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Dual superconformal symmetry from $AdS_5 \times S^5$ superstring integrability

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We discuss $2d$ duality transformations in the classical $AdS_5 \times S^5$ superstring and their effect on the integrable structure. T-duality along four directions in the Poincaré parametrization of AdS_5 maps the bosonic part of the superstring action into itself. On the bosonic level, this duality may be understood as a symmetry of the first-order (phase space) system of equations for the coset components of the current. The associated Lax connection is invariant modulo the action of an $\mathfrak{so}(2,4)$-automorphism. We then show that this symmetry extends to the full superstring, provided one supplements the transformation of the bosonic components of the current with a transformation on the fermionic ones. At the level of the action, this symmetry can be seen by combining the bosonic duality transformation with a similar one applied to part of the fermionic superstring coordinates. As a result, the full superstring action is mapped into itself, albeit in a different κ-symmetry gauge. One implication is that the dual model has the same superconformal symmetry group as the original one, and this may be seen as a consequence of the integrability of the superstring. The invariance of the Lax connection under the duality implies a map on the full set of conserved charges that should interchange some of the Noether (local) charges with hidden (nonlocal) ones and vice versa.

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I. INTRODUCTION

The integrability of string theory in $AdS_5 \times S^5$ holds major promise of a complete solution for its spectrum and thus for the spectrum of dimensions of gauge invariant operators of the dual $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The bosonic $AdS_5 \times S^5$ sigma model is classically integrable, being based on the coset $SO(2,4)/SO(1,4) \times SO(6)/SO(5)$ [1,2]. That property extends to the full Green-Schwarz (GS) superstring model based on the supercoset $PSU(2,2|4)/(SO(1,4) \times SO(5))$ [3], as follows from its special κ-symmetry structure [4].

The integrability formally implies the existence of an infinite number of conserved charges and thus of an infinite-dimensional hidden symmetry algebra of the $2d$ string sigma model. In addition to the obvious kinematic superconformal $PSU(2,2|4)$ symmetry "seen" by a point-particle limit of the string and corresponding to the standard local Noether conserved charges, there are also hidden "stringy" symmetries and the associated charges.

The integrable structure of a G/H coset sigma model is naturally defined on a phase space, i.e. in terms of a family of currents or a Lax connection whose flatness implies the complete system of first-order differential equations for the components of the current. This system includes the Maurer-Cartan part as well as a "dynamical" part which follows from the standard coset sigma model Lagrangian.

Like in many similar examples (Maxwell equations, etc.), the first-order system in a sense is more general than the usual coset model: while it does not directly follow from a local Lagrangian, it may lead to different "dual" local Lagrangian representations. The latter are found by a "phase space reduction" procedure—by solving part of the first-order equations and substituting the solution into the rest which may then follow from a dual Lagrangian. Various sigma model dualities can be understood in that way (see, e.g., [5–7]). The dual sigma models will then be classically equivalent (i.e. their classical solutions will be directly related) and will share the same integrable structure, i.e. hidden symmetries.

Those of such classical duality transformations which can be implemented by a change of variables in a path integral formulation of the theory can then be extended to the quantum level. They then define quantum-dual sigma models [7] in the sense that there is a prescription relating certain quantum correlators in one theory to certain correlators in the dual theory. The well-known example is the standard $2d$ scalar duality or T-duality in which case the path integral transformation relating the two dual theories may be formulated in terms of gauging an Abelian isometry [8,9].

In general, such sigma model dualities do not respect manifest global symmetries of the theory, that is, symmetries seen by a pointlike string. For example, the usual T-duality in x direction relates the models with target-

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space metrics $ds^2 = dr^2 + a^2(r)dx^2$ and $ds^2 = dr^2 + a^{-2}(r)dx^2$, so that in the case of S^2 when $a = \sin(r)$ and x is compact, the first model has $SO(3)$ symmetry while the second only $SO(2)$. The two sigma models are still classically equivalent, i.e. they share the same first-order formulation and integrable structure. Put differently, the corresponding $2d$ field theories (i.e. string models as opposed to their point-particle truncations) are, in a sense, equally symmetric.[1]

