to (A.4). From the (ab) components we obtain

$$\begin{aligned} d * d\lambda - \frac{1}{5} e^{8\lambda-4\phi} d\beta \wedge *d\beta + \frac{1}{20} e^{-2\lambda-4\phi} D\theta^T \wedge * \mathcal{T}^{-1} D\theta - \frac{1}{20} e^{6\lambda+12\phi} h_3^T \wedge * \mathcal{T} h_3 \\ - \frac{1}{10} e^{-12\lambda+6\phi} F_2 \wedge *F_2 - \frac{1}{10} e^{-4\lambda+2\phi} B_2 \wedge *B_2 + \frac{2}{5} e^{-4\lambda-8\phi} C_1 \wedge *C_1 \\ + g^2 \left\{ \frac{1}{20} e^{8\lambda-14\phi} (l - 2\beta^2 - 2\theta^T \theta)^2 \right. \\ \left. + \frac{1}{10} e^{-6\phi} \left[2e^{-8\lambda} - e^{12\lambda} [(\text{Tr} \mathcal{T})^2 - 2\text{Tr}(\mathcal{T} \mathcal{T})] - e^{2\lambda} \text{Tr} \mathcal{T} \right] \right. \\ \left. + \frac{1}{4} e^{-10\phi} \left[e^{10\lambda} (\theta^T \mathcal{T} \theta) - e^{-10\lambda} (\theta^T \mathcal{T}^{-1} \theta) \right] \right. \\ \left. - \frac{1}{5} e^{-2\lambda-14\phi} \beta^2 (\theta^T \mathcal{T}^{-1} \theta) + \frac{1}{20} e^{6\lambda-18\phi} (\chi^T \mathcal{T} \chi) \right\} \text{vol}_4 = 0. \end{aligned} \quad (\text{A.21})$$

From the $(\alpha\beta)$ components of (A.4), and also using (A.21), we obtain

$$\begin{aligned} D(\mathcal{T}_{\alpha\gamma}^{-1} * D(\mathcal{T}^\gamma{}_\beta)) + \frac{3}{2} e^{-4\phi-2\lambda} \left(2\mathcal{T}_{\alpha\delta}^{-1} \delta_{\beta\gamma} - \mathcal{T}_{\gamma\delta}^{-1} \delta_{\alpha\beta} \right) D\theta^\gamma \wedge *D\theta^\delta \\ - \frac{1}{2} e^{12\phi+6\lambda} (2\mathcal{T}_{\beta\delta} \delta_{\alpha\gamma} - \mathcal{T}_{\gamma\delta} \delta_{\alpha\beta}) h_3^\gamma \wedge *h_3^\delta \\ + g^2 \left\{ e^{-6\phi} \left[4e^{12\lambda} (\mathcal{T}^2)_{\alpha\beta} - 2e^{12\lambda} \mathcal{T}_{\alpha\beta} \text{Tr} \mathcal{T} - e^{12\lambda} \delta_{\alpha\beta} (2\text{Tr}(\mathcal{T} \mathcal{T}) - (\text{Tr} \mathcal{T})^2) \right. \right. \\ \left. \left. - 6e^{2\lambda} \mathcal{T}_{\alpha\beta} + 3e^{2\lambda} \delta_{\alpha\beta} \text{Tr} \mathcal{T} \right] \right. \\ \left. + \frac{3}{2} e^{-10\phi} \left[e^{10\lambda} (2\mathcal{T}_{\beta\delta} \delta_{\alpha\gamma} - \mathcal{T}_{\gamma\delta} \delta_{\alpha\beta}) - e^{-10\lambda} (2\mathcal{T}_{\alpha\delta}^{-1} \delta_{\beta\gamma} - \mathcal{T}_{\gamma\delta}^{-1} \delta_{\alpha\beta}) \right] \theta^\gamma \theta^\delta \right. \\ \left. - 6e^{-2\lambda-14\phi} \beta^2 (2\mathcal{T}_{\alpha\delta}^{-1} \delta_{\beta\gamma} - \mathcal{T}_{\gamma\delta}^{-1} \delta_{\alpha\beta}) \theta^\gamma \theta^\delta \right. \\ \left. + \frac{1}{2} e^{6\lambda-18\phi} (2\mathcal{T}_{\beta\delta} \delta_{\alpha\gamma} - \mathcal{T}_{\gamma\delta} \delta_{\alpha\beta}) \chi^\gamma \chi^\delta \right\} \text{vol}_4 = 0. \end{aligned} \quad (\text{A.22})$$

The mixed $(a\alpha)$ components of (A.4) are trivially satisfied.

