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COORBITAL RESTRICTED PROBLEM AND ITS APPLICATION IN THE DESIGN OF THE ORBITS OF THE LISA SPACECRAFT

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On the basis of many coorbital phenomena in astronomy and spacecraft motion, a dynamics model is proposed in this paper — treating the coorbital restricted problem together with method for obtaining a general approximate solution. The design of the LISA spacecraft orbits is a special 2+3 coorbital restricted problem. The problem is analyzed in two steps. First, the motion of the barycenter of the three spacecraft is analyzed, which is a planar coorbital restricted three-body problem. And an approximate analytical solution of the radius and the argument of the center is obtained consequently. Secondly, the configuration of the three spacecraft with minimum arm-length variation is analyzed. The motion of a single spacecraft is a near-planar coorbital restricted three-body problem, allowing approximate analytical solutions for the orbit radius and the argument of a spacecraft. Thus approximative expressions for the arm-length are given.

Keywords: Restricted problem; coorbital restricted problem; LISA Constellation; orbital design.

1. Introduction

As early as 1776, Lagrange proved theoretically that there are particular solutions of the equilateral triangle for the three-body problem.[1] Then he indicated that there are such particular solutions for the circular or elliptical restricted three-body problem also. Thus, this is the theoretical proof of the existence of the coorbital particular solutions. The asteroid Achilles (588),[2] which was found by Wolf in 1906, is coorbital with Jupiter and forms a nearly equilateral triangle with the Sun and Jupiter. More than ten asteroids with similar orbits were found during the 50 years hereafter, which compose the famous Trojan group. In 1918 the Japanese astronomer T. Hirayama found that among some asteroids, their orbit semi-major axis, eccentricity and inclination are very close to each other. He named these asteroids an asteroid family,[3] and doubted that they had a common origin. The asteroids in the same family are almost coorbital. Until the 1980s, J. G. Williams *et al.* at JPL have identified more than 80 asteroid families.[4] In recent years more and more asteroids in the main belt were found. As a result, it would be expected that many more families will be recognized correspondingly. The phenomenon of the coorbital motion also exists in the Kuiper belt objects (KBO),[5] which have been found since 1992. There are many coorbital phenomena in the star motion in the spiral arms of the Galaxy. In other words, the coorbital phenomenon has a certain universality in the motion of celestial bodies.

The so-called restricted problem is opposite to the general problems. Classical restricted problems include the restricted three-body problems, which were discussed in many celestial mechanics textbooks,[6,7] two fixed-center problem,[8] the Hill problem,[9] the Fatou problem,[10] and so on. In the 1980s, the American astronomer V. Szebehely proposed the concept of the general N+K restricted problem,[11] which is a problem including N bodies with big mass and K bodies with very small mass. The gravity of the big mass bodies must be considered, while the gravity of the small mass bodies to the big mass bodies can be ignored. Whether to include or ignore the gravity among the small mass bodies should be determined according to the real situation. His proposal was supported by most people.

In this paper the coorbital problem, which is a 2+3 restricted problem, is discussed by using the orbit design of the planned Laser Interferometer Space Antenna (LISA)[12] as a typical example. In this problem, the barycenter of three spacecraft moves in the orbit in which the Earth moves around the Sun, under the influence of the gravity of the Sun and the Earth (including the Moon). This is an example of a planar restricted three-body problem. In addition to that, the motion of the three spacecraft themselves under the influence of the gravity of the Sun and the Earth (including the Moon) is an example of a 2+3 near-planar restricted problem.

Both LISA and ASTROD[13] use interferometric laser ranging, and the Doppler effects on transmitted and received frequencies need to be addressed. LISA′s strategy is to minimize arm-length variation and relative velocity of the spacecraft. For ASTROD, the arm-length changes of the three spacecraft are in the same order as the distances between the three spacecraft and the relative velocities go up to 70 km/s with line-of-sight velocities varying from -20 to $+20$ km/s. For 1064 nm (532 nm) laser light, the Doppler frequency change goes up to 40 (80) GHz. For ASTROD, a different strategy, which relies on technology, is used. The recent development of optical clocks and frequency synthesizers using optical combs makes this heterodyne problem tractable.

