

# Geometrical Expression of the Angular Resolution of a Network of Gravitational-Wave Detectors and Improved Localization Methods

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**Abstract.** We report for the first time a method-independent geometrical expression for the angular resolution of an arbitrary network of interferometric gravitational wave (GW) detectors when the arrival-time of a GW is unknown. We discuss the implications of our results on how to improve angular resolutions of a GW network and on improvements of localization methods. An example of an improvement to the null-stream localization method for GWs of unknown waveforms is demonstrated.

## 1. Introduction

Several types of astrophysical sources are expected to be detectable both in gravitational waves (GWs) and in electromagnetic waves. Coincidence detections of these sources are of significant astronomical interest [1, 2]. A clear understanding of the angular resolution of an array of multiple GW detectors is vital to localizations of GW sources and to coincident detections.

A standard approach to measure how well we can localize a source is to calculate the Fisher information matrix where method-independent lower-bounds on statistical errors of estimated parameters can be obtained. Numerical results have been calculated by many authors for angular resolutions of both the ground-based and the future space GW detector [3, 4, 5, 6]. Explicit analytical expressions for the angular resolution of a network of GW detectors have been rare in the literature. We found two approximate formulae for a 3-detector network which are summarized in [7]. One is from private communication of Thorne (cited in Ref. [8]). The other is based on normalized numerical results for a 3-detector network for detections of GWs from neutron star-neutron star coalescence using the coherent approach [5]. A general expression for an arbitrary network of GW detectors have not been seen.

Localization of GW sources of unknown waveforms can be obtained by the so-called “null-stream” method [8, 9]. GWs are known to have only two polarizations. The response of an interferometric GW detector is a linear combination of the two wave polarizations. Therefore if we have data from more than two detectors, we can linearly combine the data to cancel out the GW signal. The resulting data streams are called the “null-streams” as they have null-responses

to GW signals. Localization of a GW source can be achieved by searching for sky directions where the constructed null-stream is statistically “null” [8, 9]. There are also semi-null streams [10] where in linearly combined data, signals are not exactly canceled out but are significantly reduced. We propose that localization be further improved by including information from these semi-null streams.

In this report, we summarize results of our on-going research work concerning (1) the angular resolution of an arbitrary network of interferometric GW detectors [11] and (2) localization methods for GWs of unknown waveforms [12]. An explicit geometrical expression for the angular resolution of an arbitrary network of GW detectors is presented for the first time. An improved localization method using null-streams combined with semi-null streams is demonstrated and compared to that of a straightforward null-stream-only method.

## 2. Mathematical Preliminaries

Suppose we have a network of  $N_d$  gravitational-wave detectors, each with spatial size much shorter than the GW wavelength, the observed strain of an incoming GW by the individual detector  $I$  is then a linear combination of the two wave polarizations in the transverse traceless (TT) gauge,

$$d_I(t_0 + \tau_I + t) = f_I^+ h_+(t) + f_I^\times h_\times(t), \quad 0 < t < T, \quad (1)$$

where  $t_0$  is the arrival time of the wave at the coordinate origin,  $\tau_I$  is the wave travel time from the origin to the  $I$ -th detector,  $T$  is the signal duration,  $t \in [0, T]$  is the time label of the wave. The quantities  $f^+$  and  $f^\times$  are the detector’s *antenna beam pattern* functions [13] for the two wave polarizations ( $h_+$ ,  $h_\times$ ). They depend on the relative orientation between the detector configuration and the frame in which the wave polarizations are defined (which is in turn related to the propagation direction  $\hat{\mathbf{n}}$ ).

If we assume signal duration to be short enough such that the motion of the detector array is unimportant, then in the frequency domain, and in matrix notation, we can write time-delay-shifted responses of all detectors as

$$\mathbf{d}(\Omega) = \mathbf{A}\mathbf{h}(\Omega), \quad (2)$$

where  $\Omega$  is the angular frequency. The antenna pattern  $\mathbf{A}$  is an  $N_d \times 2$  constant matrix,

$$\mathbf{A} = \begin{bmatrix} f_1^+(\hat{\mathbf{n}}) & f_1^\times(\hat{\mathbf{n}}) \\ \vdots & \vdots \\ f_{N_d}^+(\hat{\mathbf{n}}) & f_{N_d}^\times(\hat{\mathbf{n}}) \end{bmatrix}, \quad (3)$$

and  $\mathbf{h}(\Omega)$  is a 2-dimensional vector function,

$$\mathbf{h}(\Omega) = \begin{bmatrix} h_+(\Omega) \\ h_\times(\Omega) \end{bmatrix}. \quad (4)$$

