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The lightcone of Gödel-like spacetimes

G Dautcourt

MPI für Gravitationsphysik, 14476 Golm, Germany

E-mail: daut@aei.mpg.de

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Abstract
A study of the lightcone of the Gödel universe is extended to the so-called Gödel-like spacetimes. This family of highly symmetric 4D Lorentzian spaces is defined by metrics of the form $\mathrm{d}s^2 = -(\mathrm{d}t + H(x)\,\mathrm{d}y)^2 + D^2(x)\,\mathrm{d}y^2 + \mathrm{d}x^2 + \mathrm{d}z^2$, together with the requirement of spacetime homogeneity, and includes the Gödel metric. The quasi-periodic refocussing of cone generators with startling lens properties, discovered by Ozsváth and Schücking for the lightcone of a plane gravitational wave and also found in the Gödel universe, is a feature of the whole Gödel family. We discuss geometrical properties of caustics and show that (a) the focal surfaces are two-dimensional null surfaces generated by non-geodesic null curves and (b) intrinsic differential invariants of the cone attain finite values at caustic subsets.

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1. Introduction

The study of null congruences, null hypersurfaces and in particular lightcones in general relativity is complicated by the existence of caustics, i.e. points of intersection of the generating null geodesics. Since light rays become focussed in the presence of matter as well as shear, caustics occur inevitably in realistic situations as frequently encountered in astrophysics. The corresponding strong lens effect is an important astrophysical tool [48]. In numerical relativity, caustics are less welcome, and they act as a barrier for current characteristic codes [13, 19, 52]. The differential geometry of caustics in a spacetime setting is still not well developed, contrary to their mathematical classification using methods of singularity theory [3, 17, 38]. This may be due to the fairly complicated structure of these objects, involving crossings and singularities. Important steps have been taken, among others, by Friedrich and Stewart [19] and by the Newman school [20–22, 27].

A way toward a better understanding is the study of curved spacetimes with analytically known focal surfaces. Ozsváth and Schücking presented in 1962 an exact and detailed analytical picture of the lightcone of a plane gravitational wave [32]. They found a cyclic structure of the focal set, produced by a semi-periodic re-focussing of light rays. An often reproduced illustration of a similar lightcone drawn by Penrose [37] served as the starting point for investigations in global Lorentzian geometry [6, 18]. A very similar focal structure is present on the lightcone of a quite different spacetime, the rotating Gödel universe [2, 16, 28]. In view of the T-duality of higher-dimensional supersymmetric versions of the Gödel metric and pp waves [8, 25] it is perhaps not surprising that the same type of caustic is present.

For further analytical studies of caustics it makes sense to discuss spaces of high symmetry first, since here the geodesic equations can be integrated completely. Furthermore, if spacetime homogeneity applies, all lightcones have the same intrinsic geometry, independent of the vertex location. To this class of spacetimes belong metrics of the Ozsváth class III [33], which include the Gödel metric. They have been studied by Rosquist *et al* [29]. Other examples are the spacetime-homogeneous Gödel-like or Gödel-type metrics [9, 11, 39–45, 49, 51], also generalizations of the Gödel metric and admitting at least a G_5 Killing symmetry. Their lightcone, and in particular the focal subset, is the subject of this paper.

Section 2 shortly reviews the two-parameter family of Gödel-like metrics. Basic geometrical properties of these metrics depend on a dimensionless parameter k, measuring the influence of rotation on the spacetime geometry. k^2 may range from $-\infty$ to ∞, but in this paper we confine the discussion to a range of positive k^2. The k-sequence coincides with the family of $(2+1)$-dimensional geometries investigated by Rooman and Spindel [46], if one flat space dimension is added to Rooman–Spindel. Their parameter μ is our k.

The lightcone geometry of the Gödel family with $1 < k^2 < \infty$ ($k^2 = 2$ corresponds to the Gödel cosmos [5, 23, 24, 26, 34, 35]) is studied in section 3, based on a paper by Calvão, Soares and Tiomno [10]. Further sections consider briefly some limiting cases. In section 4 we treat the special case $k^2 = 1$, known as the Rebouças–Tiomno metric [40, 44]. Its subspace $z =$ const is the three-dimensional anti-de Sitter space AdS_3. The causal family $0 < k^2 < 1$ without closed timelike curves is omitted here, only the static degeneration $k^2 \to 0$ with vanishing rotation is shortly considered in section 5. The concluding section notes that a cyclic behavior of caustics on many lightcones (and on null hypersurfaces in general) may be expected as a consequence of the Sachs equations [47] for divergence and shear of the generator congruence.

2. Gödel-like metrics

Raychaudhuri and Guha Thakurta [39] have introduced as 'homogeneous spacetimes of the Gödel type' the metrics

$$ds^2 = -(dt + H(x)\,dy)^2 + D^2(x)\,dy^2 + dx^2 + dz^2, \tag{1}$$

together with the additional condition of spacetime homogeneity. Spacetime homogeneity requires at least one further Killing field, apart from the three translational Killing vectors along the axes, which evidently exist. This leads to the *necessary* conditions

$$H'/D = \text{const} = 2\Omega,\; D''/D = \text{const} = l^2, \tag{2}$$

with two parameters Ω, l. We replace Ω by the parameter $k = 2\Omega/l$ and consider the sequence of metrics labeled by k. Only real k are taken here and l is assumed non-negative. Rebouças and Tiomno found that conditions (2) are also *sufficient* for spacetime homogeneity [40]. Moreover, it was shown in [40] that for the metrics satisfying (1) and (2) a further Killing vector exists, leading to a G_5 group of motions. A re-examination of the symmetries by Teixeira, Rebouças and Åman [51], who dropped the more or less implicit assumption of time-independent Killing fields made so far, has shown that in the special case $k^2 = 1$ the group is G_7, the maximal symmetry group within the Gödel-like class of spacetimes.

We write the metric (1) in cylindrical coordinates (t, r, ϕ, z) as

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + \frac{2k}{l}\sinh^2(lr/2)\,\mathrm{d}\phi\right)^2 + \frac{\sinh^2(lr)}{l^2}\,\mathrm{d}\phi^2 + \mathrm{d}r^2 + \mathrm{d}z^2. \tag{3}$$

The numbers (k^2, l) in the two-parameter family (3) specify a metric uniquely. Members with different pairs (k^2, l) represent different spacetimes. In the limit $k \to \infty$ and $l \to 0$ such that $kl = 2\Omega$ remains finite, the function $\sinh(lr)/l$ can be replaced by r, and (3) becomes the Som–Raychaudhuri metric [49]. For $k^2 \to 2$ one recovers the Gödel metric and for $k^2 = 1$ the already mentioned G_7 metric is obtained, studied in detail by Rebouças and Tiomno [40].