A special case is provided by the AdS_n sigma model in Poincaré coordinates ($a = \mathrm{e}^r$ in the above example corresponds to AdS_2), where the formal T-duality along all $(n-1)$ translational directions combined with a simple coordinate transformation $r \mapsto -r$ gives back an equivalent sigma model on the dual AdS_n space [11]. This means that the original and the dual models happen to have equivalent global Noether symmetries $SO(2, n-1)$ but realized on different (dual) sets of variables. This "T-self-duality" property of AdS_5 was used in [12] to construct classical solutions for open strings related to the strong-coupling limit of gluon scattering amplitudes (see also [13]). It appears also to be related to the "dual conformal symmetry" of maximally helicity violating (MHV) amplitudes observed at weak coupling [14,15].[2]

Since the two dual models are classically equivalent and also share the same integrable structure, the local Noether charges of the dual model should be related to hidden (nonlocal) charges of the original model and vice versa. One can indeed express the Lax connection in terms of either original or dual variables and thus, in principle, relate the charges in the two pictures [19].[3] Given that the sigma model has an infinite number of hidden charges, the true significance of this "doubling" of a particular subset of them, i.e. of the Noether symmetry charges, still remains to be understood.

As was suggested in [19], and this will be one of the aims of the present paper, one should be able to extend this construction of the flat currents and associated charges to the dual of the full $\mathrm{AdS}_5 \times S^5$ superstring model. In the process, we will show explicitly that the Lax connections of the original and the dual sigma models are not independent but equivalent (in particular, one does not get doubling of conserved charges).

Very recently, it was suggested [21] that the dual conformal symmetry of perturbative $\mathcal{N} = 4$ SYM theory MHV amplitudes [14] has an extension to the full "dual superconformal symmetry" if one considers the full set of supergluon amplitudes. Simultaneously, it was suggested [22] that, if the T-duality transformation of the $\mathrm{AdS}_5 \times S^5$ superstring action along the four translational directions of AdS_5 [11] is followed by a similar $2d$ duality transformation applied to part of the fermionic world-sheet variables (corresponding to the Poincaré supersymmetry generators Q but not $\bar{Q}$), then the dual action will take the original form, i.e. it will be again equivalent to the $\mathrm{AdS}_5 \times S^5$ supercoset GS action. Thus, the dual Noether charges will again generate the full superconformal group $PSU(2, 2|4)$.

Below we shall extend the discussion in [19] by showing that in the bosonic AdS_n sigma model the particular $2d$ duality corresponding to the T-duality can be realized as a discrete symmetry of the phase space equations,[4] i.e. of the first-order system of equations for the components of the $SO(2, n-1)/SO(1, n-1)$ coset current. Furthermore, this duality can be reinterpreted as a (spectral parameter dependent) automorphism of the global $\mathfrak{so}(2, n-1)$ symmetry algebra (4.25). This fact then makes the original and the dual integrable structures equivalent in a precise manner.

We shall also consider the full $\mathrm{AdS}_5 \times S^5$ superstring case in the same κ-symmetry gauge as in [11][5] in which the dual action is quadratic in the fermions and explicitly performs the bosonic duality transformation in the Lax connection. We shall then follow the idea of [22] and supplement the bosonic duality by a fermionic duality transformation to discover that the resulting action (which is again quadratic in the fermions) can be identified with the original $\mathrm{AdS}_5 \times S^5$ superstring action written in a different (complex) κ-symmetry gauge (used previously in [26]).

We shall show that the necessity of the fermionic duality transformation becomes transparent in the first-order formulation in terms of the supercoset current: the discrete symmetry of the phase space system which corresponded to the bosonic T-duality should be extended to act also on the fermionic components of the currents in order to make it possible to identify the original and the dual Lax connections upon an action of an automorphism of the $\mathfrak{psu}(2, 2|4)$ superalgebra (5.23).[6] This not only implies

[1]This is similar to the fate of space-time supersymmetry under T-duality: it remains a symmetry of the underlying conformal field theory but may become nonlocally realized (see, e.g., [10]).

[2]The T-duality on string side seems to be intimately connected with the relation between the gluon scattering amplitudes and Wilson loops at strong [12,16] (see also [17]) and weak [14,15,18] coupling.

[3]For previous work on the relation between local and nonlocal charges under T-duality in a different context see, e.g., [20].