In Sec. 2, the motion equations of a body in the rotating or synodic coordinate system[14] are given. In Sec. 3, the motion of the barycenter of the LISA constellation with three spacecraft is discussed. In Sec. 4, the motion of a single spacecraft and the variation of the arm-length in the constellation is analyzed. It has been found that the inclination between the plane determined by the triangle of the three spacecraft and the ecliptic plane is a key parameter for the optimization of the orbit design of the spacecraft. And its optimum value is presented in another paper.[15]

2. Equations of Motion in the Rotating Coordinate System

The first step is to analyze the motion of a particle P with negligible mass under the influence of gravitational forces of the Sun and the earth-moon barycenter. For simplification, S and E are used to stand for the location of the Sun and earth-moon barycenter, respectively, and their masses are m_s and m_e. Setting the sun S as origin, the orbital plane, along which the earth-moon barycenter rotates around the Sun, as base plane, and the direction to the earth-moon barycenter E as x-axis, a heliocentric rotating or synodic coordinate system is set up and shown as Fig. 1. This is a non-inertial system, in which the frame vectors are $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

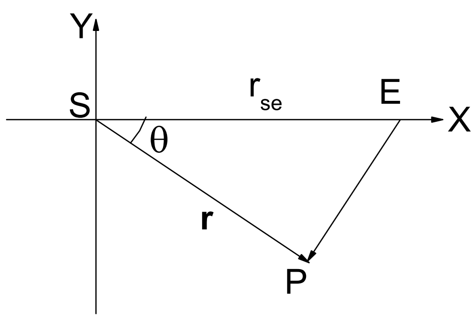

Fig. 1. Heliocentric rotating coordinate system.

For an approximation, it could be assumed that the earth-moon barycenter takes uniform circular motion around the Sun with 1 AU radius and 1 sidereal year period. Thus the motion of particle P could be analyzed on the base of the circular restricted three-body problem. The radius vector to the earth-moon barycenter $\mathbf{r}_{se} = \mathbf{i}$ is the unit vector of the x-axis. Using the astronomical unit AU as the unit of distance gives $r_{se} = |\mathbf{r}_{se}| = 1$. The angle velocity of the vector $\mathbf{r}_{se}$ rotating around the Sun is

$$\boldsymbol{\omega} = n\mathbf{k}\,, \tag{1}$$

where the mean motion[6] $n = \sqrt{G(m_e + m_s)}$ and G is the gravitational constant. It is also the angle velocity of the barycentric rotating coordinate system, which is relative to the inertial system in the definition.

Let the radius vector of particle P be $\mathbf{r} = (x, y, z)^\tau$, where τ denotes the transpose operation of a matrix. In the aforementioned coordinate system the equation of motion of the particle P is

$$\ddot{\mathbf{r}} = -n^2 \left\{ \left(\frac{1-\mu}{r^3} - 1 \right) \mathbf{r} + \mu \left[\frac{\mathbf{r}_{ep}}{r_{ep}^3} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \right\} + 2n \begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} . \tag{2}$$

Denoting $\mathbf{r}_{ep}=(x-1, y, z)^\tau$ it can be written in the following coordinate form

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = -n^2 \left\{ \left(\frac{1-\mu}{r^3} - 1 \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right. \\ \left. + \mu \left[\frac{1}{r_{ep}^3} \begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \right\} + 2n \begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} , \tag{3}$$

where

$$\mu = \frac{m_e}{m_s + m_e} . \tag{4}$$

Normalizing the unit of mass with the Sun mass gives $Gm_s = G = 0.0002959122082 85591$, $Gm_e = 8.997011346712499 \times 10^{-10}$ (gravitational constant of the earth-moon system). Hence, it can be derived that $\mu = 3.04 \times 10^{-6}$, $n = 0.0172021251$ radian per day. Let

$$\Omega = n^2 \left\{ \frac{1}{2} \left[(x-\mu)^2 + y^2 \right] + \left(\frac{1-\mu}{\sqrt{x^2+y^2+z^2}} + \frac{\mu}{\sqrt{(1-x)^2+y^2+z^2}} \right) \right\} , \tag{5}$$

then Eq. (2) can be written as

$$\ddot{\mathbf{r}} - 2n \begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} = \frac{d\Omega}{d\mathbf{r}} . \tag{6}$$

3. Motion of the Barycenter of the LISA Constellation

3.1. *Planar coorbital circular restricted three-body problem*

The motion of the barycenter C of the LISA constellation is a special example of the circular restricted three-body problem, mentioned in the last section. The initial status of point C, which is regarded as a particle with zero mass, is

$$\mathbf{r}(0) = (\cos\theta_0, \sin\theta_0, 0)^\tau . \tag{7}$$

That is to say, when $t = 0$, C is located at the orbit of the earth-moon barycenter trailing the barycenter by the angle $-\theta_0$. (See Fig. 1. Now the particle P is replaced by the barycenter C). This problem can be considered as a planar coorbital circular restricted three-body problem, namely, a planar circular restricted three-body problem with a zero-mass body coorbital with another body.