For convenience we note some properties of the metric (3). The non-vanishing components of the Ricci tensor are (we follow the conventions of [50])

$$R_0{}^0 = -k^2l^2/2,\; R_1{}^1 = R_2{}^2 = l^2(k^2-2)/2,\; R_2{}^0 = kl(k^2-1)(1-\cosh(lr)). \tag{4}$$

The eigenvalues λ determined from $\det|R_\mu{}^\nu - \lambda\delta_\mu{}^\nu| = 0$ follow as

$$\lambda_1 = 0, \quad \lambda_2 = -k^2l^2/2, \quad \lambda_{3,4} = l^2(k^2-2)/2. \tag{5}$$

The Weyltensor, also given in the coordinate form, has the non-vanishing components

$$\begin{aligned}
&C_{0101} = -C_{1313} = (k^2-1)l^2/6, \quad C_{0112} = -kl(k^2-1)\cosh(lr)/6,\\
&C_{0202} = (k^2-1)\sinh(lr)/6, \quad C_{0303} = -(k^2-1)l^2/3,\\
&C_{0323} = -kl(k^2-1)(\cosh(lr)-1)/3,\\
&C_{1212} = (k^2-1)(\cosh(lr)-1)(k^2(\cosh(lr)-1) + 2\cosh(lr) + 2)/6,\\
&C_{2323} = (k^2-1)(\cosh(lr)-1)(-2k^2(\cosh(lr)-1) - 1 - \cosh(lr))/6.
\end{aligned} \tag{6}$$

There exist several interpretations of the matter tensor as calculated from the Einstein field equations (including a cosmological constant). The $k \to \infty$ limit, the Som–Raychaudhuri metric, describes the gravitational field of a homogeneous distribution of charged rotating dust. For the Gödel family $1 < k^2 < \infty$, the combination of a perfect fluid, a scalar field and a homogeneous source-free electromagnetic field may serve as matter [40]. A perfect fluid description alone applies only to the Gödel metric $k^2 = 2$, as shown by Bampi and Zordan [4]. For more recent discussions of Gödel-like spacetimes in gravity theories derived from Lagrangians which are arbitrary functions of curvature invariants, see [12, 43].

The interpretation of the Gödel family as solutions of the Einstein or other field equations may be considered as dubious in the sense that unusual or unphysical forms of matter are involved. Therefore, these metrics (except Gödel) are not treated in the standard book on exact solutions [50]. But this aspect is not important for the geometrical discussion in this paper. The metrics are mainly interesting for their *high degree of symmetry*. All admit at least a G_5 group of motions.

3. The Gödel family $1 < k^2 < \infty$

3.1. Null geodesics

For generic Gödel-like metrics, use is made of the results by Calvão, Soares and Tiomno (CST) [10], also largely keeping their notation for comparison. The authors follow a previous paper by Novello, Soares and Tiomno dealing with the Gödel metric [30]. They give a complete discussion of timelike geodesics and also treat null geodesics. We consider only the lightlike case. The high symmetry of the metric allows us to write down a sufficient number of first integrals for the geodesic equations $\mathrm{D}x^\mu/\mathrm{D}s^2 = 0$, using the fact that for a Killing field k_μ, $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}k_\mu$ is constant along a geodesic. With the Killing translations $\partial_t, \partial_\phi, \partial_z$ and the corresponding integration constants p_t, p_ϕ, p_z or equivalently $p_t, \beta = p_z/p_t, \gamma = p_\phi/p_t$, three first integrals may be written:

$$\dot{t}/p_t = 1 + kl/2 - k^2 \sinh^2 (lr/2)/\cosh^2 (lr/2), \tag{7}$$

$$\dot{\phi}/p_t = kl/(2\cosh^2 (lr/2)) - l^2\gamma/(4\sinh^2 (lr/2)\cosh^2 (lr/2)), \tag{8}$$

$$\dot{z}/p_t = -\beta. \tag{9}$$

The dot denotes the derivative with respect to an affine parameter s. A further relation follows from $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}g_{\mu\nu} = 0$:

$$\dot{r}^2/p_t^2 = 1 - \beta^2 - \left(k\frac{\sinh (lr/2)}{\cosh (lr/2)} - \frac{l\gamma}{2\sinh (lr/2)\cosh (lr/2)}\right)^2. \tag{10}$$

As shown by CST, it is convenient to introduce instead of r another radial variable ξ, which increases monotonically with r:

$$\xi = \sinh^2 (lr/2). \tag{11}$$

Equation (10) then becomes

$$\dot{\xi}^2/p_t^2 = l^2\eta\xi^2 + l^2(1 - \beta^2 + kl\gamma)\xi - l^2\gamma^2/4 \tag{12}$$

with

$$\eta = k^2 + \beta^2 - 1. \tag{13}$$

Equations (7)–(12) refer to the class of *all* null geodesics. We are interested in the subset forming a single cone, e.g., passing through the origin of the coordinate system, $t = 0, r = 0, z = 0$. This subset is obtained by setting $\gamma = 0$. Expanding the rhs of (12) around $r = 0$ or $\xi \approx l^2r^2/4 = 0$, one obtains $(\mathrm{d}\xi/\mathrm{d}s)^2 \approx -l^4p_t^2\gamma^2/4 < 0$; hence, no geodesics with $\gamma \neq 0$ can pass the origin. On the other hand, every geodesic with $\gamma = 0$ passes the origin.

With $\gamma = 0$ the first integrals simplify considerably. The geodesic equations can be integrated completely, leading to the following parameter representation of the lightcone with the vertex at $t = 0, r = 0, z = 0$:

$$t = \frac{2k}{l}\arctan\left(\frac{k}{\sqrt{\eta}}\tan w\right) - \frac{2w(k^2-1)}{l\sqrt{\eta}}, \tag{14}$$

$$r = \frac{2\epsilon}{l}\operatorname{arsinh}\left(\frac{\sin w\sqrt{1-\beta^2}}{\sqrt{\eta}}\right), \tag{15}$$

$$\phi = \phi_0 + \arccos\left(\sqrt{1-\beta^2-\eta\xi}/(\sqrt{1-\beta^2}\sqrt{1+\xi})\right), \tag{16}$$

$$z = -\frac{2w\beta}{l\sqrt{\eta}}, \tag{17}$$

with $\epsilon = 1(-1)$ for the future (past) cone. We have introduced a new affine parameter w instead of s by

$$w = lp_t\sqrt{\eta}(s - s_0)/2 \tag{18}$$

(note $\eta > 0$, since $k^2 > 1$ is assumed, the case $k^2 = 1$ is treated separately). $w > 0(< 0)$ corresponds to the future (past) cone. Equation (16) differs from the corresponding equation (47)—restricted to $\gamma = 0$—in [10]. Both equations are correct, but refer to different initial values ϕ_0. We have replaced the CST equation in order to have $\phi = \phi_0$ at the origin $r = 0$.