[4]The usual $2d$ scalar duality $dx \mapsto *dx$ or $j \mapsto *j$ can be viewed as a phase space transformation (see, e.g., [23]) that exchanges momenta $\partial_\tau x$ with some combinations of coordinates $\partial_\sigma x$ (or $x_{k-1} - x_k$ in discrete mode representation).

[5]This "S-gauge" [24,25] is a natural choice for a comparison with the boundary gauge theory as it corresponds to setting the fermionic components associated to the superconformal generators S, $\bar{S}$ to zero. It naturally complements the parametrization of the bosonic part of the supercoset in terms of the coordinates corresponding to the translational P, dilational D, and the R-symmetry $SU(4)$ generators.

[6]In general, the duality as a symmetry of the first-order system should be understood modulo a choice of G/H coset representative, i.e. local H-symmetry gauge, and a choice of κ-symmetry gauge (and also modulo certain analytic continuation).

that the resulting "T-dual" model should have the full dual superconformal symmetry but should also allow one, in principle, to establish the duality isomorphism on the full infinite set of conserved charges. In that sense, the dual superconformal symmetry may be viewed as a consequence of integrability.[7]

From a broader perspective, this T-duality is just one particular symmetry of the first-order system for the $\mathrm{AdS}_5 \times S^5$ superstring. One may consider also other duality transformations that will lead to equivalent classical systems; for example, one may mix the fermionic and bosonic dualities in a different order, etc. These more general transformations are also worth studying (though not all of them may have path integral, i.e. quantum, counterparts) as they may further clarify the structure of this integrable theory. The special feature of T-duality is that it preserves the maximal possible global symmetry group. The existence of such transformation appears to be deeply rooted in the structure of the superconformal algebra: the possibility to choose the translational subalgebra as maximal Abelian subalgebra in $SO(2,4)$ ($[P_a, P_b] = 0$) together with its $\mathcal{N} = 4$ Poincaré supersymmetry counterpart ($\{Q^{i\alpha}, Q^{j\beta}\} = 0 = [P_a, Q^{i\alpha}]$); the invariant meaning of this $2d$ duality transformation is that it acts on the associated four bosonic and eight fermionic $2d$ fields.

Let us emphasize once more the point (already mentioned in [19]) that, given conserved Noether charges in the original sigma model, we may express them in terms of the dual variables and thus get a collection of (possibly nonlocal) conserved charges in the dual model. The existence of an additional set of conserved Noether charges in the dual model which are local in the dual variables and thus nonlocal in the original variables means that they must originate from some hidden conserved charges in the original model, and this may be viewed as a consequence of its integrability. This explains the title of the present paper.

The structure of the paper is as follows.

In Sec. II, we make some general comments on $2d$ duality transformations in the group G and the coset G/H bosonic sigma models.

In Sec. III, we review the structure of the $\mathrm{AdS}_5 \times S^5$ superstring sigma model using the supercoset construction. We shall present the equations of motion in first-order form and describe two important families of flat currents implying integrability of this model. Then we shall specialize the discussion to the standard choice of the basis of generators of the superconformal algebra, choose a parametrization of the supercoset adapted to Poincaré coordinates in AdS_5 and fix a particular κ-symmetry gauge in which the sigma model action takes a simple form.

In Sec. IV, we follow [11] and transform the superstring action by applying $2d$ duality (T-duality) to four scalar fields corresponding to translational directions of AdS_5. We shall show how to express the flat Lax connection in terms of the dual variables which implies integrability of the dual model. We shall then consider the first-order system of equations for the current and, first ignoring the fermions, show that the T-duality transformation can be understood as a symmetry of this system and of the corresponding Lax connection. This means that one does not get two copies of the integrable structure but rather an automorphism on the space of conserved charges (4.25).

In Sec. V, we show, following the suggestion of [22], that combining the bosonic duality with a similar duality transformation on half of fermions present in the κ-symmetry gauge-fixed action one gets back the same $\mathrm{AdS}_5 \times S^5$ superstring action but written in a different κ-symmetry gauge (and modulo a certain analytic continuation). This implies recovering after the duality the full global superconformal symmetry. The combined action of bosonic and fermionic duality transformations will then be understood more abstractly as a symmetry of the first-order system and the Lax connection which manifests itself as an automorphism of the superconformal algebra: under its action, original and dual Lax connections get identified (5.23) (see also Table I).