Finally, we consider the Einstein equations (A.4). The ab components of (A.4) give:

$$\begin{aligned}
 d * d\phi + \frac{1}{10} e^{8\lambda-4\phi} d\beta \wedge *d\beta + \frac{1}{10} e^{-2\lambda-4\phi} D\theta^T \wedge *T^{-1}D\theta - \frac{1}{10} e^{6\lambda+12\phi} h_3^T \wedge *T h_3 \\
 + \frac{1}{20} e^{-12\lambda+6\phi} F_2 \wedge *F_2 + \frac{1}{20} e^{-4\lambda+2\phi} B_2 \wedge *B_2 + \frac{4}{5} e^{-4\lambda-8\phi} C_1 \wedge *C_1 \\
 + g^2 \left\{ \frac{1}{2} l e^{-10\phi} - \frac{7}{80} e^{8\lambda-14\phi} (l - 2\beta^2 - 2\theta^T \theta)^2 \right. \\
 \left. + \frac{1}{20} e^{-6\phi} \left[3e^{-8\lambda} + e^{12\lambda} [(\text{Tr} T)^2 - 2\text{Tr}(TT)] + 6e^{2\lambda} \text{Tr} T \right] \right. \\
 \left. - \frac{1}{4} e^{-10\phi} \left[e^{10\lambda} (\theta^T T \theta) - 2\theta^T \theta + e^{-10\lambda} (\theta^T T^{-1} \theta) \right] \right. \\
 \left. - \frac{7}{5} e^{-2\lambda-14\phi} \beta^2 (\theta^T T^{-1} \theta) - \frac{3}{20} e^{6\lambda-18\phi} (\chi^T T \chi) \right\} \text{vol}_4 = 0. \tag{A.23}
 \end{aligned}$$

The mn components of (A.4), after using (A.23), lead to the following $D = 4$ Einstein equations (using an orthonormal frame associated with ds_4^2):

$$\begin{aligned}
 R_{mn}^{(4)} = & 30 \nabla_m \phi \nabla_n \phi + 30 \nabla_m \lambda \nabla_n \lambda + \frac{1}{4} \text{Tr}(T^{-1} D_m T T^{-1} D_n T) \\
 & + \frac{3}{2} e^{8\lambda-4\phi} \nabla_m \beta \nabla_n \beta + \frac{3}{2} e^{-2\lambda-4\phi} \text{Tr}(D_m \theta T^{-1} D_n \theta) \\
 & + \frac{1}{2} e^{-12\lambda+6\phi} \left(F_{mp} F_n{}^p - \frac{1}{4} \eta_{mn} F_{pq} F^{pq} \right) + \frac{3}{2} e^{-4\lambda+2\phi} \left(B_{mp} B_n{}^p - \frac{1}{4} \eta_{mn} B_{pq} B^{pq} \right) \\
 & + 6e^{-4\lambda-8\phi} C_m C_n + \frac{1}{4} e^{6\lambda+12\phi} \left(h_{mp_1 p_2}^T T h_n{}^{p_1 p_2} - \frac{1}{3} \eta_{mn} h_{p_1 p_2 p_3}^T T h^{p_1 p_2 p_3} \right) \\
 & - g^2 \eta_{mn} \left\{ \frac{3}{2} l e^{-10\phi} - \frac{3}{16} e^{8\lambda-14\phi} (l - 2\beta^2 - 2\theta^T \theta)^2 \right. \\
 & \left. + \frac{1}{4} e^{-6\phi} \left[3e^{-8\lambda} + e^{12\lambda} [(\text{Tr} T)^2 - 2\text{Tr}(TT)] + 6e^{2\lambda} \text{Tr} T \right] \right. \\
 & \left. - \frac{3}{4} e^{-10\phi} \left[e^{10\lambda} (\theta^T T \theta) - 2\theta^T \theta + e^{-10\lambda} (\theta^T T^{-1} \theta) \right] \right. \\
 & \left. - 3e^{-2\lambda-14\phi} \beta^2 (\theta^T T^{-1} \theta) - \frac{1}{4} e^{6\lambda-18\phi} (\chi^T T \chi) \right\}. \tag{A.24}
 \end{aligned}$$

The ma components of (A.4) are trivially satisfied.