Generally, assuming the argument angle is θ, the radius is $\mathbf{r}(0) = (\cos\theta_0, \sin\theta_0, 0)^\tau$. According to Eq. (6), the equation of motion in the polar coordinate system (r, θ) can be deduced to:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 - 2nr\dot{\theta} &= \frac{\partial\Omega}{\partial r}\,, \\ \frac{d}{dt}[r^2(\dot{\theta}+n)] &= \frac{\partial\Omega}{\partial\theta}\,. \end{aligned} \tag{8}$$

From expression (5),

$$\Omega = n^2\left\{\frac{1}{2}\left[r^2 - 2\mu r\cos\theta + \mu^2\right] + \left(\frac{1-\mu}{r} + \frac{\mu}{r_{ec}}\right)\right\}\,,$$

or

$$\Omega = \Omega_0 + \mu\Omega_1\,, \tag{9}$$

where

$$\begin{aligned} \Omega_0 &= n^2\left(\frac{1}{2}r^2 + \frac{1}{r}\right)\,, \\ \Omega_1 &= n^2\left(-r\cos\theta - \frac{1}{r} + \frac{1}{r_{ec}}\right)\,, \end{aligned} \tag{10}$$

and the length of the vector from the Earth to the barycenter C is $r_{ec} = \sqrt{1 - 2r\cos\theta + r^2}$. Because μ is a small parameter (in this case, 3.04×10^{-6}), Eq. (8) can be regarded as a perturbed problem with $\mu\Omega_1$ as the perturbation or disturbing term. It can be solved by two steps. First, regarding the equation as an unperturbed problem, i.e $\Omega = \Omega_0$, the solution is given out in Sec. 3.2. Secondly, on the base of the unperturbed solution, an approximate analytical solution is derived in Sec. 3.3.

3.2. *The unperturbed solution*

For illustrative purposes, we provisionally neglect the perturbation due to the earth-moon system, namely setting $\mu = 0$, gives $\Omega = \Omega_0$. The equation of motion can be written as

$$\begin{aligned} \ddot{r} - r(\dot{\theta}+n)^2 &= -\frac{n^2}{r^2}\,, \\ \frac{d}{dt}[r^2(\dot{\theta}+n)] &= 0\,. \end{aligned} \tag{11}$$

This is the so-called restricted two-body problem, which can be solved by means of a similar method used in the general two-body problem. Integrating Eq. (11) gives

$$r^2(\dot{\theta}+n) = h\,, \tag{12}$$

$$r = \frac{(h/n)^2}{1 + e\cos(\theta + nt + \beta)}\,, \tag{13}$$

where h, e, β are the integration constants. Obviously,

$$r = 1, \quad \theta = \theta_0 = \text{constant} \tag{14}$$

satisfies Eq. (11). Therefore Eq. (14) is a particular solution which corresponds to the initial value (7).

Hence the center C moves along the same circular orbit of earth-moon with the equivalent angle velocity in this case, and in the general solutions (12) and (13), this particular solution has the particular integration constants:

$$e = 0, \; h = n\,. \tag{15}$$

For each value of θ_0, (14) is a static point on the circular orbit of earth-moon, in the rotating coordinate system. These particular solutions are named the coorbital solutions.

3.3. *Approximate analytical solution*

When $\mu \neq 0$, $\Omega = \Omega_0 + \mu\Omega_1$ and the equation becomes

$$\begin{aligned} &\ddot{r} - r(\dot{\theta} + n)^2 = -\frac{n^2}{r^2} + \mu n^2 \left(\frac{1}{r^2} - \cos\theta - \frac{r - \cos\theta}{r_{ec}^3} \right), \\ &\frac{d}{dt}[r^2(\dot{\theta} + n)] = \mu n^2 r \sin\theta \left(1 - \frac{1}{r_{ec}^3} \right). \end{aligned} \tag{16}$$

The iterative method is used to obtain the approximate analytical solution. Replacing r in the right-hand side of Eq. (16) with the unperturbed particular solution (14), and $r_{ec} = 2\sin\frac{|\theta|}{2}$ (when $r = 1, r_{ec} = \sqrt{2 - 2\cos\theta} = 2\left|\sin\frac{\theta}{2}\right| = 2\sin\frac{|\theta|}{2}$) gives the following solution with the accuracy of first-order perturbation of μ

$$\begin{aligned} &\ddot{r} - r(\dot{\theta} + n)^2 = -\frac{n^2}{r^2} + \mu n^2 \left(2\sin^2\frac{\theta}{2} - \frac{1}{4}\csc\frac{|\theta|}{2} \right), \\ &\frac{d}{dt}[r^2(\dot{\theta} + n)] = \mu n^2 \sin\theta \left(1 - \frac{1}{8}\csc^3\frac{|\theta|}{2} \right). \end{aligned} \tag{17}$$

Putting the second equation in a different form, we have

$$\frac{dh}{dt} = \mu n^2 \sin\theta \left(1 - \frac{1}{8}\csc^3\frac{|\theta|}{2} \right). \tag{18}$$

Equation (13) is the expression of r when there is no perturbation. On the basis of the variation of arbitrary constants,[6] it is still true when there is perturbation. However h, e and β are not constants any more, and should be regarded as functions of t instead.