The first three appendices contain our notation and some technical details, while the last one contains some comments on the construction of conserved charges in the case of closed string world sheet.

TABLE I. Behavior of the Noether charges under bosonic and fermionic dualities.

the $P_{\alpha\dot\beta}$ charge becomes trivial
the $L_{\alpha\beta}$ and $L_{\dot\alpha\dot\beta}$ charges go into themselves and remain local
the $K_{\alpha\dot\beta}$ charge gets lifted and becomes nonlocal
the D charge goes into itself and remains local
the $R_i{}^j$ charge goes into itself and remains local
the $Q^{i\alpha}$ charge becomes trivial
the $\bar{Q}_i^{\dot\alpha}$ charge goes into the $\bar{S}^{i\dot\alpha}$ charge and remains local
the S_i^α charge gets lifted and becomes nonlocal
the $\bar{S}^{i\dot\alpha}$ charge goes into the $\bar{Q}_i^{\dot\alpha}$ charge and remains local

II. BOSONIC COSET MODELS AND THEIR DUALITIES

Before turning to the $\mathrm{AdS}_5 \times S^5$ superstring and specifics of T-duality, let us make few general comments about classical sigma model dualities in the context of the bosonic G/H coset model.

Let us start with the principal chiral model (PCM) based on

[7] Conformal invariance and integrability are expected to promote this classical symmetry to a quantum one. To attempt to relate it to dual conformal symmetry of the boundary gauge theory one should combine its action on the "bulk" string coordinates with action on the vertex operators (inserted at the boundary or an IR brane [12]) so that the gluon scattering amplitudes as computed on the string side become invariant.

$$L = \frac{1}{2}\,\mathrm{tr}(j \wedge *j), \quad \text{with} \quad j = g^{-1}dg \quad \text{and} \quad g \in G. \tag{2.1}$$

The corresponding equations of motion written in first-order form are

$$dj + j \wedge j = 0 \quad \text{and} \quad d*j = 0. \tag{2.2}$$

These two equations (2.2) follow from the condition of flatness of the following family of currents or Lax connection, with z as a complex spectral parameter:

$$\mathrm{j}(z) = aj + b*j, \quad \text{with} \quad a = -\frac{1}{4}(z - z^{-1})^2 \quad \text{and} \quad b = \frac{1}{4}(z^2 - z^{-2}). \tag{2.3}$$

The standard second-order PCM equation is found by solving the first (Maurer-Cartan) equation in (2.2) as $j = g^{-1}dg$ and then substituting the solution into the second equation.

Instead, we may construct a dual model [5] by first solving the second equation in (2.2) as $j = *d\chi$, where $\chi \in \mathfrak{g} := \mathrm{Lie}(G)$ is the dual field and substituting this into the Maurer-Cartan equation. The resulting equation $d*d\chi - d\chi \wedge d\chi = 0$ then follows from the dual Lagrangian

$$\tilde{L} = \frac{1}{2}\,\mathrm{tr}\Big(d\chi \wedge *d\chi + \frac{2}{3}\chi d\chi \wedge d\chi\Big). \tag{2.4}$$

Note that if we write $g = \mathrm{e}^{\eta}$ then for small η the relation between η and χ is the same as the usual $2d$ scalar duality $d\eta = *d\chi$. If we introduce the dual current $\tilde{j} = d\chi$, then the first-order system for the dual model will be[8] $d*\tilde{j} - \tilde{j} \wedge \tilde{j} = 0$ and $d\tilde{j} = 0$, i.e. it will be equivalent to the original one (2.2) under $j \mapsto *\tilde{j}$. This transformation will leave the Lax connection (2.3) invariant provided we supplement it by $z \mapsto \mathrm{e}^{\pi/4\mathrm{i}}z$ and an overall rescaling by i. Note that the Noether symmetries of the original and dual sigma models here are different.

The model (2.4) is sometimes called "pseudodual" [6,27] to reflect the fact that it is not quantum-equivalent to the original PCM. To construct the quantum-equivalent "non-Abelian dual" of the PCM [7], one has to start with

$$\bar{L} = \mathrm{tr}\Big[\frac{1}{2}j \wedge *j + \varphi(dj + j \wedge j)\Big], \tag{2.5}$$

where $\varphi \in \mathfrak{g}$ plays the role of a Lagrange multiplier, and subsequently integrate out j. The resulting dual model will again be classically equivalent to the PCM.[9] It has equivalent integrable structure but (after we solve for j) will have smaller Noether symmetry.

