

THE PLANE SYMMETRIC EINSTEIN-DUST SYSTEM WITH POSITIVE COSMOLOGICAL CONSTANT

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Abstract. The Einstein equations with a positive cosmological constant are coupled to the pressureless perfect fluid matter in plane symmetry. Under suitable restrictions on the initial data, the resulting Einstein-dust system is proved to have a global classical solution in the future time direction. Some late time asymptotic properties are obtained as well.

Keywords: Einstein-dust equations; positive Λ ; smooth solutions.

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1. Introduction

In [12], plane symmetric solutions of the Einstein–Vlasov system with positive cosmological constant were investigated. It was shown that a spacetime of this type which is initially expanding exists globally in the future when expressed in an areal time coordinate t and information was obtained about its asymptotics for $t \rightarrow \infty$. It is future geodesically complete and resembles the de Sitter solution at late times. Information is obtained on the decay rates of the components of the energy-momentum tensor.

This paper is concerned with the question, to what extent analogues of these results for the Einstein–Vlasov system hold in the case of the Einstein-dust system. There is an issue which has to be addressed right at the start. This is that of shell-crossing singularities. The intuitive idea behind this concept, as explained in [5, 9], is the following. A shell of dust particles which are related to each other by the symmetry of the spacetime moves in a coherent way. If two of these shells

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collide then the intermediate shells are trapped between them, so that the matter density is forced to blow up. For more information about shell-crossing singularities, see for instance, [7] and references therein. For general plane symmetric solutions of the Einstein-dust system, it must be expected that shell-crossing singularities develop, even from smooth initial data. For that reason global classical solutions cannot be expected to exist in general. In order to have a global existence theorem in the framework of classical solutions it is therefore likely to be necessary to make some kind of smallness assumption on the initial data. There is another fact which is essential in the following. This is the presence of a positive cosmological constant, known as a mechanism leading to solutions of the Einstein equations with exponential expansion. Results related to this idea have been obtained in [12, 10]. They are used in this paper to guess the decay rates on the geometric and matter quantities providing a basis to the bootstrap argument for the proof of the main result (Theorem 3.3).

The rest of the paper is organized as follows. In Sec. 2, the equations are derived and a local existence theorem is obtained for the corresponding Cauchy problem. In Sec. 3, the solution is shown to exist globally in the future time direction and its late-time asymptotic behavior is investigated, provided some restrictions on the initial datum are satisfied.

2. Preliminaries

2.1. The Einstein equations

Let (M, g) be a spacetime, where the manifold is assumed to be $M = I \times \mathbb{T}^3$, I is a real interval and $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ is the three-torus. The metric g and the matter fields are required to be invariant under the action of the Euclidean group E_2 on the universal cover. It is also required that the spacetime has an E_2 -invariant Cauchy surface of constant areal time. In such conditions the metric is assumed to have the form

$$ds^2 = -e^{2\eta(t,x)} dt^2 + e^{2\lambda(t,x)} dx^2 + t^2(dy^2 + dz^2), \quad (2.1)$$

where $t > 0$, η and λ are periodic in x , y and z range in $[0, 2\pi]$.

The Einstein equations read

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta}, \quad (2.2)$$

where $G^{\alpha\beta}$ is the Einstein tensor, $T^{\alpha\beta}$ the energy-momentum tensor and Λ is the cosmological constant we assume to be positive. We introduce the notation $\rho = e^{2\eta} T^{00}$, $j = e^{\lambda+\eta} T^{01}$ and $S = e^{2\lambda} T^{11}$.

After computations in plane symmetry with the previous coordinates considerations, we obtain from (2.2) the following equations where the subscripts t and x refer to partial derivatives with respect to t and x respectively:

$$e^{-2\eta}(2t\lambda_t + 1) - \Lambda t^2 = 8\pi t^2 \rho, \quad (2.3)$$

$$e^{-2\eta}(2t\eta_t - 1) + \Lambda t^2 = 8\pi t^2 S, \quad (2.4)$$

$$\eta_x = -4\pi t e^{\lambda+\eta} j, \tag{2.5}$$

$$e^{-2\lambda} (\eta_{xx} + \eta_x(\eta_x - \lambda_x)) - e^{-2\eta} \left(\lambda_{tt} + (\lambda_t - \eta_t) \left(\lambda_t + \frac{1}{t} \right) \right) + \Lambda = 0. \tag{2.6}$$