The cone generators depend on the two parameters β and ϕ_0, which represent a possible pair of transversal coordinates for the light rays. It appears more useful to introduce (primarily for the past cone, but easily extended to the full cone) the two angular coordinates θ, φ on the sky of a suitable observer at the vertex. The observer is assumed comoving with the cosmic fluid with the timelike velocity vector $u^\mu = \delta_0^\mu$ (in the case of the Gödel metric with $k^2 = 2$) or defined by the normed timelike eigenvector of the Ricci tensor in general. It then is not difficult to see (e.g. by using the method described in the second appendix in [16]) that β and ϕ_0 are related to the coordinates θ, φ (polar angle and longitude) on the observer sky by

$$\beta = \cos\theta, \qquad \phi_0 = \varphi. \tag{19}$$

We note some well known or easily accessible results. From (15) it follows that null geodesics from the origin re-converge after reaching (for rays labeled θ) a maximal radial extension r_θ, so we always have

$$r \leqslant r_\theta = \frac{2}{l}\,\mathrm{arsinh}\left(\frac{\sin\theta}{\sqrt{k^2 - \sin^2\theta}}\right). \tag{20}$$

r_θ is zero for rays along the polar axis and in the opposite (antipode) direction ($\theta = 0, \pi$) and reaches its largest value r_m for equatorial rays ($\theta = \pi/2$). The hypersurfaces $r =$ const are always timelike; in particular, $r = r_m$ is the so-called light cylinder or optical horizon. Evidently, the spacetime region $r > r_m$ cannot be reached by null geodesics from the origin.

At the horizon $r = r_m$ the coefficient of $\mathrm{d}\phi^2$ in (3) is zero; thus, the ϕ-coordinate lines become closed lightlike (for $r > r_{\max}$, timelike) lines. They are not geodesics, however. A theory of *non-geodesic null curves* in a Minkowski spacetime was developed by Bonnor [7]. His approach translates immediately to curved spacetimes. A calculation shows that the closed null curves on the optical horizon are *null helices* with constant Bonnor curvatures $k_1 = 1, k_2 = l(1 + k^2)/(4k), k_3 = 0$.

3.2. Focal subsets and inner metric

Equations (14)–(17) supplemented by (19) map the intrinsic coordinates (w, θ, φ) of the lightcone to the spacetime coordinates (t, r, ϕ, z). The critical points of this map are those where the Jacobian matrix does not have the maximal rank 3. This happens if close cone generators intersect. For the critical or focal points, all four subdeterminants of the Jacobian must vanish simultaneously:

$$\frac{\partial(r,\phi,z)}{\partial(w,\theta,\varphi)} = 0, \qquad \frac{\partial(\phi,z,t)}{\partial(w,\theta,\varphi)} = 0, \qquad \frac{\partial(z,t,r)}{\partial(w,\theta,\varphi)} = 0, \qquad \frac{\partial(t,r,\phi)}{\partial(w,\theta,\varphi)} = 0. \tag{21}$$

A straightforward calculation of (21) leads to the condition $f(w, \theta) = 0$ for the focal set, where

$$f(w,\theta) \equiv k^2 \sin w \cos^2\theta + (k^2 - 1) w \cos w \sin^2\theta. \tag{22}$$

Another way to find singularities is to look for higher degeneration of the induced lightcone metric. Numbering the inner coordinates as $y^1 = w, y^2 = \theta, y^3 = \varphi$, the intrinsic three-dimensional cone metric is determined by $(i, k = 1, \ldots, 3)$

$$\gamma_{ik} = \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^k} g_{\mu\nu}. \tag{23}$$

γ_{ik} is already degenerate of rank 2. A direct calculation shows that the only nonvanishing independent components are

$$\gamma_{22} = \frac{4(f^2 - 2fk^2\cos^2\theta\sin^3 w + k^2 q\cos^2\theta\sin^4 w)}{l^2\eta^3\cos^2 w}, \tag{24}$$

$$\gamma_{23} = \frac{4k\sin\theta\cos\theta\sin^2 w(f - q\sin w)}{l^2\eta^{5/2}\cos w}, \tag{25}$$

$$\gamma_{33} = \frac{4q\sin^2\theta\sin^2 w}{l^2\eta^2}. \tag{26}$$

To obtain these compact expressions we have introduced—besides the focal function $f(w,\theta)$—a non-negative function $q(w,\theta)$:

$$q(w,\theta) = (k^2 + \cos^2\theta - 1)\cos^2 w + k^2\cos^2\theta\sin^2 w. \tag{27}$$

For later use we note that q is zero on some closed φ-coordinate lines, defined by $\theta = \pi/2$, $w = (2n-1)\pi/2, n = 0, \pm1, \pm2, \pm3, \ldots$ and arbitrary φ in the range $(0, 2\pi)$.

The determinant of the two-dimensional metric (24)–(26) can be written as a square of a function $h(w,\theta)$:

$$\gamma_{22}\gamma_{33} - \gamma_{23}^2 = h^2, \tag{28}$$

$$h(w,\theta) = 4f(w,\theta)\sin\theta\sin w/(l^2\eta^2). \tag{29}$$

Higher order degeneration of the metric requires $h = 0$ and is therefore given by

(i) the set of focal points $f(w,\theta) = 0$, where neighboring light rays intersect,

(ii) the set of points with $w = n\pi$, n integer, called keel singularities in [16], where all rays with equal θ and different φ meet in a point on the nth keel, a spacelike line of finite length, and

(iii) the pole rays $\theta = 0, \pi$, resulting from the $\sin\theta$-factor, i.e. from the singularity of the polar coordinate system.

3.3. Newman–Penrose coefficients on the cone

Additionally to the intrinsic metric, the geometry of a null hypersurface may be described by some of the Newman–Penrose spin coefficients, mainly by divergence and shear and their change along a ray.