[8]We assume Minkowski signature on the world sheet.

[9]Indeed, while the classical equations that follow from (2.5) will involve an extra field φ, one can easily see that they imply (2.2). One finds that $*j = -\nabla\varphi$ and $dj + j \wedge j = 0$ but then $\nabla^2 = 0$ which leads to $d*j = 0$.

Let us now turn to the case of the G/H symmetric space coset model given by

$$L = \frac{1}{2}\,\mathrm{tr}(j_{(2)} \wedge *j_{(2)}), \quad \text{with} \quad j = g^{-1}dg = j_{(0)} + j_{(2)} \equiv A + j_{(2)}, \tag{2.6}$$

where we split the current according to the $\mathbb{Z}_2$-decomposition of the Lie algebra $\mathfrak{g} \cong \mathfrak{g}_{(0)} + \mathfrak{g}_{(2)} \equiv \mathfrak{h} + \mathfrak{g}_{(2)}$. The corresponding first-order system may be written as ($\nabla := d + A$)

$$dA + A \wedge A + j_{(2)} \wedge j_{(2)} = 0, \qquad \nabla j_{(2)} = 0 \quad \text{and} \quad \nabla *j_{(2)} = 0, \tag{2.7}$$

where the first two equations are the $\mathfrak{h}$ and $\mathfrak{g}_{(2)}$ components of the Maurer-Cartan equation. These equations follow from the flatness of a Lax connection similar to the one in (2.3)

$$\mathrm{j}(z) = A + aj_{(2)} + b*j_{(2)}, \quad \text{with} \quad a = \frac{1}{2}(z^2 + z^{-2}) \quad \text{and} \quad b = -\frac{1}{2}(z^2 - z^{-2}). \tag{2.8}$$

Here we observe a formal duality symmetry of this phase space system and its integrable structure under $j_{(2)} \mapsto \mathrm{i}*j_{(2)}$ and $z \mapsto \mathrm{e}^{\pi/4\mathrm{i}}z$. To relate the coset fields, we may define a nonlocal map $g \mapsto \tilde{g}$ such that $(g^{-1}dg)_{(2)} = *(\tilde{g}^{-1}d\tilde{g})_{(2)}$.[10]

One may also consider here an analog of the non-Abelian duality transformation in the PCM that can be performed at the path integral level by adding the Maurer-Cartan equations with the Lagrange multiplier fields to the action and then solving for the current components (for an example in the S^2 case, see [29]).

In addition to this formal symmetry, there may be other "dualities," i.e. linear transformations of the current components that map this first-order system into itself and respect its integrable structure. The T-duality that we are going to discuss below in the special case of $\mathrm{AdS}_5 \cong SO(2,4)/SO(1,4)$ is one of them that has a remarkable property of being a "self-duality": it maps the system into an equivalent one with the *same* $SO(2,4)$ global symmetry.

III. REVIEW OF $\mathrm{AdS}_5 \times S^5$ SUPERSTRING SIGMA MODEL

We will begin this section with a summary of the supercoset formulation of the type IIB superstring action on $\mathrm{AdS}_5 \times S^5$. We will then move on to the discussion of some aspects of its classical integrability by reviewing the construction of flat currents. We will also explicitly construct the Noether currents for the supercoset model in the parametrization adapted to the standard basis of the

[10]One may consider similar formal Hodge duality transformation also in the superstring case [28].

superconformal group. These currents will later be our starting point in the construction of a family of gauge-invariant flat currents for the T-dual model.