2.2. The equations for dust

We consider a pressureless perfect fluid with energy density $\mu := \mu(t, x) > 0$ and 4-velocity U^α . The latter is normalized to be of unit length $U^\alpha U_\alpha = -1$. The plane symmetry allows us to set $U^\alpha := \xi(e^{-\eta}, e^{-\lambda}u, 0, 0)$ where $\xi = (1 - u^2)^{-1/2}$ is the relativistic factor, $u := u(t, x)$ being the scalar velocity that satisfies $|u| < 1$. The energy momentum tensor for a pressureless perfect fluid is

$$T^{\alpha\beta} = \mu U^\alpha U^\beta,$$

that is

$$T^{00} = e^{-2\eta} \frac{\mu}{1 - u^2} =: e^{-2\eta} \rho$$

$$T^{01} = e^{-\lambda-\eta} \frac{\mu u}{1 - u^2} =: e^{-\lambda-\eta} j$$

$$T^{11} = e^{-2\lambda} \frac{\mu u^2}{1 - u^2} =: e^{-2\lambda} S,$$

the other components being zero.

The equations for dust are given by

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{2.7}$$

The components $\nabla_\alpha T^{\alpha 2}$ and $\nabla_\alpha T^{\alpha 3}$ vanish identically. Computing the remaining two components gives

$$(e^\lambda \rho)_t + (e^\eta j)_x = -\lambda_t e^\lambda S - \eta_x e^\eta j - \frac{2}{t} e^\lambda \rho, \tag{2.8}$$

$$(e^\lambda j)_t + (e^\eta S)_x = -\lambda_t e^\lambda j - \eta_x e^\eta \rho - \frac{2}{t} e^\lambda j, \tag{2.9}$$

and expressing Eqs. (2.8)–(2.9) in terms of the variables μ and u gives

$$\begin{aligned} (1 - u^2)[\mu_t + u e^{\eta-\lambda} \mu_x] + \mu[2uu_t + (1 + u^2)e^{\eta-\lambda} u_x] \\ = -\mu(1 - u^2)[(1 + u^2)\lambda_t + 2t^{-1} + 2u\eta_x e^{\eta-\lambda}], \end{aligned} \tag{2.10}$$

$$\begin{aligned} (1 - u^2)u[\mu_t + u e^{\eta-\lambda} \mu_x] + \mu[(1 + u^2)u_t + 2u e^{\eta-\lambda} u_x] \\ = -\mu(1 - u^2)[2u(\lambda_t + t^{-1}) + (1 + u^2)\eta_x e^{\eta-\lambda}]. \end{aligned} \tag{2.11}$$

Adding and subtracting (2.10) and (2.11) yields

$$\begin{aligned} (1 - u)[\mu_t + u e^{\eta-\lambda} \mu_x] + \mu[u_t + e^{\eta-\lambda} u_x] \\ = -\mu[(1 - u^2)(\lambda_t + \eta_x e^{\eta-\lambda}) + 2(1 - u)t^{-1}], \end{aligned} \tag{2.12}$$

$$\begin{aligned} (1 + u)[\mu_t + u e^{\eta-\lambda} \mu_x] + \mu[-u_t + e^{\eta-\lambda} u_x] \\ = -(1 + u)\mu[(1 - u)\lambda_t + 2t^{-1} - (1 - u)\eta_x e^{\eta-\lambda}], \end{aligned} \tag{2.13}$$

and the linear combinations (2.12) + (2.13) and $(1 + u)(2.12) + (u - 1)(2.13)$ lead to

$$D\mu = -\mu [(1 - u^2)\lambda_t - 2t^{-1} + e^{\eta-\lambda}u_x], \tag{2.14}$$

$$Du = (1 - u^2) [-u\lambda_t - e^{\eta-\lambda}\eta_x], \tag{2.15}$$

where the derivative is

$$D := \partial_t + ue^{\eta-\lambda}\partial_x.$$

This is called the characteristic derivative and we denote the corresponding integral curve by (t, γ) . This means that this curve satisfies the differential equation $\gamma_t = ue^{\eta-\lambda}$, and on the curve $D = \frac{d}{dt}$ so that we can, for instance, rewrite (2.14) in the form

$$\frac{d}{dt}\mu(t, \gamma(t)) = -\mu[(1 - u^2)\lambda_t - 2t^{-1} + e^{\eta-\lambda}u_x](t, \gamma(t)).$$

2.3. *The Cauchy problem and local existence*

The object of our study is the plane symmetric Einstein-dust system (2.3)–(2.6), (2.14)–(2.15) with unknowns λ, η, μ and u . The initial data are prescribed at some time $t_0 \geq 1$. To analyze the solutions of these equations, the first step is to obtain a local existence theorem. The method, which has been used, for instance, in [8] for the Einstein–Vlasov system in plane symmetry, consists on constructing an iteration and proving its convergence. In the present investigation, we follow another approach, which is described in [11]. There are several steps and it is not convenient to keep track of the differentiability in the process. For this reason, only the case of C^∞ initial data will be treated in this paper. There is a general (without symmetry) local existence theorem for the Einstein-dust system by Choquet–Bruhat [1].