We denote  $d_I$  as the data from the  $I$ -th GW detector and the corresponding noise spectral density is  $S_I$ , we define a whitened data set of

$$\hat{d}_I(\Omega) = S_I^{-\frac{1}{2}}(\Omega)d_I(\Omega). \quad (5)$$

Note that  $\hat{\mathbf{d}}(\Omega)$  corresponds to the whitened data set at each frequency. Correspondingly, we denote  $\hat{\mathbf{A}}$  as a  $N_d \times 2$  response matrix weighted by noise,

$$\hat{\mathbf{A}}(\Omega) \equiv \begin{bmatrix} \frac{f_1^+(\hat{\mathbf{n}})}{\sqrt{S_1(\Omega)}} & \frac{f_1^\times(\hat{\mathbf{n}})}{\sqrt{S_1(\Omega)}} \\ \vdots & \vdots \\ \frac{f_{N_d}^+(\hat{\mathbf{n}})}{\sqrt{S_{N_d}(\Omega)}} & \frac{f_{N_d}^\times(\hat{\mathbf{n}})}{\sqrt{S_{N_d}(\Omega)}} \end{bmatrix}, \quad (6)$$

so that we have  $\hat{\mathbf{d}} = \hat{\mathbf{A}}\mathbf{h}$ . For simplicity, we keep the  $\Omega$ -dependence in the notation only when it is necessary for clarity.

### 3. Geometrical Expression of Angular Resolution

The angular resolutions are calculated by applying the Fisher information matrix to obtain method-independent lower limits on the statistical errors in estimating the direction of a GW source. The limits are for unbiased estimators and Gaussian noise (for cautions in using these limits, see [14]). The covariance matrix of the “best estimated” angular direction of a GW source can be obtained from the corresponding sub-matrix of the inverse of the Fisher matrix for all unknown parameters. We show in [11] that in case the initial arriving time  $t_0$  of the wave is unknown, the lower bounds of one-sigma error area of angular parameters estimated using data from an arbitrary network of GW detectors can be written in a compact geometrical form. Here we only summarize the result without showing derivations. We present also only cases for short-duration GWs where antenna beam patterns of GW detectors are treated as constant. Similar expressions for continuous GWs for ground-based detectors and for the space GW detector LISA can be found in [11].

We have defined the error solid angle to be twice the area of the  $1\text{-}\sigma$  error ellipse (measured in *srad*) in angular parameters of  $\theta$  (latitude-like) and  $\phi$  (longitude-like),

$$\Delta\Omega = 2\pi|\cos\theta|\sqrt{\langle\Delta\theta^2\rangle\langle\Delta\phi^2\rangle - \langle\Delta\theta\Delta\phi\rangle^2}, \quad (7)$$

we have found that for an arbitrary network of GW detectors,

$$\Delta\Omega = \frac{4\sqrt{2}\pi c^2}{\sqrt{\sum_{J,K,L,M} \Delta_{JK}\Delta_{LM} |(\mathbf{r}_{KJ} \times \mathbf{r}_{ML}) \cdot \hat{\mathbf{n}}|^2}}, \quad (8)$$

where  $\mathbf{r}_{KJ}$  is the displacement vector from detector K to detector J.

For the worst-case scenario where nothing is known about the initial arrival time  $t_0$  or the waveform of a GW, we found that

$$\Delta_{KJ} = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\Omega \Omega^2 \hat{d}_J^* \hat{d}_K P_{KJ} \quad \text{for } K \neq J, \quad (9)$$

where matrix  $\mathbf{P} = \hat{\mathbf{A}}(\hat{\mathbf{A}}^\dagger \hat{\mathbf{A}})^{-1} \hat{\mathbf{A}}^\dagger$  ( $\hat{\mathbf{A}}$  is defined in Eq. 6). Note that only  $J \neq K$  terms contribute in Eq. 8.

For the best-case scenario where the GW waveform is known and the only unknowns are the initial wave arrival time  $t_0$  and sky directions, we found

$$\Delta_{KJ} = \frac{\xi_K \xi_J}{\sum_I \xi_I} \quad \text{for } J \neq K \quad (10)$$

where we have defined

$$\xi_J \equiv 2 \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \Omega^2 |\hat{d}_J|^2. \quad (11)$$

Note that  $\xi_J$  corresponds to the noise-weighted GW energy flux coupled to the  $J$ th detector.

#### 3.1. Implication

Here we note the clear geometrical meaning of  $|(\mathbf{r}_{KJ} \times \mathbf{r}_{ML}) \cdot \hat{\mathbf{n}}|$  in Eq. 8, which is twice the area of the quadrangle formed by the projections of detectors  $J$ ,  $K$ ,  $L$  and  $M$  onto the plane orthogonal to the wave propagation direction. We also note that, in the worst-case scenario

where the waveform is unknown, the angular resolution is inversely proportional to the weighted correlation of responses between detectors (Eq. 9). In the best-case scenario, it is inversely proportional to the fractional GW energy flux coupled to each detector (Eq. 10, Eq. 11).