A. Superstring action

As was shown in [3], the type IIB Green-Schwarz superstring action on $\mathrm{AdS}_5 \times S^5$ can be understood as a sigma model-type action on the coset superspace

$$G/H = PSU(2,2|4)/(SO(1,4) \times SO(5)) \tag{3.1}$$

with the bosonic part being

$$SO(2,4)/SO(1,4) \times SO(6)/SO(5) \cong \mathrm{AdS}_5 \times S^5. \tag{3.2}$$

The coset (3.1) admits a $\mathbb{Z}_4$-grading in the sense that the subgroup $H = SO(1,4) \times SO(5)$ of $G = PSU(2,2|4)$ arises as the fixed point set of an order 4 automorphism of G [30]. Concretely, this means that at the Lie algebra level $\mathfrak{g} := \mathrm{Lie}(G)$ we have $(m, n = 0, \ldots, 3)$

$$\mathfrak{g} \cong \bigoplus_{m=0}^{3} \mathfrak{g}_{(m)}, \quad \text{with} \quad \mathfrak{g}_{(0)} \equiv \mathfrak{h} := \mathrm{Lie}(H) \quad \text{and} \quad [\mathfrak{g}_{(m)}, \mathfrak{g}_{(n)}\} \subset \mathfrak{g}_{(m+n)}. \tag{3.3}$$

Here $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(2)}$ are generated by bosonic generators while $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(3)}$ by fermionic ones, respectively (for more details, see Sec. III D below).

To define the superstring action, we consider the map $g\colon \Sigma \to G$, where Σ is a world-sheet surface (with an arbitrary Lorentzian $2d$ metric) and introduce the current

$$j = g^{-1}dg = j_{(0)} + j_{(1)} + j_{(2)} + j_{(3)}, \quad \text{with} \quad j_{(0)} := A \in \mathfrak{h} \quad \text{and} \quad j_{(m)} \in \mathfrak{g}_{(m)}. \tag{3.4}$$

The dynamical $2d$ fields (string coordinates) will take values in the coset superspace $G/H := \{gH | g \in G\}$. The action that describes them should simultaneously be invariant under the global (left) G-transformations of the form

$$g \mapsto g_0 g \quad \text{for } g_0 \in G, \tag{3.5a}$$

and the local (right) H-transformations of the form

$$g \mapsto gh \quad \text{for } h \in H. \tag{3.5b}$$

By construction, the current j is invariant under (3.5a). Under (3.5b), the A part of j in (3.4) transforms as a connection, $A \mapsto h^{-1}Ah + h^{-1}dh$, while the $j_{(m)}$s with $m = 1, 2, 3$ transform covariantly, $j_{(m)} \mapsto h^{-1}j_{(m)}h$.

The superstring action can be written as a sum of kinetic and Wess-Zumino (WZ) terms [3,26,30],

$$S = -\frac{T}{2}\int_\Sigma \mathrm{str}[j_{(2)} \wedge *j_{(2)} + \kappa j_{(1)} \wedge j_{(3)}], \tag{3.6}$$

where $T = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension, $*$ is the Hodge star on Σ and str denotes the supertrace on $\mathfrak{g}$ compatible with the $\mathbb{Z}_4$-grading,

$$\mathrm{str}(V_m V_n) = 0, \qquad V_m \in \mathfrak{g}_{(m)}, \qquad m + n \neq 0 \,\mathrm{mod} 4. \tag{3.7}$$

The κ-symmetry condition requires that $\kappa = \pm 1$; in what follows we shall assume that (the opposite sign choice is related by parity transformation on Σ)

$$\kappa = 1. \tag{3.8}$$

Note that regardless of the requirement of κ-symmetry, the superstring action (3.6) is integrable [4] only for the same choice of $\kappa = \pm 1$. This is not totally surprising since (i) the bosonic coset model is classically integrable [1] and (ii) it is local κ-symmetry that relates bosons to fermions and thus extends this property to the fermionic GS generalization of the bosonic coset model.[11]

B. Equations of motion

Starting with the Maurer-Cartan equation for the current (3.4)

$$dj + j \wedge j = 0 \tag{3.9}$$

and splitting it according to the $\mathbb{Z}_4$-grading of the algebra gives [cf. (2.7)]

$$\begin{aligned} dA + A \wedge A + j_{(1)} \wedge j_{(3)} + j_{(2)} \wedge j_{(2)} + j_{(3)} \wedge j_{(1)} &= 0, \\ \nabla j_{(1)} + j_{(2)} \wedge j_{(3)} + j_{(3)} \wedge j_{(2)} &= 0, \\ \nabla j_{(2)} + j_{(1)} \wedge j_{(1)} + j_{(3)} \wedge j_{(3)} &= 0, \\ \nabla j_{(3)} + j_{(1)} \wedge j_{(2)} + j_{(2)} \wedge j_{(1)} &= 0. \end{aligned} \tag{3.10}$$