In fact, in that reference the author writes the Einstein-dust system (2.2)–(2.7) in the equivalent form

$$u^\nu \nabla_\nu G_{\alpha\beta} = -8\pi\mu u_\alpha u_\beta \nabla_\nu u^\nu, \tag{2.16}$$

$$u^\nu \nabla_\nu u_\beta = 0, \tag{2.17}$$

$$\nabla_\nu (\mu u^\nu) = 0, \tag{2.18}$$

in order to prove that the equations are hyperbolic in the Leray sense. Then by the Leray–Dionne theory [3, 6], this implies existence and uniqueness of the solution to the Cauchy problem for Einstein-dust equations. For more details, see also [2].

We can apply this general result to the case with symmetry. Consider a plane symmetric compact C^∞ initial data for the Cauchy problem. Then the symmetry is inherited by the corresponding solutions. (See [4, Sec. 5.6] for a discussion of this.) Next areal coordinates can be introduced in the spacetime. The conclusion is a local existence and uniqueness theorem for the plane symmetric Einstein-dust system. The solution, like the initial data, is C^∞ .

3. Global Existence to the Future and Asymptotics

This section is concerned with the main result of this paper. We first prove two lemmas dealing with bounds on the unknowns as well as all their derivatives.

In what follows, C will denote a positive constant estimating functions that are uniformly bounded, and it may change from line to line.

Let $P_n(t) := \|\partial_x^n \lambda(t)\|_\infty + \|\partial_x^n \mu(t)\|_\infty + \|\partial_x^n w(t)\|_\infty$

Lemma 3.1. *Consider a C^∞ plane symmetric solution of Einstein-dust system on a time interval $[0, T)$ such that the following estimates hold, where $w := e^{-\lambda}u_x$:*

$$|\lambda_t - t^{-1}| \leq Ct^{-4}, \quad |\Lambda t^2 e^{2\eta} - 3| \leq Ct^{-3}, \quad |\lambda_{tx}| \leq Ct^{-4} \tag{3.1}$$

$$|\eta_x| \leq Ct^{-3}, \quad |\eta_{xx}| \leq Ct^{-3}, \quad \mu \leq Ct^{-3} \tag{3.2}$$

$$|u| \leq Ct^{-1}, \quad |\mu_x| \leq Ct^{-3}, \quad |u_x| \leq Ct^{-1}, \quad |w_x| \leq Ct^{-2}. \tag{3.3}$$

If all derivatives with respect to x of order up to n of the quantities λ, μ, w, η and u are bounded then all derivatives with respect to x of order up to $n + 1$ of the same quantities are bounded.

Proof. Note that from the hypotheses of the lemma, it follows that the quantities λ, μ, w, η and u , as well as all their first order derivatives are bounded on $[0, T)$.

By definition $w = e^{-\lambda}u_x$, which implies

$$u_{xx} = e^\lambda w_x + \lambda_x u_x, \tag{3.4}$$

and from (2.5),

$$\eta_{xx} = -4\pi t e^{\eta+\lambda} j_x + \eta_x (\eta_x + \lambda_x). \tag{3.5}$$

Differentiating u_{xx} and η_{xx} $n - 1$ times with respect to x , the boundedness of $\partial_x^{n+1}\eta$ and $\partial_x^{n+1}u$ follows immediately from the hypotheses of the lemma.

The expression for λ_{tx} is

$$\lambda_{tx} = \eta_x e^{2\eta} (\Lambda t + 8\pi t \rho) + 4\pi t \rho_x e^{2\eta}. \tag{3.6}$$

Differentiating this n times with respect to x gives a linear equation for $\partial_t(\partial_x^{n+1}\lambda)$ with coefficients which are known to be bounded, except for terms involving $\partial_x^{n+1}\rho$. But the latter can be estimated in terms of $\partial_x^{n+1}\mu$ so that the following holds

$$|\partial_t(\partial_x^{n+1}\lambda)| \leq C(1 + |\partial_x^{n+1}\mu|). \tag{3.7}$$