We have thus demonstrated that the KK truncation ansatz is consistent. Any solution of the $D = 4$ equations of motion given in (A.11)–(A.24) gives rise to a solution of $D = 11$ supergravity after uplifting first to $D = 7$ via (3.1), (3.3), (3.4) and (3.5), and then to $D = 11$ through (2.5), (2.6).

We end this appendix by noting that these $D = 4$ equations of motion remain inert

under the scaling

$$\begin{aligned}
 g_{mn} &\rightarrow L^2 g_{mn}, \\
 F_{mn} &\rightarrow L F_{mn}, \\
 B_{mn} &\rightarrow L B_{mn}, \\
 C_m &\rightarrow C_m, \\
 h_{mnp} &\rightarrow L^2 h_{mnp}, \\
 g &\rightarrow L^{-1} g.
 \end{aligned}
 \tag{A.25}$$

B Simplified B_2, C_1 system

Consider the following Lagrangian in flat space

$$\mathcal{L} = -\frac{1}{2} B_2 \wedge *B_2 + \frac{m^2}{2} C_1 \wedge *C_1 + C_1 \wedge dB_2.
 \tag{B.1}$$

This describes, somewhat unconventionally, a massive vector field. The equations of motion are

$$dC_1 = *B_2,
 \tag{B.2}$$

$$m^2 *C_1 + dB_2 = 0.
 \tag{B.3}$$

We can solve the B_2 equation of motion (B.2) for B_2 , $B_2 = -*dC_1$, and then substitute into (B.3) to get

$$d*dC_1 - m^2 *C_1 = 0,
 \tag{B.4}$$

which is the usual equation for a massive spin 1 field. Note also that we can substitute into the Lagrangian (B.1) to get

$$\mathcal{L} = -\frac{1}{2} dC_1 \wedge *dC_1 + \frac{m^2}{2} C_1 \wedge *C_1,
 \tag{B.5}$$

which leads to the same equation of motion (B.4).

Alternatively, from (B.2) we deduce that $d*B_2 = 0$ which we can solve by writing

$$B_2 = -*d\tilde{B}_1
 \tag{B.6}$$

and we observe that \tilde{B}_1 is only defined up to a gauge transformation $\tilde{B}_1 \rightarrow \tilde{B}_1 + d\Lambda$. Equation (B.2) can then be written $dC_1 = d\tilde{B}_1$, which is solved via

$$C_1 = \tilde{B}_1 + db
 \tag{B.7}$$

and notice that this maintains the gauge invariance provided that $b \rightarrow b - \Lambda$. In terms of these variables (B.3) can be written as

$$d*d\tilde{B}_1 - m^2 *(\tilde{B}_1 + db) = 0.
 \tag{B.8}$$

Note that this comes from a Lagrangian which can be obtained by substituting (B.6), (B.7) into (B.1), namely

$$\mathcal{L} = -\frac{1}{2}\tilde{H}_2 \wedge *\tilde{H}_2 + \frac{m^2}{2}(\tilde{B}_1 + db) \wedge *(\tilde{B}_1 + db), \quad (\text{B.9})$$

where $\tilde{H}_2 = d\tilde{B}_1$. This is the standard Stueckelberg Lagrangian.

C The hypermultiplet moduli space

Here we fix our conventions for the Lie algebra \mathfrak{g}_2 of the group G_2 , and give details of the construction of the hypermultiplet moduli space $G_{2(2)}/\text{SO}(4)$. This is the eight-dimensional quaternionic-Kähler, symmetric space associated to the split, maximally noncompact real form $\mathfrak{g}_{2(2)}$. It is this real form we will be referring to when we write \mathfrak{g}_2 below.