Our formula is consistent with the known concept that a larger network is advantageous for a better angular resolution. For instance, inclusion of the future Australian AIGO detector can improve dramatically the angular resolution of the network [1]. It further indicates that angular resolutions can be improved by optimizing values of  $\Delta_{IJ}$ . For instance, building more detectors of correlated response is advantageous for localizing GWs of unknown waveforms. Similarly, a specific localization method can be improved to approach the intrinsic angular resolution by selecting data contributing significantly to fractional energy flux (best-case scenario) or to correlations of data between detectors (worst-case scenario). In other words, a localization method can be improved by proper treatments of data corresponding to weak responses.

#### 4. Ranking Network Responses by Singular Value Decomposition Method

In this and the next section, we demonstrate how one can improve the null-stream localization method for GWs of unknown waveforms. We show how to apply the singular value decomposition (SVD) method [15] to recombine data from a network of GW detectors to form new data streams with characterized sensitivity to GWs from a sky direction. Specifically, we use the SVD to construction signal streams, generalized null-streams that have null responses to GWs, and semi-null streams that have weak responses to signals. The SVD of  $\hat{\mathbf{A}}$  yields (see also [10])

$$\hat{\mathbf{A}} = \mathbf{U}\mathbf{S}\mathbf{V}^\dagger, \mathbf{S} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \quad (12)$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are unitary matrices of dimensions of  $N_d \times N_d$  and  $2 \times 2$  respectively at each frequency, i.e.,  $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$  and  $\mathbf{V}\mathbf{V}^\dagger = \mathbf{I}$ ,  $s_1 \geq s_2 \geq 0$  are the so-called singular values. Note they are all frequency-dependent.

We then construct new data streams by inserting this decomposition into equation  $\hat{\mathbf{d}} = \hat{\mathbf{A}}\mathbf{h}$ , and have

$$\mathbf{U}^\dagger \hat{\mathbf{d}} = \begin{pmatrix} s_1(\mathbf{V}^\dagger \mathbf{h})_1 \\ s_2(\mathbf{V}^\dagger \mathbf{h})_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (13)$$

It is evident that the first two components of the new data streams contain signal information, the last  $N_d - 2$  terms are null-streams as they have zero-response to signal. In general, null-streams can be written as

$$\mathbf{N}(\Omega) = \begin{pmatrix} (\mathbf{U}^\dagger \hat{\mathbf{d}})_3 \\ (\mathbf{U}^\dagger \hat{\mathbf{d}})_4 \\ \vdots \\ (\mathbf{U}^\dagger \hat{\mathbf{d}})_{N_d} \end{pmatrix}. \quad (14)$$

Naturally in case  $s_2 = 0$ ,  $(\mathbf{U}^\dagger \hat{\mathbf{d}})_2$ -term should also be included as a null stream. We assume for now that  $s_2 \neq 0$ . The sensitivity level of each signal stream to a GW can be ranked by its singular value. Suppose  $s_{\max} = \max_{i,\Omega} s_i(\Omega)$  ( $i = 1, 2$ ), we define (tentatively) semi-null

streams as signal streams with corresponding singular values much less than  $s_{\max}$ .

$$\mathbf{SN}(\Omega) = \begin{pmatrix} (\mathbf{U}^\dagger \hat{\mathbf{d}})_i \\ \dots \end{pmatrix} \quad \text{if } s_i \ll s_{\max} \quad i = 1, 2. \quad (15)$$

Although responses in semi-null streams may not be zero, they can be insignificant compared to dominating signal streams.

### 5. Improved Localization Strategy Using Semi-Null Streams

It has been demonstrated [8, 9] that null-streams  $\mathbf{N}(\Omega)$  can be used to localize a source by searching through sky directions for minimum statistic of

$$P_N = \int d\Omega \sum_{I'} |N_{I'}(\Omega)|^2. \quad (16)$$

It has also been proposed [10] that semi-null streams can be included to improve the angular resolution. One possible new statistic is

$$P_{SN} = \int d\Omega \left( \sum_{I'} |N_{I'}(\Omega)|^2 + \sum_{I'} |SN_{I'}(\Omega)|^2 \right). \quad (17)$$