To illustrate this we first shortly consider a fairly known example, a generic null hypersurface in a Minkowski spacetime. Here the real divergence and complex shear evolve along a given ray according to the Penrose equations [36]

$$\rho = \left(\rho_0 + w\left[\sigma_0\bar\sigma_0 - \rho_0^2\right]\right)/f, \qquad \sigma = \sigma_0/f \tag{30}$$

with the focal function

$$f = 1 - 2w\rho_0 + w^2\left(\rho_0^2 - \sigma_0\bar\sigma_0\right). \tag{31}$$

ρ_0 and σ_0 depend on the two transversal parameter fixing a ray. From (30) we have

$$\rho^2 - |\sigma|^2 = \left(\rho_0^2 - \sigma_0\bar\sigma_0\right)/f. \tag{32}$$

This equation shows that a parabolic point (a point with $\rho^2 = |\sigma|^2$) on a Minkowskian ray implies that the whole ray consists of parabolic points—provided the denominator f in (32) does not vanish. The denominator vanishes and thus both ρ and $|\sigma|$ diverge, if the affine parameter w takes one of the two values

$$w_f = 1/(\rho_0 \pm |\sigma_0|). \tag{33}$$

At each w_f a focal surface is passed, and the sign of $\rho^2 - |\sigma|^2$ changes. The quotient $j = \rho/|\sigma|$ remains finite, more exactly, $j \to \pm 1$ at a focal point. One also notes that a focal point can be considered as a degenerate parabolic point.

A similar behavior of the first-order invariant j at caustics holds for the Ozsváth–Schücking plane wave lightcone [1] and was found in [16] for the lightcone of the Gödel metric. The difference is only that in both cases one meets an *unlimited* number of focal points if one moves along a ray. It is easy to see that this holds for Gödel-like metrics in general: if one starts from a cone metric γ_{AB} (A, B always run 2, 3) with $h = \sqrt{\det|\gamma_{AB}|}$ and w as the running (not necessarily affine) parameter on the generating rays, divergence and shear amount can be calculated from

$$\rho = -\frac{1}{2h}\frac{\partial h}{\partial w}, \qquad |\sigma|^2 = \rho^2 - \det\left(\frac{\partial \gamma_{AB}}{\partial w}\right) \Big/ (4h^2). \tag{34}$$

Explicitly we find for the metric (24)–(26)

$$\rho = -\cot 2w - \frac{q}{2f\cos w}, \tag{35}$$

$$|\sigma|^2 = \rho^2 + \frac{k^2\cos^2\theta}{\eta} + \frac{\eta - 2q}{f\sin w}. \tag{36}$$

It is seen that both ρ and $|\sigma|$ diverge at focal points $f = 0$. The quantity $1/j^2 = |\sigma|^2/\rho^2$ measures the anisotropic part of distance change to neighboring null geodesics along a given ray. It can be written as

$$j^{-2} = 1 + \frac{4f(\eta - 2q)\sin w\cos^2 w}{(f\cos 2w + q\sin w)^2} + \frac{4f^2k^2\cos^2\theta\sin^2 w\cos^2 w}{\eta(f\cos 2w + q\sin w)^2}. \tag{37}$$

$1/j^2$ evidently goes to 1 for $f \to 0$; the same limit is reached at keel singularities $w = n\pi$.

3.4. Geometry of caustics

The affine parameter w gives rise to a foliation of the cone, but spacelike surfaces $w = \text{const}$ have no invariant meaning since w is not uniquely determined. There exist however invariantly defined two-surfaces on the cone, e.g. the spacelike 'zero divergence' surfaces. Here $\rho = 0$, and from (35) one obtains their equation as

$$-\frac{\tan 2w}{2w} = \frac{(k^2 - 1)\sin^2\theta}{k^2 - 1 + (k^2 + 1)\cos^2\theta}. \tag{38}$$

Since the rhs is not negative for the metrics considered here, such surfaces can only occur at points where $\tan 2w/(2w)$ is negative or null, that is, in the range $(2m - 1)\pi/2 \leqslant 2w \leqslant m\pi$, $m = \pm 1, \pm 2, \pm 3, \ldots$.

Other invariantly defined subsets of the cone are the focal surfaces $\mathcal{F}$ (described as 'points of the second kind' in [32]). Their equation $f = 0$ can be written as

$$-\frac{\tan w}{w} = \frac{k^2 - 1}{k^2}\tan^2\theta; \tag{39}$$

thus, focal surfaces occur at points with $(2n - 1)\pi/2 \leqslant w \leqslant n\pi$, $n = \pm 1, \pm 2, \pm 3, \ldots$, where $\tan w/w$ is negative. Contrary to the zero-divergence surfaces, focal surfaces are two-dimensional (finite and, as will be argued, non-geodesic) *null surfaces*. Solving (39) for θ and introducing this function $\theta_f(w)$ in (14)–(17), we obtain a parametric representation of $\mathcal{F}$. The intrinsic metric of the focal surface follows from (24) to (26) as

$$ds^2 = f_{ww}\,dw^2 + 2f_{w\varphi}\,dw\,d\varphi + f_{\varphi\varphi}\,d\varphi^2 \tag{40}$$

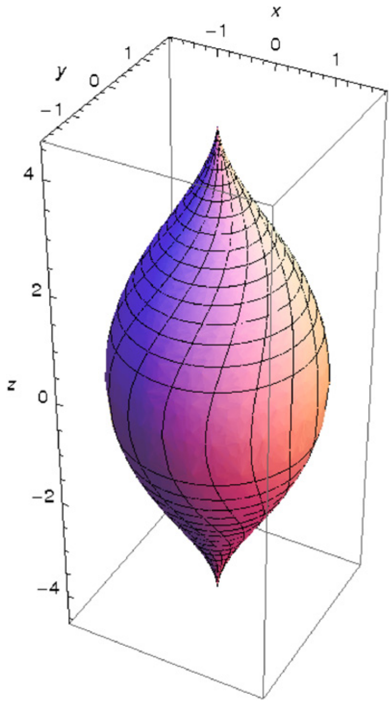

Figure 1. The first focal surface $\mathcal{F}_1$ of the $l = 1$, $k^2 = 2$ (Gödel) lightcone is shown as a projection into the 3-space $t = \text{const}$ using spacetime coordinates $x = r\cos\phi$, $y = r\sin\phi$, z. The cusps at top and bottom lie on the exceptional rays and are the intersection points with the first keel. The projected keel is the part of the z-axis between top and bottom. We add a comment of caution. The 3-space $t = \text{const}$ with the metric tensor g_{ik} is a curved Riemannian V^3, while the plot must use the Euclidean R^3. Thus, distances and angles cannot be represented correctly. In particular, distortions can occur if the map between V^3 and R^3 becomes singular, e.g. at points where $\det|g_{ik}| = 4(\xi + (1-k^2)\xi^2)/l^2$ is zero. This happens at $\xi = 1/(k^2-1)$, corresponding to the 'equatorial line' of $\mathcal{F}_1$, the intersection of $\mathcal{F}_1$ with the plane $z = 0$. The plot gives the wrong impression that this line has nonzero length, while the true V^3-length is zero, cf (44).