C.1 G_2 conventions

We find it convenient to choose the following set of positive roots for \mathfrak{g}_2 ,

$$\begin{aligned} \vec{\alpha}_1 &= (0, 2), & \vec{\alpha}_2 &= \left(-\frac{1}{\sqrt{3}}, -1\right), \\ \vec{\alpha}_3 &= \left(-\frac{1}{\sqrt{3}}, 1\right) = \vec{\alpha}_1 + \vec{\alpha}_2, & \vec{\alpha}_4 &= \left(-\frac{2}{\sqrt{3}}, 0\right) = \vec{\alpha}_1 + 2\vec{\alpha}_2, \\ \vec{\alpha}_5 &= \left(-\sqrt{3}, -1\right) = \vec{\alpha}_1 + 3\vec{\alpha}_2, & \vec{\alpha}_6 &= \left(-\sqrt{3}, 1\right) = 2\vec{\alpha}_1 + 3\vec{\alpha}_2, \end{aligned} \quad (\text{C.1})$$

with $\vec{\alpha}_1, \vec{\alpha}_2$ as the simple roots.⁸ We collectively denote the two Cartan generators H_1, H_2 as \vec{H} , and the six positive and six negative root generators as $E_i \equiv E_{\vec{\alpha}_i}$ and $F_i \equiv E_{-\vec{\alpha}_i}$, $i = 1, \dots, 6$, respectively. The canonical commutation relations read

$$\begin{aligned} [H_1, H_2] &= 0, & [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] &= \frac{1}{2}\vec{\alpha} \cdot \vec{H}, \\ [\vec{H}, E_{\vec{\alpha}}] &= \vec{\alpha} E_{\vec{\alpha}}, & [E_{\vec{\alpha}}, E_{\vec{\beta}}] &= N_{\vec{\alpha}, \vec{\beta}} E_{\vec{\alpha} + \vec{\beta}}, \end{aligned} \quad (\text{C.2})$$

where the non-vanishing structure constants $N_{\vec{\alpha}, \vec{\beta}}$ are given by

$$N_{\vec{\alpha}_1, \vec{\alpha}_2} = N_{\vec{\alpha}_1, \vec{\alpha}_5} = -N_{\vec{\alpha}_2, \vec{\alpha}_4} = -N_{\vec{\alpha}_3, \vec{\alpha}_4} = 1, \quad N_{\vec{\alpha}_2, \vec{\alpha}_3} = -\frac{2}{\sqrt{3}}, \quad (\text{C.3})$$

together with the relations

$$N_{\vec{\alpha}, \vec{\beta}} = -N_{\vec{\beta}, \vec{\alpha}} = -N_{-\vec{\alpha}, -\vec{\beta}} = N_{\vec{\beta}, -\vec{\alpha} - \vec{\beta}} = N_{-\vec{\alpha} - \vec{\beta}, \vec{\alpha}}. \quad (\text{C.4})$$

These \mathfrak{g}_2 commutation relations are all that is needed to compute all the quantities we are interested in. For calculational purposes, however, it proves helpful to have an explicit relation of the generators of \mathfrak{g}_2 . Calling E_{ij} the 7×7 matrix with 1 in the i -th row and

⁸The convenience of this choice can be seen from the Lagrangian (4.31): it is this set of roots that governs the couplings of the dilatons to the axion kinetic terms in the hypermultiplet sector of our theory.

j -th column and 0 elsewhere, an explicit realisation of the \mathfrak{g}_2 generators in the fundamental representation is given by