Replacing θ in the perturbation terms with the unperturbed particular solution (14), and putting

$$k = \mu \sin\theta_0 \left(1 - \frac{1}{8}\csc^3\frac{|\theta_0|}{2} \right) \tag{19}$$

in Eq. (18) gives

$$\frac{dh}{dt} = kn^2 \,. \tag{20}$$

If $h_0 = n$ in the case of initial value (14), the particular solution of Eq. (20) is

$$\frac{h}{n} \equiv 1 + \tau = 1 + knt \,, \tag{21}$$

where $\tau = knt$.

Next, the approximate solution for the argument angle θ will be presented. Taking $e = 0$, from Eqs. (12) and (13)

$$\dot{\theta} = \frac{h}{r^2} - n = n\left[\left(\frac{h}{n}\right)^{-3} - 1\right] . \tag{22}$$

Substituting the approximate solution (21) into (22) and considering that k contains a small factor μ, yields $\dot{\theta} = -3kn^2 t$. Integrating it we have

$$\Delta\theta \equiv \theta - \theta_0 = -\frac{3}{2}k(nt)^2 \,. \tag{23}$$

The comparison of Expression (23) with an accurate numerical integration is shown in Fig. 2. At the end of the mission lifetime, the variation of θ has been more than 1 degrees. The accuracy of this expression still does not meet our requirement. Iteration of θ is needed.

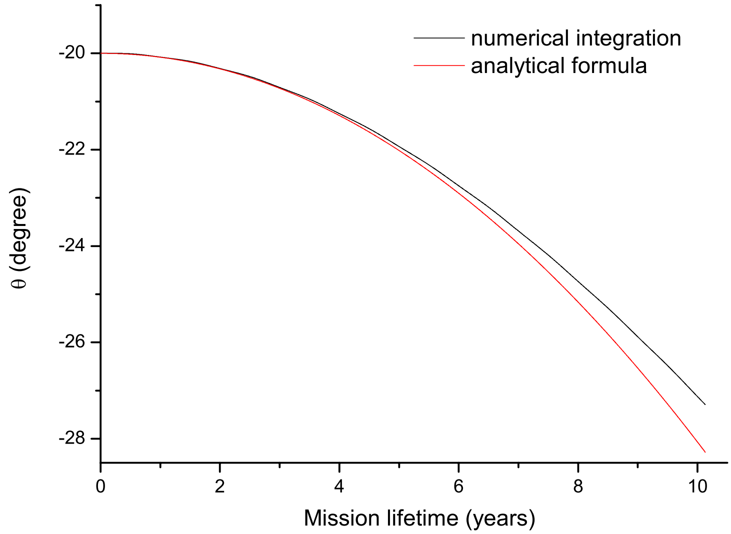

Fig. 2. Comparison of Eq. (23) with numerical integration

3.4. *A useful solution for the LISA mission*

Next, we find a more accurate expression for θ. The quantity τ, given by (21), is solved at first. Because of $\frac{dh}{dt} = \dot{\theta}\frac{dh}{d\theta}$, from Eqs. (18) and (22), we have

$$\left[\left(\frac{h}{n}\right)^{-3} - 1\right] d\left(\frac{h}{n}\right) = \mu\cos\frac{\theta}{2}\left(2\sin\frac{\theta}{2} - \frac{1}{4}\csc\frac{\theta}{2}\csc\frac{|\theta|}{2}\right) d\theta\,. \tag{24}$$

Integrating it

$$\frac{1}{2}\left(\frac{h}{n}\right)^{-2} + \frac{h}{n} = \frac{3}{2} - \mu\left(\cos\theta_0 - \frac{1}{2}\csc\frac{|\theta_0|}{2}\right) + \mu\left(\cos\theta - \frac{1}{2}\csc\frac{|\theta|}{2}\right). \tag{25}$$

Taking note of $h/n = 1 + \tau$, the solution with the accuracy of terms of τ^2 is

$$\tau = \left\{\frac{2\mu}{3}\left[\left(\cos\theta - \frac{1}{2}\csc\frac{|\theta|}{2}\right) - \left(\cos\theta_0 - \frac{1}{2}\csc\frac{|\theta_0|}{2}\right)\right]\right\}^{\frac{1}{2}}. \tag{26}$$