(This figure is in colour only in the electronic version)

with

$$f_{ww} = \frac{T\sin^2 w(1+T\cos^2 w)^3}{l^2(k^2-1)^4\cos^4 w(1+T)^3(k^2(1+T)-1)}, \tag{41}$$

$$f_{w\varphi} = -\frac{2T\sin^2 w(1+T\cos^2 w)^2}{l^2(k^2-1)^{5/2}(1+T)^{5/2}\cos^2 w\sqrt{k^2(1+T)-1}}, \tag{42}$$

$$f_{\varphi\varphi} = \frac{4T\sin^2 w(1+T\cos^2 w)}{l^2(k^2-1)(1+T)^2} \tag{43}$$

and $T = -\tan w/w$. The range of the coordinates w, φ for the nth focal surface is $0 \leqslant \varphi \leqslant 2\pi, (2n-1)\pi/2 \leqslant w \leqslant n\pi$. Since $f_{ww}f_{\varphi\varphi} - f_{w\varphi}^2 = 0$, we have the metric of two-dimensional null surfaces, with metric components depending only on w.

We consider the first focal surface $n = 1$ in more detail. $\mathcal{F}_1$ is shown in figure 1 as projection into the 3-space $t = \text{const}$, using $(x = r\cos\phi, y = r\sin\phi, z)$ as spacetime coordinates. The surface is smooth except at $w = \pi/2$ and $w = \pi$. At $w = \pi/2$, $f_{\varphi\varphi}$

tends to zero, f_{ww} to infinity, their product $f_{ww}f_{\varphi\varphi}$ $\left(\text{or } f_{w\varphi}^2\right)$ is finite and equal to the constant $\pi^4/(4l^4k^2(k^2-1))$.

We may try to explain this geometrically. As noted above, the sign of the metric component $g_{\phi\phi}$ in (1) decides whether the closed coordinate lines of ϕ are spacelike or timelike. Calculated on the lightcone, $g_{\phi\phi}$ becomes γ_{33}, and calculated on a focal surface on the cone, γ_{33} becomes $f_{\varphi\varphi}$; the ϕ-coordinate lines are φ-coordinate lines on $\mathcal{F}_1$. Contrary to $g_{\phi\phi}$, γ_{33} and $f_{\varphi\varphi}$ cannot become negative, they reach zero only at isolated points or lines. Apart from coordinate singularities, the zeros of γ_{33} are found at keels $w = n\pi$ and as zeros of q. q becomes zero only for the φ-coordinate line at $\theta = \pi/2$, $w = \pi/2$ (or $w = (2n-1)\pi/2$, $n = 0, \pm 1, 2, 3, \ldots,$ at other focal surfaces $\mathcal{F}_n$). This particular line is a closed null line, with zero length also from the viewpoint of the cone geometry (it is one of the closed null curves on the optical horizon). The other φ-coordinate lines on $\mathcal{F}_1$ (the 'parallels' in figure 1) are spacelike for $\pi/2 < w < \pi$. Their total length

$$L(w) = \int_0^{2\pi} \mathrm{d}\varphi \sqrt{f_{\varphi\varphi}} = \frac{8\pi \cos w \sin^3 w (\sin w \cos w - w)}{l^2(k^2-1)(w\cos w - \sin w)^2} \tag{44}$$

increases for $w \geqslant \pi/2$ from zero to a maximum and declines to zero for $w \to \pi$, when the keel is reached. Here $\mathcal{F}_1$ shrinks to the two cusp points, and all components $f_{ww}, f_{w\varphi}, f_{\varphi\varphi}$ vanish.

Instead of slicing by φ-lines we can represent $\mathcal{F}_1$ by the lines $y^i = (w, \theta_f(w), \varphi = \text{const})$, the twisted 'meridians' in figure 1. Their tangent vector is $\frac{\mathrm{d}y^i}{\mathrm{d}w} = \left(1, \frac{\mathrm{d}\theta_f}{\mathrm{d}w}, 0\right)$, with a norm given by $(\mathrm{d}\theta_f/\mathrm{d}w)^2\gamma_{22} = f_{ww} \geqslant 0$. Thus, these lines are spacelike except at the endpoints $w = \pi/2, w = \pi$.

One can construct a focal surface $\mathcal{F}$ in a still different way. A two-dimensional null surface always admits a foliation by null lines. The equation $f_{AB}f^B = 0$ has solutions f^A different from zero since $\det|f_{AB}| = 0$. The tangent lines to these directions can be taken as u-coordinate lines of a new (u, v) coordinate system on $\mathcal{F}$. In (u, v) coordinates the inner metric of a two-dimensional null surface is represented by the normal form

$$\mathrm{d}s^2 = F(u, v)\mathrm{d}v^2. \tag{45}$$

Explicitly, the transformation from (w, φ) to (u, v) is given for $\mathcal{F}_1$ by

$$u = w, \tag{46}$$

$$v = \varphi - \int_w^{\pi} \frac{\mathrm{d}w(w - \sin w \cos w)}{\cos w\sqrt{\sin w - w\cos w}\sqrt{k^2(\sin w - w\cos w) + w\cos w}} \tag{47}$$

and the metric function $F(u,v)$ depends only on $u = w$:

$$F = \frac{4T\sin^2 w(1 + T\cos^2 w)}{l^2(k^2-1)(1+T)^2}. \tag{48}$$

The u-coordinate lines on $\mathcal{F}_1$ have zero lengths and can therefore be denoted as null lines, but they are different from the null geodesic generators of the lightcone. From the four-dimensional viewpoint they are non-geodesic null curves. It should not be too difficult to develop a theory of such curves on null hypersurfaces, analogously to Bonnor's theory in [7].

A way to visualize focal surfaces on the past cone is to locate them on the observer sky. We may think of radiation emitted from different parts of the focal surface. If we walk down the cone into the past with an increasing affine parameter $|w|$, after passing the first zero-divergence surface at $|w| = \pi/4$, the first focal surface ($n = 1$) starts at $|w| = \pi/2$. Radiation from caustic points at this epoch would appear to the observer as a luminous ring along the celestial equator $\theta = \pi/2$. For larger $|w|$ the focal surface radiation comes in as two luminous parallels, moving from the equator toward the poles. The pole ($\theta = 0$) and its antipode ($\theta = \pi$) are reached for $|w| = \pi$, marking the two singular endpoints of the focal surface (seen as cusps in figure 1). The cusps are also intersections of the focal surface with the two exceptional rays on the past cone (these rays are in the Gödel case related to the rotation direction and its antipode direction [16], and present also for $k^2 \neq 2$).