$$\begin{aligned}
H_1 &= \frac{1}{\sqrt{3}}(E_{11} - E_{22} + 2E_{33} - 2E_{55} + E_{66} - E_{77}) & H_2 &= E_{11} + E_{22} - E_{66} - E_{77} \\
E_1 &= -2E_{16} - 2E_{27} & F_1 &= -\frac{1}{2}(E_{61} + E_{72}) \\
E_2 &= \frac{1}{2\sqrt{3}}(2E_{41} - E_{52} - E_{63} + 2E_{74}) & F_2 &= \frac{2}{\sqrt{3}}(E_{14} - E_{25} - E_{36} + E_{47}) \\
E_3 &= \frac{1}{\sqrt{3}}(E_{13} - 2E_{24} + 2E_{46} - E_{57}) & F_3 &= \frac{1}{\sqrt{3}}(E_{31} - E_{42} + E_{64} - E_{75}) \\
E_4 &= -\frac{1}{\sqrt{3}}(E_{21} + E_{43} + E_{54} + E_{76}) & F_4 &= -\frac{1}{\sqrt{3}}(E_{12} + 2E_{34} + 2E_{45} + E_{67}) \\
E_5 &= \frac{1}{2}(-E_{51} + E_{73}) & F_5 &= -2E_{15} + 2E_{37} \\
E_6 &= -E_{23} - E_{56} & F_6 &= -E_{32} - E_{65}
\end{aligned} \tag{C.5}$$

It can be checked that this set of 7-dimensional matrices satisfy the commutation relations (C.2) with (C.3), (C.4).

A couple of operations that are needed for our analysis are the Cartan involution τ and the generalised transpose \sharp . The former is the Lie algebra automorphism defined through its action on the Cartan-Weyl basis as

$$\tau(\vec{H}) = -\vec{H}, \quad \tau(E_i) = -F_i, \quad \tau(F_i) = -E_i, \quad i = 1, \dots, 6. \tag{C.6}$$

The τ -invariant subalgebra, spanned by the generators $K_i = E_i - F_i$, $i = 1, \dots, 6$ is the maximal compact subalgebra $\mathfrak{so}(4)$. Indeed the combinations J_x, L_x , $x = 1, 2, 3$,

$$\begin{aligned}
J_1 &= \frac{1}{4}(K_5 - \sqrt{3}K_3), & L_1 &= \frac{1}{4}(3K_5 + \sqrt{3}K_3), \\
J_2 &= \frac{1}{4}(K_6 + \sqrt{3}K_2), & L_2 &= \frac{1}{4}(3K_6 - \sqrt{3}K_2), \\
J_3 &= \frac{1}{4}(K_1 - \sqrt{3}K_4), & L_3 &= \frac{1}{4}(3K_1 + \sqrt{3}K_4),
\end{aligned} \tag{C.7}$$

can be checked to satisfy the canonical $SO(4) \approx SO(3) \times SO(3)$ commutation relations

$$[J_x, J_y] = \epsilon_{xyz} J_z, \quad [J_x, L_y] = 0, \quad [L_x, L_y] = \epsilon_{xyz} L_z. \tag{C.8}$$

The generalised transpose \sharp can be defined, at the Lie algebra level, as $\sharp = -\tau$:

$$\sharp(\vec{H}) = \vec{H}, \quad \sharp(E_i) = F_i, \quad \sharp(F_i) = E_i, \quad i = 1, \dots, 6. \tag{C.9}$$

We are also interested in the action of the generalised transpose at the group level, which can be defined through the exponential map: for $X \in \mathfrak{g}_2$ with group element $g = e^X \in G_2$, we can define $g^\sharp = e^{\sharp(X)}$. Relevant properties of \sharp are $(g^\sharp)^\sharp = g$, which follows from its definition at the Lie algebra level, and $(g_1 g_2)^\sharp = g_2^\sharp g_1^\sharp$ for all $g_1, g_2 \in G_2$, which can be shown with the help of the Baker-Campbell-Hausdorff formula.

C.2 The $G_{2(2)}/\text{SO}(4)$ coset space

The Iwasawa decomposition (see e.g. [50]) of \mathfrak{g}_2 can be invoked to construct a coset representative of the maximally non-compact space $G_{2(2)}/\text{SO}(4)$ via the exponentiation of the Borel subalgebra of Cartan and positive root generators:

$$\mathcal{V} = e^{\frac{1}{2}\vec{\varphi}\cdot\vec{H}} e^{\zeta\mathbf{E}_1} e^{\sqrt{3}(-\theta_1\mathbf{E}_2+\theta_2\mathbf{E}_3)} e^{\xi_2\mathbf{E}_6} e^{2\sqrt{3}a\mathbf{E}_4-\xi_1\mathbf{E}_5}, \quad (\text{C.10})$$

where

$$q^u \equiv (\varphi_1, \varphi_2, \zeta, \theta_1, \theta_2, a, \xi_1, \xi_2), \quad u = 1, \dots, 8, \quad (\text{C.11})$$

are coordinates on $G_{2(2)}/\text{SO}(4)$, and the numerical factors in the exponentials have been chosen to make contact with the main text. In (C.10) we have defined $\vec{\varphi} = (\varphi_1, \varphi_2)$, and have used a dot to denote the usual Euclidean scalar product.