Here, for α being a Lie algebra valued p-form on Σ, we defined

$$\nabla\alpha := d\alpha + A \wedge \alpha - (-)^p \alpha \wedge A. \tag{3.11}$$

The variation of (3.6) over g together with (3.10) then yields the following field equations:

$$\begin{aligned} \nabla *j_{(2)} + j_{(3)} \wedge j_{(3)} - j_{(1)} \wedge j_{(1)} &= 0, \\ j_{(2)} \wedge (j_{(1)} + *j_{(1)}) + (j_{(1)} + *j_{(1)}) \wedge j_{(2)} &= 0, \\ j_{(2)} \wedge (j_{(3)} - *j_{(3)}) + (j_{(3)} - *j_{(3)}) \wedge j_{(2)} &= 0. \end{aligned} \tag{3.12}$$

Equations (3.10) and (3.12) constitute the full system of superstring equations in first-order form, i.e. the equations for the superalgebra valued one-form j. This system is invariant under the bosonic H-gauge transformations and the fermionic κ-gauge symmetry[12] (and also $2d$ reparametrizations). This invariance will be important to keep in mind when discussing the duality transformations later on.

[11] The same applies also to similar lower-dimensional GS models constructed in [31].

[12] Under κ-symmetry we have $\delta_\kappa j = d\epsilon + [j, \epsilon]$ where $\epsilon = \epsilon_+ + \epsilon_-$ is a certain combination of self-dual and anti-self-dual fermionic vector parameters with $j_{(2)}$ and also $\delta(\sqrt{-g}g^{ab}) \sim \epsilon_+ j_{(1)} + \epsilon_- j_{(3)}$ (for details see [3,32]).

Equations (3.12), understood as second-order equations on g, imply and also are implied by the conservation condition

$$d *J_N = 0 \tag{3.13}$$

for the Noether current J_N associated with the global G-symmetry (3.5a) of the action. As follows from the action (3.6), J_N is given by

$$J_N = g\left[j_{(2)} - \frac{1}{2} *(j_{(1)} - j_{(3)})\right]g^{-1}. \tag{3.14}$$

Note that, like the action itself, J_N is invariant under the H-gauge transformations (3.5b).

To study the "physical" string dynamics, one needs to take care of the gauge symmetries. The local H-symmetry (3.5b) can be fixed by making a particular choice of the coset representative (i.e. the explicit choice of g in terms of the independent string coordinates); one should also choose a κ-symmetry gauge. We will discuss some particular choices below. In general, one needs also to add the equations of motion for the $2d$ metric (i.e. the Virasoro constraints) and to fix a $2d$ reparametrization gauge but this will not be required for the aims of the present paper.[13]

C. One-parameter families of flat currents

As was shown in [4], the $\mathbb{Z}_4$-grading of the above G/H supercoset allows for the construction of one-parameter families of flat currents.[14] These (related) families of flat currents allow in turn for the construction of infinitely many nonlocal conserved charges *à la* Lüscher and Pohlmeyer [1].

Indeed, one may verify that the following combination of the components of the current in (3.4)

$$\begin{aligned} \mathrm{j}(z) &= A + zj_{(1)} + \frac{1}{2}(z^2 + z^{-2})j_{(2)} + z^{-1}j_{(3)} \\ &\quad - \frac{1}{2}(z^2 - z^{-2}) *j_{(2)}, \end{aligned} \tag{3.15}$$

where z is a complex spectral parameter [34] so that $\mathrm{j}(1) = j$, satisfies the flatness condition

$$d\mathrm{j}(z) + \mathrm{j}(z) \wedge \mathrm{j}(z) = 0. \tag{3.16}$$

And vice versa, imposing this flatness condition leads to the full system (3.10) and (3.12) of first-order equations for the current j.