Keels (denoted as 'points of the first kind' in [32]) are another example of invariantly defined subsets on the cone. The keels $w = n\pi$, parametrized by θ, are pieces of spacelike lines connecting the singular endpoints of the corresponding focal surface. At each keel point labeled with θ all rays with the same θ and different φ intersect. The lightcone metric further degenerates at keels and becomes a matrix of rank 1; only $\gamma_{22} = 4f^2/(l^2\eta^3)$ differs in general from zero. At the two endpoints, where the keel meets the corresponding focal surface, the matrix rank of γ_{ik} is zero; all components γ_{ik} vanish here. The keel appears as a second vertex in representations which suppress the z-coordinate, e.g. in the well-known figure of the Gödel cone in the Hawking–Ellis monograph [26]. But taking all dimensions into account, the keel is an extended spacelike line, which shrinks to a point only in the $k^2 \to 1$ limit of the Gödel family. The invariant length of the nth keel is given by

$$\int_0^\pi \mathrm{d}\theta \sqrt{\gamma_{22}} = \frac{2n\pi\sqrt{k^2-1}}{l}\left(E\left(\frac{1}{1-k^2}\right) - K\left(\frac{1}{1-k^2}\right)\right), \tag{49}$$

with E and K as complete elliptic integrals.

As known from the Gödel universe or the lightcone of the Ozsváth–Schücking anti-Mach metric [32], focal surfaces, keels and zero-divergence surfaces occur as *quasi-periodic*, due to the fact that the focal function $f(w, \theta)$ is not strictly periodic in w, while all other functions are *circular* functions of the affine parameter.

3.5. Differential invariants

Another quantitative description of null hypersurfaces is provided by their intrinsic differential invariants. The quantity j defined as a quotient of divergence and shear is already an invariant; it is the only invariant depending exclusively on the *first* derivatives of the cone metric. A comment is necessary here. While $|\sigma|$ is always not negative by definition, ρ changes the sign when the ray passes a caustic (focal surface or keel), and j is +1 or -1 before and behind this point. Thus, our formal definition produces jumps in j at these points, as written down without further explanation in [16] for the Gödel cone. This suggests us to redefine the first-order invariant as $\tilde{j} = \lambda\rho/|\sigma|$, $\lambda^2 = 1$, with appropriately chosen $\lambda = f/|f|$ or $\lambda = \mathrm{sgn}(\rho^2 - |\sigma|^2)$, to ensure that $\tilde{j}$ is a continuous function through caustics. For example, for the Minkowski space null hypersurfaces we have $\tilde{j} = j_0 + w\big(1 - j_0^2\big)|\sigma_0|$ as a smooth function of w, while $j = \rho/|\sigma|$ shows the unnatural discontinuity at the two focal surfaces. Nevertheless we keep j as the abbreviation for $\rho/|\sigma|$.

Besides j there exist higher order invariants [14, 15, 31]. For their calculation we use a triad formalism [16]. The degenerate inner metric of the cone can be represented by

$$\gamma_{ik} = t_i\bar{t}_k + \bar{t}_i t_k, \tag{50}$$

where t_i is a complex covariant vector intrinsic to the cone. The generator direction ϵ^i satisfies $\gamma_{ik}\epsilon^k = 0$. To obtain a complete co- and contravariant triad on the cone we add further vectors t^i, $\bar{t}^i$, γ_i such that

$$t_i t^i = 0, \qquad t_i\bar{t}^i = 1, \qquad \gamma_i t^i = 0, \qquad \gamma_i\epsilon^i = 1. \tag{51}$$

We use adapted inner cone coordinates $y^1 = w$, $y^2 = \theta$, $y^3 = \varphi$ with $\epsilon^i = \delta^i_1$. The degenerate metric γ_{ik} then reduces to the two-dimensional metric γ_{AB}. Comparison with (24)–(26) gives (together with $t_1 = 0$, and up to a rotation)

$$t_2 = \mathrm{i}\frac{f}{l\eta\sqrt{q/2}} + \frac{k\cos\theta\sin w}{l\eta^{3/2}\cos w\sqrt{q/2}}(f - q\sin w), \tag{52}$$

$$t_3 = \sin\theta\sqrt{2q}\sin w/(l\eta). \tag{53}$$

The contravariant components t^i are calculated from $t_i = \gamma_{ik}t^k$. The result is (besides $t^1 = 0$)

$$t^2 = \mathrm{i}l\eta\sqrt{2q}/(4f), \tag{54}$$

$$t^3 = \frac{l\eta}{4\sin\theta\sin w\sqrt{q/2}} + \mathrm{i}\frac{kl\cos\theta\sqrt{2\eta}}{4f\sin\theta\cos w\sqrt{q}}(q\sin w - f). \tag{55}$$

Rotation coefficients related to this triad and of relevance here can now be obtained from

$$\rho + \mathrm{i}\nu = -t^2\bar{t}_{2,1} - t^3\bar{t}_{3,1}, \tag{56}$$

$$\sigma = -\bar{t}^2\bar{t}_{2,1} - \bar{t}^3\bar{t}_{3,1}, \tag{57}$$

$$\tau = (\bar{t}^2 t^3 - t^2\bar{t}^3)(\bar{t}_{2,3} - \bar{t}_{3,2}). \tag{58}$$

One may verify that the expressions for ρ and $|\sigma|$ obtained from (56), (57) agree with (35) and (36). As noted, the components t_i, t^i are not uniquely determined. This affects some rotation coefficients, but not ρ, $|\sigma|$ and also not the invariants. The freedom could (but will not here) be used to reach, e.g., $\nu = 0$ in (56). For our choice of the triad the real and imaginary part of the complex shear is given by

$$\mathfrak{Re}(\sigma) = \frac{2\eta\cos^2 w - q(1 + 2\cos^2 w)}{q\sin 2w} + \frac{q}{2f\cos w}, \tag{59}$$

$$\mathfrak{Im}(\sigma) = \frac{k(k^2 - 1)\cos\theta\sin^2\theta\sin^2 w}{q\sqrt{\eta}}. \tag{60}$$

A null hypersurface has in general four second-order differential invariants of the inner geometry, written as complex quantities I and J and conveniently expressed in terms of rotation coefficients [14]. The quantity I is linear in the second derivatives of the metric, with derivatives only along the generators and, like j, dimensionless:

$$I = \frac{\mathrm{i}}{|\sigma|}\left(\frac{D\rho}{\rho} - \frac{D\sigma}{\sigma}\right) + 2\frac{\nu}{|\sigma|}. \tag{61}$$

Explicitly, we find for the real part I_1

$$|\sigma|^3 I_1 = \frac{2k(k^2 - 1)\cos\theta\sin^2\theta(\eta\sin w - f)}{f\eta^{3/2}}. \tag{62}$$