A metric on $G_{2(2)}/\text{SO}(4)$ can be constructed as follows. First introduce the right-invariant one-forms \mathcal{F}_1^i , $i = 1, \dots, 6$, defined as

$$\begin{aligned} \mathcal{F}_1^1 &= d\zeta, \\ \mathcal{F}_1^2 &= \sqrt{3} d\theta^1, \\ \mathcal{F}_1^3 &= \sqrt{3} (d\theta^2 - \zeta d\theta^1), \\ \mathcal{F}_1^4 &= 2\sqrt{3} G_1 = 2\sqrt{3} \left(da + \frac{1}{2}(\theta^1 d\theta^2 - \theta^2 d\theta^1) \right), \\ \mathcal{F}_1^5 &= F_1^1 = d\xi^1 - 6\theta^1 da - \theta^1(\theta^1 d\theta^2 - \theta^2 d\theta^1), \\ \mathcal{F}_1^6 &= F_1^2 - \zeta F_1^1 \\ &= d\xi^2 - 6\theta^2 da - \theta^2(\theta^1 d\theta^2 - \theta^2 d\theta^1) - \zeta (d\xi^1 - 6\theta^1 da - \theta^1(\theta^1 d\theta^2 - \theta^2 d\theta^1)). \end{aligned} \quad (\text{C.12})$$

The right-invariant Maurer-Cartan form associated to the coset representative (C.10) takes values in the Borel subalgebra. Using the one-forms (C.12), it can be written as

$$\begin{aligned} d\mathcal{V} \mathcal{V}^{-1} &= \frac{1}{2} d\vec{\varphi} \cdot \vec{H} + e^{\frac{1}{2}\vec{\alpha}_1 \cdot \vec{\varphi}} \mathcal{F}_1^1 \mathbf{E}_1 - e^{\frac{1}{2}\vec{\alpha}_2 \cdot \vec{\varphi}} \mathcal{F}_1^2 \mathbf{E}_2 + e^{\frac{1}{2}\vec{\alpha}_3 \cdot \vec{\varphi}} \mathcal{F}_1^3 \mathbf{E}_3 + e^{\frac{1}{2}\vec{\alpha}_4 \cdot \vec{\varphi}} \mathcal{F}_1^4 \mathbf{E}_4 \\ &\quad - e^{\frac{1}{2}\vec{\alpha}_5 \cdot \vec{\varphi}} \mathcal{F}_1^5 \mathbf{E}_5 + e^{\frac{1}{2}\vec{\alpha}_6 \cdot \vec{\varphi}} \mathcal{F}_1^6 \mathbf{E}_6. \end{aligned} \quad (\text{C.13})$$

Next, introduce the \mathfrak{g}_2 -valued one-form

$$P = \frac{1}{2} \left(d\mathcal{V} \mathcal{V}^{-1} + (d\mathcal{V} \mathcal{V}^{-1})^\sharp \right), \quad (\text{C.14})$$

and the quadratic form

$$\mathcal{M} = \mathcal{V}^\sharp \mathcal{V}, \quad (\text{C.15})$$

which are related by

$$\frac{1}{2} \mathcal{M}^{-1} d\mathcal{M} = \mathcal{V}^{-1} P \mathcal{V}. \quad (\text{C.16})$$

A globally G_2 right-invariant and locally $SO(4)$ left-invariant metric on the coset space is finally given by either of the two equivalent expressions

$$\begin{aligned} h_{uv}dq^u dq^v &= \frac{1}{4}\text{Tr}(PP) \\ &= \frac{1}{16}\text{Tr}(\mathcal{M}^{-1}d\mathcal{M}\mathcal{M}^{-1}d\mathcal{M}) . \end{aligned} \tag{C.17}$$