Simplifying it by using $\theta = \theta_0 + \Delta\theta$ and $\Delta\theta = -\frac{3}{2}\mu k(nt)^2$, and putting

$$k_1 = \mu\left[8\cos\theta_0 + \csc\frac{|\theta_0|}{2}\left(1 + 2\cot^2\frac{\theta_0}{2}\right)\right], \tag{27}$$

gives

$$\tau = knt\left(1 - \frac{3}{64}k_1(nt)^2\right). \tag{28}$$

Then from Eq. (22), $\dot{\theta} = n(-3\tau + 6\tau^2)$. Substituting (28) into it, we have

$$\dot{\theta} = n\left[-3k(nt) + 6k^2(nt)^2 + \frac{9}{64}kk_1(nt)^3\right]. \tag{29}$$

Integrating it we find

$$\Delta\theta = -\frac{3}{2}k(nt)^2 + 2k^2(nt)^3 + \frac{9}{256}kk_1(nt)^4\,. \tag{30}$$

Finally, back to the radius expression (13), substitute the approximate expressions of h/n, θ into it. As to the parameters e and β, considering the initial values gotten from Eq. (14) and comparing with numerical integration, we have $e = 5 \times 10^{-5}$, $\beta = \frac{3\pi}{2} - \theta_0$. The final result is

$$r = \frac{1 + 2knt\left(1 - \frac{3}{64}k_1(nt)^2\right)}{1 + e\sin\left(\Delta\theta + nt\right)}. \tag{31}$$

The comparison of Eqs. (30), (31) with an accurate numerical integration is shown in Fig. 3. Obviously, the accuracy of this expression can meet our requirement during the mission lifetime.

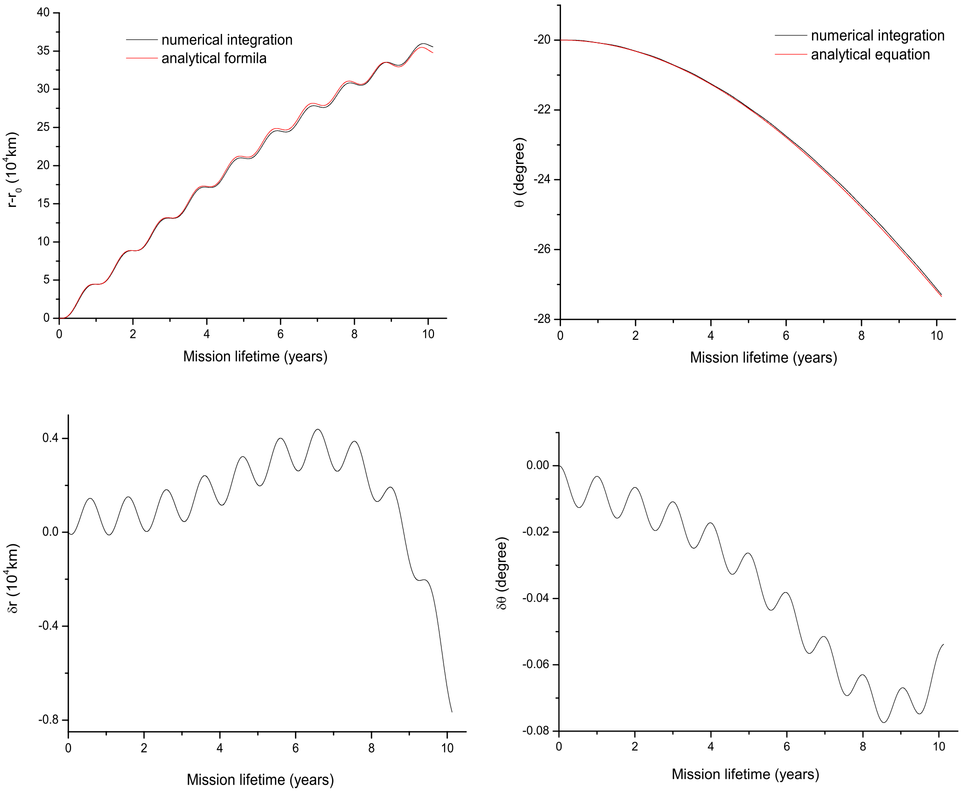

Fig. 3. Comparison of Eqs. (30), (31) with numerical integration, the upper two diagrams show the (almost identical) curves themselves in different colors (color online), the lower two diagrams show the differences between the equations and numerical integration.

4. Variation of the Arm-Length of the LISA Constellation

4.1. *Motion of a single spacecraft*

First, the motion of a single spacecraft will be analyzed. The formation of the constellation and the variation of the arm-length will be discussed subsequently. It is evident that the motion of a spacecraft in the rotating coordinate system can still be described by Eq. (6). The difference from the barycenter C is that the spacecraft, which moves around the Sun, is not restricted to the ecliptic plane. In other words, it is no longer a planar problem. However, because the orbit inclination of a LISA spacecraft is small (less than 0.02 radian), its motion can be regarded as a near-planar coorbital circular restricted three-body problem, which will be proved later.