Integrating this in time implies that

$$\|\partial_x^{n+1}\lambda(t)\|_\infty \leq \|\partial_x^{n+1}\lambda(t_0)\|_\infty + C \int_{t_0}^t (1 + P_{n+1}(s)) ds. \tag{3.8}$$

Now recall that

$$\begin{aligned} D\mu_x &= -\mu_x [(1 - u^2)\lambda_t - 2t^{-1} + 2e^{\eta-\lambda}u_x + u(\eta_x - \lambda_x)e^{\eta-\lambda}] \\ &\quad - \mu [-2uu_x\lambda_t + (1 - u^2)\lambda_{tx} + (\eta_x - \lambda_x)e^{\eta-\lambda}u_x + e^{\eta-\lambda}u_{xx}], \end{aligned} \tag{3.9}$$

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and

$$\begin{aligned}
 D(w_x) = & -\Lambda t u(1 - u^2)(\eta_{xx} + 2\eta_x^2 - \lambda_x \eta_x)e^{2\eta-\lambda} - 2\eta_x w e^\eta [w + t\Lambda(1 - 3u^2)e^\eta] \\
 & + 3t^{-1} u u_x w (\Lambda t^2 e^{2\eta} - 1) - u w [\eta_{xx} + \eta_x(\eta_x - \lambda_x)]e^{\eta-\lambda} - \lambda_{tx} w \\
 & - \left[\lambda_t + [2u_x + u(2\eta_x - \lambda_x)]e^{\eta-\lambda} + \frac{t^{-1}(1 - 3u^2)}{2} (\Lambda t^2 e^{2\eta} - 1) \right] w_x.
 \end{aligned}
 \tag{3.10}$$

Differentiating (3.9) n times with respect to x and using (3.5) and (3.6) shows that $D(\partial_x^{n+1}\mu)$ depends linearly on $\partial_x^{n+1}\lambda$, $\partial_x^{n+1}\mu$ and $\partial_x^{n+2}u$ with bounded coefficients. But $\partial_x^{n+2}u$ can be estimated in terms of $\partial_x^{n+1}w$ and $\partial_x^{n+1}\lambda$. It then follows that

$$|D(\partial_x^{n+1}\mu)| \leq C(1 + |\partial_x^{n+1}\mu| + |\partial_x^{n+1}\lambda| + |\partial_x^{n+1}w|), \tag{3.11}$$

and integrating this along the characteristic γ implies that

$$\|\partial_x^{n+1}\mu(t)\|_\infty \leq \|\partial_x^{n+1}\mu(t_0)\|_\infty + C \int_{t_0}^t (1 + P_{n+1}(s)) ds. \tag{3.12}$$

Likewise taking the x -derivative n times in (3.10) leads to

$$|D(\partial_x^{n+1}w)| \leq C(1 + |\partial_x^{n+1}\mu| + |\partial_x^{n+1}\lambda| + |\partial_x^{n+1}w|), \tag{3.13}$$

and integration along γ implies

$$\|\partial_x^{n+1}w(t)\|_\infty \leq \|\partial_x^{n+1}w(t_0)\|_\infty + C \int_{t_0}^t (1 + P_{n+1}(s)) ds. \tag{3.14}$$

Putting (3.8), (3.12) and (3.14) together implies

$$P_{n+1}(t) \leq P_{n+1}(t_0) + C \int_{t_0}^t (1 + P_{n+1}(s)) ds. \tag{3.15}$$

By Gronwall's inequality, it follows that P_{n+1} is bounded and thus so are $\partial_x^{n+1}\lambda$, $\partial_x^{n+1}\mu$ and $\partial_x^{n+1}w$. This completes the proof of the lemma. \square

Lemma 3.2. *If the hypotheses of Lemma 3.1 are satisfied and if all derivatives of the quantities λ , η , μ and u of the form $\partial_t^k \partial_x^n$ with n arbitrary and $k \leq m$ are bounded then the derivatives of the form $\partial_t^{m+1} \partial_x^n$ of the same quantities are bounded.*

Proof. From the evolution equations we have

$$\lambda_t = \frac{1}{2}(\Lambda t e^{2\eta} - t^{-1}) + 4\pi t e^{2\eta} \rho, \tag{3.16}$$

$$\eta_t = \frac{1}{2}(t^{-1} - \Lambda t e^{2\eta}) + 4\pi t e^{2\eta} S, \tag{3.17}$$

$$\mu_t = -u e^{\eta-\lambda} \mu_x - \mu [(1 - u^2)\lambda_t - 2t^{-1} + e^{\eta-\lambda} u_x], \tag{3.18}$$

$$u_t = -u e^{\eta-\lambda} u_x + (1 - u^2) [-u \lambda_t - e^{\eta-\lambda} \eta_x]. \tag{3.19}$$

Differentiating (3.16)–(3.19) n times with respect to x and m times with respect to t allows $\partial_t^{m+1}\partial_x^n\lambda$, $\partial_t^{m+1}\partial_x^n\eta$, $\partial_t^{m+1}\partial_x^n\mu$ and $\partial_t^{m+1}\partial_x^n u$ to be bounded. \square

We can now prove the main result of the present investigation.