In terms of the forms (C.12), this is equivalent to

$$h_{uv}dq^u dq^v = \frac{1}{4}d\vec{\varphi} \cdot d\vec{\varphi} + \frac{1}{4}\sum_{i=1}^6 e^{\vec{\alpha}_i \cdot \vec{\varphi}} (\mathcal{F}_1^i)^2 , \tag{C.18}$$

and, in terms of the explicit coordinates (C.11) and the one-forms G_1, F_1^α , this is⁹

$$\begin{aligned} h_{uv}dq^u dq^v &= \frac{1}{4}(d\varphi_1)^2 + \frac{1}{4}(d\varphi_2)^2 + \frac{1}{4}e^{2\varphi_2}(d\zeta)^2 + \frac{3}{4}e^{-\frac{1}{\sqrt{3}}\varphi_1 - \varphi_2}(d\theta^1)^2 \\ &\quad + \frac{3}{4}e^{-\frac{1}{\sqrt{3}}\varphi_1 + \varphi_2}(d\theta^2 - \zeta d\theta^1)^2 + 3e^{-\frac{2}{\sqrt{3}}\varphi_1}(G_1)^2 + \frac{1}{4}e^{-\sqrt{3}\varphi_1 - \varphi_2}(F_1^1)^2 \\ &\quad + \frac{1}{4}e^{-\sqrt{3}\varphi_1 + \varphi_2}(F_1^2 - \zeta F_1^1)^2 . \end{aligned} \tag{C.19}$$

The Riemannian manifold $G_{2(2)}/SO(4)$ equipped with this metric is a quaternionic-Kähler space and, therefore, Einstein. Indeed, the Ricci tensor of (C.19) can be checked to give (-8) times the metric (C.19) itself. Being further a symmetric space, its holonomy group coincides with the isotropy group $SO(4)$, which should be thought of as a subgroup of the holonomy group $Sp(1) \times Sp(2)$ of a generic eight-dimensional quaternionic-Kähler space. The $Sp(1)$ factor of the holonomy is related to the existence of a triplet of complex structures $J^x, x = 1, 2, 3$, satisfying the quaternion algebra

$$J^x J^y = -\delta^{xy} + \epsilon^{xyz} J^z . \tag{C.20}$$

We now explicitly elucidate the $SO(4)$ holonomy of our quaternionic Kähler manifold $G_{2(2)}/SO(4)$, and in particular the canonical $Sp(1)$ factor giving rise to (C.20). To this end we first introduce the obvious orthonormal frame $e^{\bar{u}}$, where here $\bar{u} = 1, \dots, 8$, are tangent space indices, for the metric (C.19):

$$e^1 = \frac{1}{2}d\varphi_1 , \quad e^2 = \frac{1}{2}d\varphi_2 , \quad e^{i+2} = \frac{1}{2}e^{\frac{1}{2}\vec{\alpha}_i \cdot \vec{\varphi}} \mathcal{F}_1^i , \quad i = 1, \dots, 6. \tag{C.21}$$

The corresponding spin connection w , satisfying $de^{\bar{u}} + w^{\bar{u}}_{\bar{v}} \wedge e^{\bar{v}} = 0$, is a one-form valued in the Lie-algebra $\mathfrak{so}(8)$, the holonomy algebra for a generic orientable eight-dimensional manifold. Introducing the $\mathfrak{so}(8)$ generators $M_{\bar{u}\bar{v}} = E_{\bar{u}\bar{v}} - E_{\bar{v}\bar{u}}$, where $E_{\bar{u}\bar{v}}$ is here the 8×8 matrix with 1 in the (\bar{u}, \bar{v}) position and 0 elsewhere, we find that w can be written as

$$w = e^3 M_3 + e^4 N_2 - e^5 N_1 + e^6 N_3 + e^7 M_1 + e^8 M_2 , \tag{C.22}$$

⁹We find agreement with [51], up to an overall factor of $1/2$, with the identifications $\chi_1 = \zeta, \chi_2 = \sqrt{3}\theta^1, \chi_3 = \sqrt{3}\theta^2, \chi_4 = 2\sqrt{3}a, \chi_5 = \xi^1, \chi_6 = \xi^2$.