In cylindrical coordinates

$$\begin{aligned} x &= \hat{r}\cos\theta\,, \\ y &= \hat{r}\sin\theta\,, \\ z &= z\,, \end{aligned} \tag{32}$$

Eq. (6) becomes

$$\begin{aligned} \ddot{\hat{r}} - \hat{r}\dot{\theta}^2 - 2n\hat{r}\dot{\theta} &= \frac{\partial\Omega}{\partial\hat{r}}\,, \\ \frac{d}{dt}(\hat{r}^2(\dot{\theta}+n)) &= \frac{\partial\Omega}{\partial\theta}\,, \\ \ddot{z} &= \frac{\partial\Omega}{\partial z}\,, \end{aligned} \tag{33}$$

where $\Omega = \Omega_0 + \mu\Omega_1$ and

$$\begin{aligned} \Omega_0 &= n^2\left[\frac{1}{2}\hat{r}^2 + \frac{1}{\sqrt{\hat{r}^2+z^2}}\right], \\ \Omega_1 &= n^2\left[-\hat{r}\cos\theta - \frac{1}{\sqrt{\hat{r}^2+z^2}} + \frac{1}{\sqrt{\hat{r}^2+z^2-2\hat{r}\cos\theta+1}}\right]. \end{aligned} \tag{34}$$

Due to z being a small quantity, z was allowed to be set to zero approximately in the first two equations of (33). The two equations could be regarded as the motion equations of the projection point of the spacecraft $(\hat{r}, \theta)$ on the xy plane

$$\begin{aligned} \ddot{\hat{r}} - \hat{r}\dot{\theta}^2 - 2n\hat{r}\dot{\theta} &= \frac{\partial\hat{\Omega}}{\partial\hat{r}}\,, \\ \frac{d}{dt}\left(\hat{r}^2(\dot{\theta}+n)\right) &= \frac{\partial\hat{\Omega}}{\partial\theta}\,, \end{aligned} \tag{35}$$

where $\hat{\Omega} = \hat{\Omega}_0 + \mu\hat{\Omega}_1$ and

$$\begin{aligned} \hat{\Omega}_0 &= n^2\left(\frac{1}{2}\hat{r}^2 + \frac{1}{\hat{r}}\right), \\ \hat{\Omega}_1 &= n^2\left(-\hat{r}\cos\theta - \frac{1}{\hat{r}} + \frac{1}{\sqrt{\hat{r}^2-2\hat{r}\cos\theta+1}}\right). \end{aligned} \tag{36}$$

The form of Eqs. (35) and (36) is completely identical to that of Eqs. (8) and (10), except that the initial value (14) is no longer true. Hence the conclusion on the planar problem in the previous section requires further detailed analysis, when applied to the motion of the projected point of the spacecraft onto the ecliptic plane.

For the non-perturbed motion of $\mu = 0$, Eqs. (12) and (13) are still valid if only the heliocentric distance r is replaced with its projection $\hat{r}$ on the ecliptic plane.

$$\hat{r}^2(\dot{\theta}+n) = \hat{h}\,, \tag{37}$$

$$\hat{r} = \frac{(\hat{h}/n)^2}{1+e\cos(\theta+nt+\beta)} = \frac{(\hat{h}/n)^2}{1+e\cos f}\,, \tag{38}$$

where

$$f = \theta + nt + \beta \tag{39}$$

and β is an integration constant that can be determined by the initial position of the spacecraft, as will be discussed in detail in Sec. 4.2 later. The curve (38) is the projection of the spacecraft orbit onto the ecliptic plane. It is an ellipse in the $(\hat{r}, f)$ coordinate with $a\cos I$ as the semi-major axis and $a(1+e)\cos I$ as the aphelic distance, where a, I are the semi-major axis and inclination of the spacecraft orbit, respectively, so that the eccentricity e remains constant. Regarding f as the true anomaly of the projection point, the mean anomaly M and eccentric anomaly[6] E could be defined in the same way as dealing with the two-body problem:

$$\begin{aligned} M &= M_0 + nt = E - e\sin E\,, \\ \hat{r}\cos f &= \cos E - e\,, \\ \hat{r}\sin f &= \sqrt{1-e^2}\sin E\,. \end{aligned} \tag{40}$$