Theorem 3.3. *Consider any C^∞ solution of Einstein-dust system with positive cosmological constant in plane symmetry written in areal coordinates with C^∞ initial data. Let δ be a positive constant and suppose the following inequalities hold:*

$$|\lambda_t(t_0) - t_0^{-1}| \leq \delta, \quad |\Lambda t_0^2 e^{2\eta(t_0)} - 3| \leq \delta, \quad |\eta_x(t_0)| \leq \delta, \quad |\lambda_{tx}(t_0)| \leq \delta, \quad (3.20)$$

$$\begin{aligned} |\eta_{xx}(t_0)| \leq \delta, \quad \mu(t_0) \leq \delta, \quad |u(t_0)| \leq \delta, \quad |\mu_x(t_0)| \leq \delta, \\ |u_x(t_0)| \leq \delta, \quad |w_x(t_0)| \leq \delta. \end{aligned} \quad (3.21)$$

Then if δ is sufficiently small, the corresponding solution exists on $[t_0, \infty)$. Moreover, for this solution the following properties hold at late times:

$$|\lambda_t - t^{-1}| = O(t^{-4}), \quad |\Lambda t^2 e^{2\eta} - 3| = O(t^{-3}), \quad |\eta_x| = O(t^{-3}), \quad (3.22)$$

$$|\eta_{xx}| = O(t^{-3}), \quad |\lambda_{tx}| = O(t^{-4}), \quad \mu = O(t^{-3}), \quad (3.23)$$

$$|\mu_x| = O(t^{-3}), \quad |u| = O(t^{-1}), \quad |u_x| = O(t^{-1}), \quad |w_x| = O(t^{-2}). \quad (3.24)$$

Proof. The proof proceeds by a bootstrap argument.

By continuity it follows from the hypothesis (3.20)–(3.21) that

$$|\lambda_t(t) - t^{-1}| \leq 2\delta, \quad |\Lambda t^2 e^{2\eta(t)} - 3| \leq 2\delta, \quad |\eta_x(t)| \leq 2\delta, \quad |\eta_{xx}(t)| \leq 2\delta,$$

$$|\lambda_{tx}(t)| \leq 2\delta, \quad \mu(t) \leq 2\delta, \quad |u(t)| \leq 2\delta, \quad |\mu_x| \leq 2\delta, \quad |u_x| \leq 2\delta, \quad |w_x| \leq 2\delta,$$

for t close to t_0 .

Let C_1 and ε be constants for $0 < C_1 < 1$ and $0 < \varepsilon < 1/2$. We can reduce δ if necessary so that $2\delta < C_1 t_0^{-4+\varepsilon}$. Then there exists some time interval on which the solution of the Einstein-dust system exists and the following bootstrap assumption is satisfied

$$|\lambda_t - t^{-1}| \leq C_1 t^{-4+\varepsilon}, \quad |\Lambda t^2 e^{2\eta} - 3| \leq C_1 t^{-3+\varepsilon}, \quad |\lambda_{tx}| \leq C_1 t^{-4+\varepsilon}, \quad (3.25)$$

$$|\eta_x| \leq C_1 t^{-3+\varepsilon}, \quad |\eta_{xx}| \leq C_1 t^{-3+\varepsilon}, \quad \mu \leq C_1 t^{-3+\varepsilon}, \quad (3.26)$$

$$|u| \leq C_1 t^{-1+\varepsilon}, \quad |\mu_x| \leq C_1 t^{-3+\varepsilon}, \quad |u_x| \leq C_1 t^{-1+\varepsilon}, \quad |w_x| \leq C_1 t^{-2+\varepsilon}. \quad (3.27)$$

Consider the maximal interval $[t_0, t_*)$ on which the solution of the full system (2.3)–(2.6), (2.14)–(2.15) exists and (3.25)–(3.27) hold. Suppose t_* is finite.