where we have defined the combinations of $\mathfrak{so}(8)$ generators

$$\begin{aligned}
M_1 &\equiv \sqrt{3}M_{17} + M_{27} + M_{38} + M_{46} \\
M_2 &\equiv \sqrt{3}M_{18} - M_{28} + M_{37} + M_{56} \\
M_3 &\equiv -2M_{23} - M_{45} - M_{78} \\
N_1 &\equiv -\frac{1}{\sqrt{3}}M_{15} + M_{25} - M_{34} + \frac{2}{\sqrt{3}}M_{46} + M_{68} \\
N_2 &\equiv \frac{1}{\sqrt{3}}M_{14} + M_{24} + M_{35} + \frac{2}{\sqrt{3}}M_{56} - M_{67} \\
N_3 &\equiv \frac{2}{\sqrt{3}}M_{16} - \frac{2}{\sqrt{3}}M_{45} + M_{47} + M_{58}.
\end{aligned} \tag{C.23}$$

These six generators close into the Lie algebra $\mathfrak{so}(4) \approx \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This can be seen by taking the further combinations

$$J_x^{(8)} = \frac{1}{4}(M_x - \sqrt{3}N_x), \quad L_x^{(8)} = \frac{1}{4}(3M_x + \sqrt{3}N_x), \quad x = 1, 2, 3, \tag{C.24}$$

and verifying that they satisfy the canonical commutation relations (C.8). We have thus demonstrated that the spin connection corresponding to the metric (C.19) takes values in $\mathfrak{so}(4) \subset \mathfrak{so}(8)$, which shows that the holonomy of $G_{2(2)}/\text{SO}(4)$ is indeed $\text{SO}(4) \approx \text{SU}(2) \times \text{SU}(2)$. The two $\mathfrak{su}(2)$ components can be seen by writing

$$w = \omega^x J_x^{(8)} + \Delta^x L_x^{(8)}, \tag{C.25}$$

where we have defined the one-forms $\omega^x, \Delta^x, x = 1, 2, 3$, as

$$\omega^1 = \sqrt{3}e^5 + e^7, \quad \omega^2 = -\sqrt{3}e^4 + e^8, \quad \omega^3 = e^3 - \sqrt{3}e^6, \tag{C.26}$$

and

$$\Delta^1 = -\frac{1}{\sqrt{3}}e^5 + e^7, \quad \Delta^2 = \frac{1}{\sqrt{3}}e^4 + e^8, \quad \Delta^3 = e^3 + \frac{1}{\sqrt{3}}e^6. \tag{C.27}$$

It turns out that the ω^x are the components of the canonical $Sp(1)$ part of the connection related to (C.20). To see this we calculate the curvature of ω^x , defined by

$$-2K^x = d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z, \tag{C.28}$$

to find

$$\begin{aligned}
K^1 &= \frac{1}{2} \left(e^{15} + \sqrt{3}e^{17} - \sqrt{3}e^{25} + e^{27} + \sqrt{3}e^{34} + e^{38} - e^{46} - \sqrt{3}e^{68} \right), \\
K^2 &= \frac{1}{2} \left(-e^{14} + \sqrt{3}e^{18} - \sqrt{3}e^{24} - e^{28} - \sqrt{3}e^{35} + e^{37} - e^{56} + \sqrt{3}e^{67} \right), \\
K^3 &= \frac{1}{2} \left(-2e^{16} - 2e^{23} + e^{45} - \sqrt{3}e^{47} - \sqrt{3}e^{58} - e^{78} \right),
\end{aligned} \tag{C.29}$$

where $e^{15} = e^1 \wedge e^5$, etc. Some algebra now shows that $(J^x)^{\bar{u}\bar{v}} = \delta^{\bar{u}\bar{v}}(K^x)_{\bar{u}\bar{v}}$, $x = 1, 2, 3$, is indeed a triplet of complex structures satisfying the quaternion algebra (C.20). Finally, we note that it can be checked that the curvature of the $Sp(1) \subset Sp(2)$ connection Δ does not lead to a quaternionic structure.

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