The relation (39) can be written as

$$\theta = f - (nt+\beta) = f - (nt + f_0 - \theta_0) = (f - M) + (f_0 - M_0) + \theta_0\,, \tag{41}$$

where the subscript 0 denotes the value of the variable when $t = 0$. Substituting the equation of the center[6]

$$\delta \equiv f - M = \left(2e - \frac{e^3}{4}\right)\sin M + \frac{5}{4}e^2\sin 2M + \frac{13}{12}e^3\sin 3M + \cdots \tag{42}$$

into (41) yields

$$\Delta\theta \equiv \theta - \theta_0 = \delta - \delta_0\,. \tag{43}$$

This is the expression of the increment of the argument angle. In Eq. (38), when $f{=}\pi$, $\hat{r}$ reaches its maximum value $1+e$. Therefore, $h/n = \sqrt{1-e^2}$. Consequently the expression of the radius is

$$\hat{r} = \frac{1-e^2}{1+e\cos f}\,. \tag{44}$$

Furthermore, when $\mu \neq 0$, the second equation of (16) now becomes

$$\frac{d}{dt}[\hat{r}^2(\dot{\theta}+n)] = \mu n^2 \hat{r}\sin\theta\left(1 - \frac{1}{r_e^3}\right)\,, \tag{45}$$

where r_e is the distance of the spacecraft to the Earth, which can be replaced by the distance of the barycenter to the Earth $r_{ec} = 2\sin\frac{|\theta_c|}{2}$. Setting $\hat{r} = 1$ and $\theta = \theta_c = \theta_{c0}$, Eq. (45) degenerates into the form of Eq. (18), and the solution (21) becomes

$$\hat{h} = \hat{h}_0 + kn^2 t\,, \tag{46}$$

where the parameter k is given by definition (19), in which the symbol θ_0 should be written as θ_{c0} now.

In the following, the expressions of radius $\hat{r}$ and argument angle θ will be deduced. From expressions (37) and (46),

$$\dot{\theta} + n = \frac{\hat{h}}{\hat{r}^2} = \frac{\hat{h}_0 + kn^2 t}{\hat{r}^2}. \tag{47}$$

Setting $\hat{h} = \hat{h}_0 = n$ approximately, the equation can be simplified to $\dot{\theta} + n = \hat{h}^4 \hat{r}^{-2} (\hat{h}_0 + kn^2 t)^{-3} = n(1 - 3knt)$, namely

$$\dot{\theta} = -3kn^2 t. \tag{48}$$

From (43) and (48),

$$\theta = \theta_0 + \Delta\theta - \frac{3}{2} k (nt)^2. \tag{49}$$

Substituting relation (41) into (38), the final result is

$$\hat{r} = \frac{1 + 2knt\left(1 - \frac{3}{64} k_1 (nt)^2\right)}{1 + e\cos(\theta + nt + \beta)}. \tag{50}$$

Iteration with Eqs. (47) and (50) will increase the accuracy further.

As to z, the distance from the spacecraft to the ecliptic plane, can be solved as the two-body problem:

$$z = (\cos E - e)\sin I. \tag{51}$$

4.2. *Formation of the constellation and variation of arm-length*

As shown in Fig. 4, three spacecraft form the LISA constellation, and its barycenter C is in coorbital motion along the Earth orbit. In this paper, we assume that the masses of the three spacecraft are identical, so the barycenter of the constellation coincides with its geometrical center. The inclined ellipse is the osculating orbit of the spacecraft SC1 at the initial epoch. Assuming ϕ is the inclination between the constellation plane, on which the spacecraft move, and the ecliptic plane, the important relationship[16] between the eccentricity e of the spacecraft orbit, the inclination I to the ecliptic plane, the arm-length l of the constellation, and the angle ϕ is shown as Fig. 4.

$$e = \sqrt{1 + \frac{l^2}{3} + \frac{2l}{\sqrt{3}}\cos\phi} - 1, \tag{52}$$

$$\sin I = \frac{l}{\sqrt{3}\,(1+e)}\sin\phi, \tag{53}$$

$$\cos I = \frac{\sqrt{3} + l\cos\phi}{\sqrt{3}\,(1+e)}. \tag{54}$$

As mentioned in another paper,[15] the inclination ϕ is one of the key parameters to the optimization of the orbits design of the spacecraft.

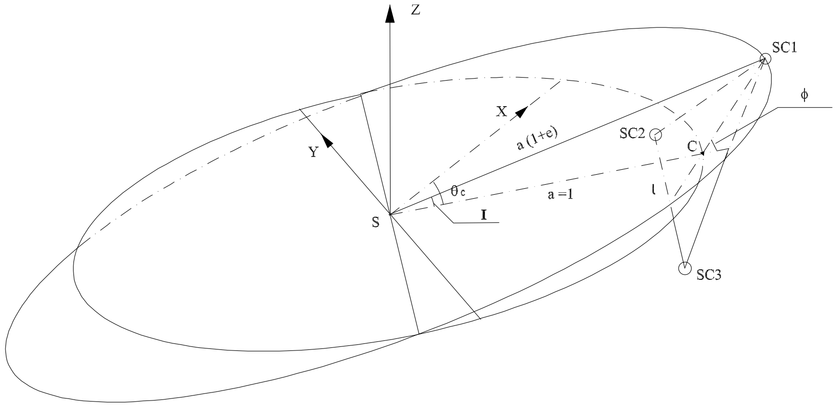

Fig. 4. The geometrical relationship among the inclination I, arm-length l and angle ϕ.