Putting inequalities (3.25)–(3.27) into equations coming from the system (2.3)–(2.6), (2.14)–(2.15) allows new estimates to be derived. We first derive an estimate

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for u . For this purpose an evolution equation for tu can be obtained from (2.15), using the field Eqs. (2.3)–(2.4) involving λ_t and η_x . The result is

$$D(tu) = u^3 - \frac{u}{2}(1 - u^2)(\Lambda e^{2\eta}t^2 - 3). \tag{3.28}$$

Using the bootstrap assumption on $|u|$ and $|\Lambda t^2 e^{2\eta} - 3|$, integrating the resulting inequality along the integral curve γ and keeping the worst powers, it follows from (3.28) that

$$|u(t)| \leq [t_0|u(t_0)| + C_1^3 + C_1^2]t^{-1} =: C_2t^{-1}. \tag{3.29}$$

Next we derive an estimate for μ . Using (2.14), an evolution equation for $t^3\mu$ follows:

$$D(t^3\mu) = t^3\mu[t^{-1}u^2 - (1 - u^2)(\lambda_t - t^{-1}) - e^{\eta-\lambda}u_x]. \tag{3.30}$$

An estimate for $e^{2\eta}$, $e^{\eta-\lambda}$ and $e^{\eta+\lambda}$ will be also required.

$$\begin{aligned} e^{2\eta} &= \Lambda^{-1}t^{-2}(\Lambda t^2 e^{2\eta}) \\ &\leq \Lambda^{-1}t^{-2}[(\Lambda t^2 e^{2\eta} - 3) + 3] \\ &\leq 3\Lambda^{-1}t^{-2} + C_1\Lambda^{-1}t^{-5+\varepsilon}. \end{aligned} \tag{3.31}$$

On the other hand, by assumption $|\lambda_t - t^{-1}| \leq C_1t^{-4+\varepsilon}$, and integrating this in time implies that $e^{-\lambda} \leq e^{C_1-\lambda(t_0)}t_0t^{-1}$ and $e^\lambda \leq e^{C_1+\lambda(t_0)}t_0^{-1}t$. Thus

$$\begin{aligned} e^{\eta-\lambda} &\leq \Lambda^{-1/2}(\sqrt{3} + C_1^{1/2})t_0e^{C_1-\lambda(t_0)}t^{-2}, \quad \text{and} \\ e^{\eta+\lambda} &\leq \Lambda^{-1/2}(\sqrt{3} + C_1^{1/2})t_0^{-1}e^{C_1+\lambda(t_0)}. \end{aligned} \tag{3.32}$$

Using this, the bootstrap assumption, integration along γ and keeping the worst powers it follows from (3.30) that

$$\mu \leq [t_0^3\mu + C_1C_2 + C_1^2 + C_1\Lambda^{-1/2}(\sqrt{3} + C_1^{1/2})t_0e^{C_1-\lambda(t_0)}]t^{-3} =: C_3t^{-3}. \tag{3.33}$$

Now estimates for $|\Lambda t^2 e^{2\eta} - 3|$ and $|t\lambda_t - 1|$ will be derived. Estimates for the matter quantities S and ρ are needed for this purpose. From the definition of S and ρ and the estimates for μ and u obtained above, we obtain

$$\rho \leq \frac{C_3}{1 - C_2}t^{-3}, \quad S \leq \frac{C_3C_2^2}{1 - C_2}t^{-5}. \tag{3.34}$$

From (2.4), we have

$$\partial_t \left[-\frac{1}{3}te^{-2\eta}(\Lambda t^2 e^{2\eta} - 3) \right] = -8\pi t^2 S,$$

so that using (3.31), (3.34), integration and keeping the worst powers gives

$$\begin{aligned} |\Lambda t^2 e^{2\eta} - 3| &\leq [3\Lambda^{-1} + C_1\Lambda^{-1}] \left[t_0|\Lambda t_0^2 - 3e^{-2\mu(t_0)}| + \frac{24\pi C_3C_2^2}{1 - C_2} \right] t^{-3} \\ &=: C_4t^{-3}. \end{aligned} \tag{3.35}$$

From (2.3), we have

$$t\lambda_t - 1 = \frac{1}{2}(\Lambda e^{2\eta}t^2 - 3) + 4\pi t^2 e^{2\eta}\rho,$$

and using (3.31), (3.34), (3.35) and keeping the worst powers yields

$$|t\lambda_t - 1| \leq \left[C_4 + 4\pi\Lambda^{-1}(3 + C_1)\frac{C_3}{1 - C_2} \right] t^{-3} =: C_5 t^{-3}. \tag{3.36}$$

An estimate for η_x will be derived. Recalling that

$$\eta_x = -4\pi t e^{\lambda + \eta} j \quad \text{and} \quad j = \frac{\mu u}{1 - u^2},$$

it follows from (3.29) and (3.32)–(3.33) that

$$\begin{aligned} |\eta_x| &\leq 4\pi\Lambda^{-1/2}(C_1^{1/2} + \sqrt{3})e^{C_1 + \lambda(t_0)}\frac{C_2 C_3}{1 - C_2}t^{-3} \\ &=: C_6 t^{-3}. \end{aligned} \tag{3.37}$$

The following are estimates for u_x and μ_x which will be needed in order to get better estimates for λ_{tx} and η_{xx} .