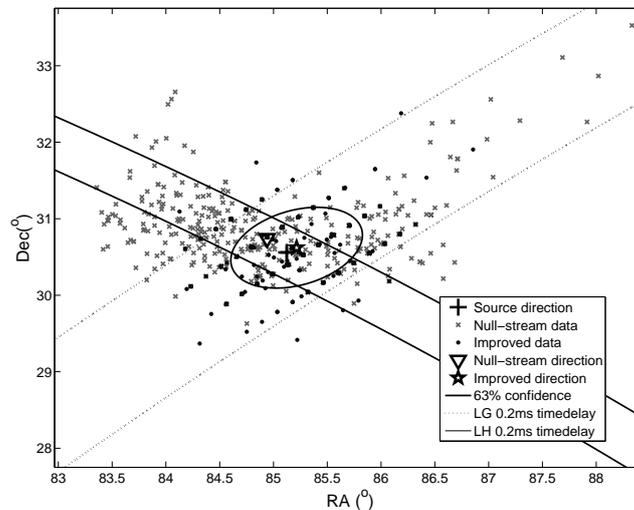
Instead of searching through sky directions for minimum statistic of Eq. 16 as discussed in [8, 9], we now search for minimum statistic in Eq. 17. The tricky part is how to set the threshold at which semi-null streams are to be included [12].

#### 5.1. Numerical Example

In this section, results from a Monte-Carlo simulation is presented to illustrate how the null-stream localization method can be improved by including the semi-null streams. As a proof-of-principle example, we have simply included all semi-null streams in Eq. 17 that satisfied an empirical threshold of  $s_i(\Omega)/s_{\max} \leq 0.01$  (where  $s_i$  are singular values defined in Eq. 12). Localization is then obtained by searching through sky directions for minimum statistic of (1) null-stream-only statistic (Eq. 16) and (2) semi-null-stream statistic (Eq. 17) respectively. Results are then compared.

We have used simulated signal and noise. For the signal, we used a Sin-Gaussian wave form of  $h_+(t) = h_\times(t) = h_0 \sin(2\pi f_0 t) \exp(-t^2/\tau^2)$  with polarization angle chosen arbitrarily to be  $\psi = 0$ , signal duration  $T = 7$  ms, sampling rate = 16 kHz, central frequency  $f_0 = 700$  Hz and  $\tau = 2$  ms. The arrival time of the GW wavefront at LIGO Livingston (L1) is chosen arbitrarily to be at 0.00 hr, March, 18, 2004. The source direction was chosen to be near that of the maximum sensitivity of L1 (right ascension RA= 85.1235<sup>0</sup> and declination Dec= 30.56<sup>0</sup>) at the chosen time. We have chosen an optimal network signal-to-noise ratio of SNR= 20. Location information of different GW observatories were obtained from [16] and references therein. For the noise, we have adopted the designed noise spectral densities for initial LIGOs (at Livingston, L1, and at Hanford, H1) [17] and for GEO [18] at 500 Hz tuning. The simulated GW signal is then injected into a total of 500 sets of randomly generated Gaussian noise. For each of the simulated data of noise plus signal, we use the Nelder-Mead method [19] to search through sky directions for minimum statistics of Eq. 16 and Eq. 17 respectively. All searches start from the source direction to shorten the search time which is adequate for the purpose of proof of principle.

The results are shown in Fig. 1. Source directions obtained using the null-stream-only method (Eq. 16) for different noise realizations are plotted in cross symbols. Filled circles are those from



**Figure 1.** Comparison of localization methods.

the improved localization method where semi-null streams are included (Eq. 17). The error ellipse is for data from the improved method at a 63% confidence level assuming a bi-variate normal distribution of angular parameters. The star symbol indicates the average direction in the improved method. The actual source direction is indicated with a cross. The triangle symbol indicates the average direction from the null-stream-only method. We also plot the time-delay lines for L1-G (gray dotted lines) and L1-H1 (solid lines) at a 0.2 ms interval.

It is evident that in this particular example, for a direction where sources are most likely to be detected by the LHG-network, inclusion of semi-null streams can improve the source localization significantly. The scatter of angular directions obtained from the null-stream-only (gray crosses) method is much larger (and therefore worse localization) than that when semi-null streams are also included (filled circles). Note that our choice of the semi-null stream is not optimal, further improvements are expected when optimal search methods are constructed [12].

## 6. Conclusion

We have reported for the first time a compact geometrical expression for an arbitrary network of GW detectors when the initial arrival time of a GW is unknown. Our results demonstrate the known geometrical elements, as well as the role of energy fluxes and correlation between responses of different detectors that determine the intrinsic angular resolution of a GW detector array. In the second part of this paper, we show an example where localization of a GW source can be improved by including semi-null streams which are linear combination of data that have weak response to a GW signal than that of null-stream-only method. We show how the Singular-Value-Decomposition method, besides providing a vehicle for generalized optimizations of detection methods and for construction of generalized null-streams [10], can also be used to identify semi-null-streams and help improve the GW source direction determination.

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