The imaginary part I_2 has a more complicated structure:

$$|\sigma|^3 I_2 = \frac{i_0 + i_1 f + i_2 f^2 + i_3 f^3}{2\eta f^2\sin^2 w\cos w(f\cos 2w + q\sin w)}. \tag{63}$$

A tedious but straightforward calculation shows that

$$i_0 = \eta q^2\sin^2 w(2q - \eta), \tag{64}$$

$$i_1 = 2\sin w(-q^3 + \eta q^2(\sin^2 w - 3) + \eta^2 q(5 - 4\sin^2 w) - 2\eta^3\cos^2 w), \quad (65)$$

$$i_2 = 4q^2 - 2q\eta\cos 2w - \eta^2, \quad (66)$$

$$i_3 = -2k^2\cos^2\theta\sin w. \quad (67)$$

$I_2 = Dj/\rho$ describes the change of the first-order quantity j along the rays. I_1 is a measure for the rotation of the shear directions (i.e. directions where the distance change to neighboring rays is a maximum or minimum) relative to the generator congruence. If I_1 is zero (as for null hypersurfaces in a Minkowski or conformally related spacetime), the shear directions always point to the *same* neighboring null rays if one follows a ray.

The complex invariant J has the dimension $(\text{length})^{-1}$, is nonlinear in the second-order derivatives of the inner metric and involves additionally transversal derivatives [14]. J describes changes of the null surface geometry in transversal directions but is considerably more complicated than I and will be discussed elsewhere.

The behavior of invariants at and in the neighborhood of focal singularities is of interest. While the rotation coefficients ρ and σ show singularities, the invariants tend to have finite values. We have already noted $j \to \pm 1$ at focal points and keels. Expanding I near $f = 0$ in powers of f leads to

$$I_1 = \frac{16k(k^2-1)\cos\theta\sin^2\theta\sin w\cos^3 w}{q^3\sqrt{\eta}} f^2 + o(f^3), \quad (68)$$

$$I_2 = \frac{4\cos^2 w(\eta - 2q)}{q^2\sin w} f + o(f^2). \quad (69)$$

Remarkably, the second-order differential invariants I_1, I_2 vanish at focal points. This also holds at keels $w = n\pi$. Writing $w - n\pi = x$, one obtains for small x

$$I_1 = 16(-1)^{n+1}k(k^2-1)\eta^{-3/2}\cos\theta\sin^2\theta x^3 + o(x^4), \quad (70)$$

$$I_2 = \frac{4(-1)^n\eta x}{n\pi(k^2-1)\sin^2\theta} + o(x^2). \quad (71)$$

For comparison, we note that null hypersurfaces in a Minkowski spacetime satisfy $I_2 = 1/j - j$; thus, I_2 vanishes at caustics, and I_1 is already zero everywhere.

3.6. *Comments on the Gödel case as treated in [16]*

For the Gödel cone ($k^2 = 2$) some differential invariants have been calculated in [16]. This paper uses different four-dimensional coordinates as well as different transversal lightcone coordinates y^A. The latter is motivated by the topological fact that no coordinate system can cover the whole sphere without singularity. The coordinates u, v in [16] avoid a singularity in the direction of the rotation axes; they become singular in equator directions instead. The polar angles θ, ϕ here avoid the equator singularities but show the usual pole singularities. The relation between both systems of transversal coordinates is given by $\cos\phi = (1-u^2)/(1+u^2)$, $\sin\theta = \sqrt{2}(v^2-1)/(v^2+1)$. For $k^2 = 2$, equation (39) thus becomes the focal equation $-\tan w/w = (v^2-1)^2/(6v^2-1-v^4)$, equation (81) in [16]. We note a misprint in equation (48) of [16]: the denominator should read $f_2 + 4(1 + f_1)$ instead of f_2.

4. The Rebouças–Tiomno G_7 metric $k^2 = 1$

The case $k^2 = 1$ was excluded so far, so we treat it separately. Rebouças and Tiomno introduced this special case as 'the first exact Gödel-type solution of Einstein's equations describing a completely causal spacetime-homogeneous rotating universe' [40]. The lightcone becomes very simple in this model. Since the Weyl tensor vanishes for $k^2 = 1$ (see (6)), the spacetime metric is conformal to the Minkowski spacetime; thus, also the lightcone metric is conformal to the Minkowski cone metric. One obtains in the limit $k^2 \to 1$, $f \to \sin w \cos^2\theta$, $\eta \to \cos^2\theta$, $q \to \cos^2\theta$ of the preceding formulas

$$\gamma_{22} = \frac{4\sin^2 w}{l^2\cos^2\theta}, \qquad \gamma_{23} = 0, \qquad \gamma_{33} = \frac{4\sin^2 w}{l^2}\tan^2\theta. \tag{72}$$

The square root h of the determinant $|\gamma_{AB}|$,

$$h = 4\sin\theta\sin^2 w/(l^2\cos^2\theta), \tag{73}$$

vanishes at the points $w = n\pi$ (the only other zeros correspond to the coordinate singularity). All light rays from the vertex $w = 0$ meet again at the points $w = n\pi$ (n integer), which are also vertices. Thus, every pair of focal surface and keel in the $k^2 > 1$ family of metrics has collapsed into a single vertex in the limit $k^2 \to 1$. The shear of the cone vanishes, and only the divergence differs from zero:

$$\rho = -\cot w. \tag{74}$$

ρ increases from $-\infty$ at $w = 0$ to zero at $w = \pi/2$ and decreases again until $-\infty$ at the next vertex $w = \pi$. The lightcone belongs to a type of null hypersurfaces characterized by $\rho \neq 0, |\sigma| = 0$ and denoted as 'type 5' in the classification of [15]. There exist no second-order inner differential invariants for this class.

The high symmetry of the Rebouças–Tiomno metric is reflected by the existence of a symmetry group G_7 [51], see also [44] for further discussions.