An evolution equation for tu_x can be derived by taking the x -derivative in (3.28):

$$\begin{aligned} D(tu_x) &= (3u^2 - 1)u_x(t\lambda_t - 1) + 3u_x u^2 + t(\eta_x + \lambda_x)u u_x e^{\eta - \lambda} - tu_x^2 e^{\eta - \lambda} \\ &\quad - t(1 - u^2)[u\lambda_{tx} + (\eta_x - \lambda_x)\eta_x e^{\eta - \lambda} + \eta_{xx} e^{\eta - \lambda}], \end{aligned} \tag{3.38}$$

and using the bootstrap assumption, some estimates obtained above as well as integration along γ leads to

$$\begin{aligned} |u_x| &\leq [t_0|u_x(t_0)| + 4C_1^2 + 4t_0C_1\Lambda^{-1/2}(\sqrt{3} + C_1^{1/2})e^{C_1 - \lambda(t_0)}(2C_1 + |\lambda_x(t_0)|)]t^{-1} \\ &=: C_7 t^{-1}. \end{aligned} \tag{3.39}$$

Now taking the x -derivative in (2.14) allows us to obtain an evolution equation for $t^3\mu_x$:

$$\begin{aligned} D(t^3\mu_x) &= -t^2(t\lambda_t - 1)[(1 - u^2)\mu_x - 2\mu u u_x] - 2t^3\mu_x u_x e^{\eta - \lambda} \\ &\quad - t^3(\eta_x - \lambda_x)u\mu_x e^{\eta - \lambda} + t^2(u^2\mu_x + 2\mu u u_x) - t^3(1 - u^2)\mu\lambda_{tx} \\ &\quad - t^3(\eta_x - \lambda_x)\mu u_x e^{\eta - \lambda} - t^3\mu u_{xx} e^{\eta - \lambda}. \end{aligned} \tag{3.40}$$

So far we know how to estimate all the terms on the right-hand side in (3.40) except the last one in u_{xx} which needs to be worked out carefully. We can obtain an evolution equation for u_{xx} by taking the x -derivative of Du_x . This does not give a satisfactory result because terms containing λ_{xx} occur, and we do not know how to estimate them. This difficulty can be overcome if we rather take the x -derivative of $D(e^{-\lambda}u_x)$, which will lead to an estimate for w_x (where $w = e^{-\lambda}u_x$). An estimate for u_{xx} will then be deduced from the following relation which is obtained by differentiating the equality $w = e^{-\lambda}u_x$ in x :

$$u_{xx} = e^\lambda w_x + \lambda_x u_x. \tag{3.41}$$

Note that the factor $e^{-\lambda}$ allows us to eliminate bad terms such as λ_{xx} . This device has been used in another context [13].

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The term w_x will now be estimated. An evolution equation for w is

$$Dw = -\lambda_t w - w^2 e^\eta - u \eta_x w e^{\eta-\lambda} - t^{-1} w (1 - 3u^2) - \frac{t^{-1} w (1 - 3u^2)}{2} (\Lambda t^2 e^{2\eta} - 3) - \Lambda t \eta_x u (1 - u^2) e^{2\eta-\lambda},$$

so that differentiating this in x implies

$$\begin{aligned} D(t^2 w_x) &= -\Lambda t^3 u (1 - u^2) (\eta_{xx} + 2\eta_x^2 - \lambda_x \eta_x) e^{2\eta-\lambda} - 2t^2 \eta_x w e^\eta [w + t\Lambda(1 - 3u^2) e^\eta] \\ &\quad + 6t u u_x w + 3t u u_x w (\Lambda t^2 e^{2\eta} - 3) - t^2 u w (\eta_{xx} + \eta_x (\eta_x - \lambda_x)) e^{\eta-\lambda} \\ &\quad - t^2 \lambda_{tx} w - \left[t(t\lambda_t - 1) - 3tu^2 + t^2 [2u_x + u(2\eta_x - \lambda_x)] \right] e^{\eta-\lambda} \\ &\quad + \frac{t(1 - 3u^2)}{2} (\Lambda t^2 e^{2\eta} - 3) w_x. \end{aligned} \tag{3.42}$$