The discussion regarding a single spacecraft in Sec. 4.1, especially the expression of argument (49), the expression of radius (50) and the expression of distance (51) are all valid for any of the three spacecraft in the constellation. For convenience, in this section the mechanical variables of the given spacecraft are marked with the subscripts $i, j = 1, 2, 3$, i.e., $\mathbf{r}_i$, $\hat{r}_i$, θ_i, etc; the mechanical variables of the barycenter of the constellation are marked with the subscript c, i.e. $\mathbf{r}_c, \theta_c$ etc; and the initial values of these variables are still marked with subscript 0, i.e. θ_{c0} etc.

The next step is to determine the integration constant in relation (39). Figure 5 shows the projection of the LISA constellation onto the ecliptic plane. The x-axis points to the direction of SE (Sun-Earth), and θ_c is the argument angle between the barycenter C of the constellation and the x-axis. It is evident from the figure that the angle $\nu \approx 6l/(12 - \sqrt{3}l) = 0^\circ.962048$. At the initial epoch $t = 0$, $\theta_{10} = \theta_{c0}, \theta_{20} = \theta_{c0} + \nu$, and $\theta_{30} = \theta_{c0} - \nu$. From relation (39),

$$\beta_i = f_{i0} - \theta_{i0} \tag{55}$$

at $t = 0$, and thus $f_{10} = 180^\circ$, $f_{20} = 60^\circ.963335$ and $f_{30} = -60^\circ.963335$ could be derived.[15] Substituting these parameters into relation (55), the integration constants β_i become $\beta_1 = 180^\circ - \theta_c,\ \beta_2 = 60^\circ.001287 - \theta_c,\ \beta_3 = -60^\circ.001287 - \theta_c$.

From the previous section, the position vectors of the spacecraft are

$$\mathbf{r}_i = (\hat{r}_i \cos\theta_i,\ \hat{r}_i \sin\theta_i,\ z_i)^\tau\ , \tag{56}$$

where the variables are given by expressions (49), (50) and (51), respectively. The vector of the arm between any two spacecraft is

$$\mathbf{r}_{ij} = (\hat{r}_j \cos\theta_j - \hat{r}_i \cos\theta_i,\ \hat{r}_j \sin\theta_j - \hat{r}_i \sin\theta_i,\ z_j - z_i)^\tau\ . \tag{57}$$

Hence the square of the arm-length is

$$r_{ij}^2 = \mathbf{r}_{ij} \cdot \mathbf{r}_{ij} = (\hat{r}_j \cos\theta_j - \hat{r}_i \cos\theta_i)^2 + (\hat{r}_j \sin\theta_j - \hat{r}_i \sin\theta_i)^2 + (z_j - z_i)^2\ . \tag{58}$$

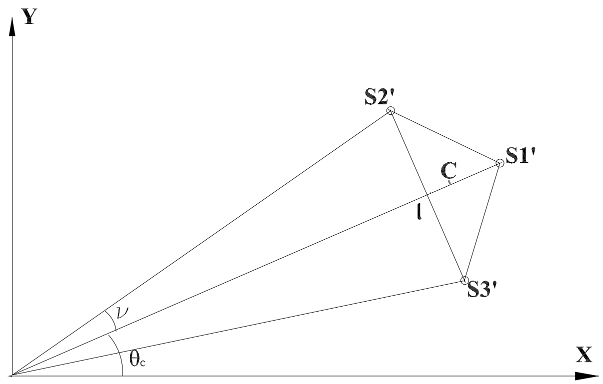

Fig. 5. The projection of LISA constellation on the ecliptic plane.

5. Discussion

A dynamical model — coorbital restricted problem together with the method for obtaining a general approximate solution — is proposed in this paper and applied to the design of the LISA spacecraft orbits, which is a special 2+3 coorbital restricted problem. This is a simplified approximate model. Besides the gravitational force of the Sun, only the perturbation of the gravitational force of the Earth-Moon system which is moving along a circle orbit for simplification, is considered. In fact, there are many very complicated perturbation effects in the LISA spacecraft motion. The perturbation of the gravitational force of other planets will be considered to optimize the orbits of LISA spacecraft to the accuracy of thousands of kilometers in another paper.[15] For further accurate research other important perturbation effects need to be taken into account carefully, including the perturbation from asteroids and general relativity, the non-gravitational effects on the orbits,[17,18] the solar-radiation pressure,[18–20] which depends on the reflectivities, the shapes, and the orientations of various surfaces of the spacecraft,[21,22] and the back-reaction from blackbody radiation emitted from the spacecraft.[23]

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