5. Static degeneration $k^2 \to 0, l$ finite

The limit $k \to 0$, keeping l finite, requires $\Omega \to 0$. It represents the static degeneration of the Gödel family and has the simple line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \frac{\sinh^2(lr)}{l^2}\,\mathrm{d}\theta^2 + \mathrm{d}z^2. \tag{75}$$

Teixera, Rebouças and Åman have shown that this metric admits a six-parameter group of motions G_6 [51]. The only nonvanishing components of the Ricci tensor are $R_{rr} = -l^2$ and $R_{\theta\theta} = -\sinh^2(lr)$ with a constant Ricci scalar $R = -l^2$, and the Riemann tensor has only one independent nonvanishing component $R_{r\theta r\theta} = -\sinh^2(lr)$. Thus, the metric cannot easily be interpreted as the solution of the field equations; it is nevertheless interesting geometrically because of its high symmetry. The null geodesics starting at the origin $r = 0, z = 0$ are given by (here $w > 0$ corresponds to the past cone)

$$t = -w, \qquad r = w\sin\theta, \qquad \phi = \phi_0, \qquad z = w\cos\theta. \tag{76}$$

The equations show that the cone generators do not re-converge as generally for the Gödel family but extend to null infinity as in the Minkowski spacetime (Minkowski is included for $l \to 0$). No focal surface or keel exist for finite w. Nevertheless the cone geometry and in particular the asymptotic behavior of the cone significantly differ from Minkowski. The cone metric with the intrinsic coordinates $y^1 = w$, $y^2 = \theta$, $y^3 = \varphi$ is given by

$$\gamma_{22} = w^2, \qquad \gamma_{23} = 0, \qquad \gamma_{33} = \frac{\sinh^2(lw\sin\theta)}{l^2}; \tag{77}$$

thus, the determinant $|\gamma_{AB}|$ is the square of the function

$$h = \frac{w}{l}\sinh(lw\sin\theta). \tag{78}$$

Hence, apart from the vertex $w = 0$ and the coordinate singularity on the symmetry axis, there exist no further singularities on the cone. Divergence and shear of the rays follow as

$$\rho = -\frac{1}{2w} - \frac{l\sin\theta}{2}\coth(lw\sin\theta), \tag{79}$$

$$\sigma = \bar{\sigma} = \frac{1}{2w} - \frac{l\sin\theta}{2}\coth(lw\sin\theta). \tag{80}$$

The divergence increases from $-\infty$ at the vertex $w = 0$ to $-l\sin\theta/2$ for $w \to \infty$, if one goes down the past lightcone; thus, it always remains negative. The (real) shear starts with zero at the vertex, becomes negative for increasing w and reaches the same negative limit $-l\sin\theta/2$ as the divergence for $w \to \infty$. One also has

$$\rho^2 - |\sigma|^2 = \frac{l\sin\theta}{w}\coth(lw\sin\theta), \tag{81}$$

which is always positive; thus, the lightcone consists exclusively of elliptic points.

The second-order invariants I_1 and J are zero, but I_2 is different from zero and given by

$$I_2 = \frac{4lwS\sin\theta(CS - lw\sin\theta)}{(lwC\sin\theta + S)(lwC\sin\theta - S)^2} \tag{82}$$

with $S \equiv \sinh(lw\sin\theta)$, $C \equiv \cosh(lw\sin\theta)$. One verifies that asymptotically, for $w \to \infty$, $j \to -1$, $I_2 \to 0$, which are standard limiting values at caustics.

6. Cyclic structure on general null hypersurfaces

At the end of their pioneering paper [32], Ozsváth and Schücking ask for the origin of the periodicity structure. A certain answer can be given by going back to the Sachs equations [47], the differential equations governing divergence and shear on a null hypersurface $\mathcal{N}$ in terms of Ricci and Weyl tensor projections into $\mathcal{N}$:

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \omega, \tag{83}$$

$$D\sigma = 2(\rho - \mathrm{i}\nu)\sigma + \psi, \tag{84}$$

with

$$\begin{aligned} D &= p^\mu\partial_\mu = \frac{\partial}{\partial w}, \qquad \rho = -p_{\mu;\nu}\bar{t}^\mu t^\nu, \qquad \nu = \mathrm{i}\bar{t}_{\mu;\nu}\bar{t}^\mu p^\nu, \\ \sigma &= -p_{\mu;\nu}\bar{t}^\mu\bar{t}^\nu, \qquad \tau = -\bar{t}_{\mu;\nu}\bar{t}^\mu\bar{t}^\nu, \end{aligned} \tag{85}$$

$$\omega = \frac{1}{2}R_{\mu\nu}p^\mu p^\nu, \qquad \psi = C_{\mu\nu\rho\sigma}p^\mu\bar{t}^\nu p^\rho\bar{t}^\sigma. \tag{86}$$

The null vector p^μ is the direction of the cone generators; the complex null vector t^μ spans spacelike directions in $\mathcal{N}$ orthogonal to p^μ. It should be stressed that ω and ψ—in spite of their origin as projections of four-dimensional quantities—depend only on the metric γ_{ik}—they are *objects of the inner geometry of* $\mathcal{N}$. The rotation coefficients ρ, ν, σ, τ defined in (85) agree with those calculated from (56) to (58) for the Gödel family.

The Sachs equations are the first equations to be solved on $\mathcal{N}$ for a characteristic initial value problem based on the Newman–Penrose formalism and starting from $\mathcal{N}$. Thus in a sense they can be considered as the Einstein field equations in a nutshell, being nonlinear and ruling the influence of matter (ω) on the null surface geometry. The Penrose equations (30) follow as a solution of the Sachs equations in the absence of matter and for a vanishing Weyl tensor ($\omega = \psi = 0$). For the lightcone of the Gödel family, calculation of ω and ψ gives

$$\omega = 2 - \frac{k^2 \cos^2\theta}{(k^2 - \sin^2\theta}), \tag{87}$$

$$\mathfrak{Re}(\psi) = \frac{2(k^2-1)\sin^2\theta}{q\eta}(-q + 2\eta\cos^2 w), \tag{88}$$

$$\mathfrak{Im}(\psi) = \frac{4k(k^2-1)\cos\theta\sin^2\theta\cos w\sin w}{q\sqrt{\eta}}. \tag{89}$$

The solution of the Sachs equations with these right-hand sides is given by (35) and (59), (60).

From the Sachs equations one can derive a differential equation for an area distance r (not to be confused with the radial coordinate r), introduced by $Dr = -\rho r$. One obtains with $Q = |\sigma|^2 + \omega$ the Jacobi equation

$$DDr + Qr = 0. \tag{90}$$

The caustics are found as zeros of r. For the Gödel family from (87) $\omega > 0$; hence, $Q > 0$.

The existence of cyclic focal features on many null hypersurfaces (not only cones) may then be considered as the property of the Jacobi equation or, more concretely, as the property of the function Q. The linear second-order differential equation (90) belongs to the most widely studied equations in applied mathematics. Starting with the classical papers by Sturm, Liouville and Kneser in the 19th century, there exist many theorems which indeed prove a cyclic or oscillatory behavior (with arbitrarily large numbers of zeros of r) for certain functions $Q > 0$. For $Q < 0$ the solutions are non-oscillatory, but this also holds for some $Q > 0$, e.g. for the Minkowski space caustics. The precise dependence of the oscillation feature on properties of Q is an open mathematical problem, see [53].

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