We use the bootstrap assumption, some estimates obtained above and integration along γ to get

$$\begin{aligned} |w_x| &\leq [t_0^2 |w_x(t_0)| + 4C_7 C_1 t_0 e^{C_1 - \lambda(t_0)} (1 + |\lambda_x(t_0)|) \\ &\quad + 4C_1^2 (1 + \Lambda^{-1/2}) (1 + |\lambda_x(t_0)|)] t^{-2} \\ &=: C_8 t^{-2}. \end{aligned} \tag{3.43}$$

It then follows from (3.41) that

$$|u_{xx}| \leq [C_8 t_0^{-1} e^{C_1 + \lambda(t_0)} + C_7 (C_1 + |\lambda_x(t_0)|)] t^{-1} =: C_{u_{xx}} t^{-1}. \tag{3.44}$$

This together with (3.40) imply that

$$\begin{aligned} |\mu_x| &\leq [t_0^3 |\mu_x(t_0)| + 7C_1^2 \\ &\quad + (3C_1 + 3C_{u_{xx}} + |\lambda_x(t_0)|) (\sqrt{3} + C_1^{1/2}) \Lambda^{-1/2} t_0 C_1 e^{C_1 - \lambda(t_0)}] t^{-3} \\ &=: C_9 t^{-3}. \end{aligned} \tag{3.45}$$

Estimates for ρ_x and j_x are required in order to derive estimates for λ_{tx} and η_{xx} . We have

$$\rho_x = \frac{\mu_x}{1 - u^2} + \frac{2u u_x \mu}{(1 - u^2)^2};$$

using estimates established above gives

$$|\rho_x| \leq \frac{C_9 + 2C_2 C_3 C_8}{(1 - C_1)^2} t^{-3},$$

and recalling that

$$\lambda_{tx} = \eta_x e^{2\eta} (\Lambda t + 8\pi t \rho) + 4\pi t \rho_x e^{2\eta},$$

it follows that

$$\begin{aligned} |\lambda_{tx}| &\leq \left[\Lambda^{-1} C_6 (3 + C_1) \left(\Lambda + \frac{8\pi C_3}{1 - C_1} + 4\pi \frac{C_9 + 2C_2 C_3 C_8}{(1 - C_1)^2} \right) \right] t^{-4} \\ &=: C_{10} t^{-4}. \end{aligned} \tag{3.46}$$

Now

$$j_x = \frac{\mu_x u + \mu u_x}{1 - u^2} + \frac{2u^2 u_x \mu}{(1 - u^2)^2},$$

so that using estimates obtained above gives

$$|j_x| \leq \frac{C_2 C_9 + C_3 C_8 (2C_2^2 + 1)}{(1 - C_1)^2} t^{-4} =: C_{j_x} t^{-4}. \tag{3.47}$$

Taking the spatial derivative of (2.5) gives

$$\eta_{xx} = -4\pi t e^{\eta + \lambda} j_x + \eta_x (\eta_x + \lambda_x). \tag{3.48}$$

It then follows that

$$\begin{aligned} |\eta_{xx}| &\leq [4\pi \Lambda^{-1/2} (C_1^{1/2} + \sqrt{3}) e^{C_1 + \lambda(t_0)} C_{j_x} + C_4 (C_4 + C_1 + |\lambda_x(t_0)|)] t^{-3} \\ &=: C_{11} t^{-3}. \end{aligned} \tag{3.49}$$

The constants $C_2 - C_{11}$ appearing along the proof are all less than or equal to $C \times (g(\delta) + C_1^2)$, with C a positive constant and $g(\delta)$ a positive function of δ tending to 0 as δ tends to 0. Therefore it is always possible to choose C_1 and δ small enough in such a way that $CC_1 \leq 1/2$ and $Cg(\delta) \leq C_1/2$, and so the constants $C_2 - C_{11}$ are all less than C_1 . This together with Lemmas 3.1–3.2 show that all derivatives of λ , η , μ and u are bounded on $[0, t_*)$. This means that the solution can be extended to a time interval $[t_0, t_1)$ on which (3.25)–(3.27) hold, with $t_1 > t_*$. This contradicts the maximality of the interval $[t_0, t_*)$. Therefore, $t_* = \infty$ and the proof of the theorem is complete. □

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