Note that, like j itself, the family of currents $\mathrm{j}(z)$ is not invariant under the H-gauge transformations (3.5b), i.e. it depends on a particular choice of representative of the coset G/H. At the same time, starting with j one may also construct another family of flat currents

$$\begin{aligned} \mathrm{J}(z) &:= g[\mathrm{j}(z) - \mathrm{j}(1)]g^{-1} = g[\mathrm{j}(z) - j]g^{-1} \\ &= g\mathrm{j}(z)g^{-1} + gdg^{-1} \end{aligned} \tag{3.17}$$

that is invariant with respect to (3.5b).[15] That requires, however, the explicit use of g, related to j by $j = g^{-1}dg$, so that J itself is nonlocal once expressed in terms of j. Expanding $\mathrm{J}(z)$ in powers of $w := -2\log(z)$ around zero (i.e. around $z = \pm 1$), we get

$$\mathrm{J}(z) = \sum_{k=0}^{\infty} w^k c_k = wc_1 + \mathcal{O}(w^2), \tag{3.18}$$

where c_1 is, in fact, the Hodge dual of the Noether current (3.14),

$$c_1 = *J_N = g\left[*j_{(2)} - \frac{1}{2}(j_{(1)} - j_{(3)})\right]g^{-1}. \tag{3.19}$$

Hence, the flatness of $\mathrm{J}(z)$

$$d\mathrm{J}(z) + \mathrm{J}(z) \wedge \mathrm{J}(z) = 0 \tag{3.20}$$

implies the conservation law $d *J_N = 0$ and thus also the second-order equations of motion (3.12) for the superstring.

Since $\mathrm{J}(z)$ is flat, we may write

$$\begin{aligned} &\mathrm{J}(z) = W^{-1}(z)dW(z) \\ \Rightarrow \quad &W(z;\sigma,\tau;\sigma_0,\tau_0) = P\exp\left(\int_{\mathcal{C}} \mathrm{J}(z)\right), \end{aligned} \tag{3.21}$$

where $\mathcal{C}$ is a contour on the world sheet Σ running from some reference point (σ_0, τ_0) to (σ, τ) and P is the path-ordering symbol. Provided that appropriate boundary conditions at spatial infinity can be chosen, one can use the path-ordered exponential W to build an infinite number of conserved nonlocal charges [1][16] for the superstring.

Let us point out that the flatness conditions (3.16) and (3.20) are invariant under formal G-gauge transformations (with parameter g), so that, e.g., $\mathrm{J}(z)$ is unique up to

$$\mathrm{J}(z) \mapsto \mathrm{J}'(z) = g^{-1}\mathrm{J}(z)g + g^{-1}dg, \quad \text{for } g \in G. \tag{3.22}$$

The power series expansion of $\mathrm{J}'(z)$ in $w = -2\log(z)$ around zero is then[17]

$$\mathrm{J}'(z) = g^{-1}dg + wg^{-1} *J_N g + \mathcal{O}(w^2) \tag{3.23}$$

so that, to zeroth order in w, Eq. (3.20) is automatically satisfied while to first order we again find $d *J_N = 0$ or the equations of motion. Below we shall use this gauge freedom to achieve a particularly simple form of the currents suitable for expressing them in terms of the T-dual variables.

[13] For a discussion of integrability of the superstring model with the gauges fixed and the Virasoro constraints imposed, see [32].

[14] See [33] for the extension to $\mathbb{Z}_m$-graded coset (super)spaces.

[15] The bosonic part of the current $\mathrm{J}(z)$ is analogous to the Lax connection of the PCM given in Eq. (2.3) and will reduce to it in the limit $G/H \to G$ in which H becomes trivial.

[16] For more details, see, e.g., the review in [35].

[17] Here g is assumed not to depend on the spectral parameter; if it does, such a transformation may be interpreted as a "dressing" transformation.

D. Standard choice of the superconformal algebra basis

Let us now make a specific choice of the basis of generators of the superconformal algebra $\mathfrak{g} = \mathfrak{psu}(2,2|4)$ adapted to the Poincaré parametrization of AdS_5 and thus a natural one for comparison with boundary conformal gauge theory in $\mathbb{R}^{1,3}$ (see Appendix B for more details):

$$\mathfrak{psu}(2,2|4) = \mathrm{span}\{P_a, L_{ab}, K_a, D, R_i{}^j | Q^{i\alpha}, \bar{Q}_i^{\dot\alpha}, S_i^\alpha, \bar{S}^{i\dot\alpha}\}, \tag{3.24}$$

where $a, b = 0, \ldots, 3$, $\alpha, \beta = 1, 2$, $\dot\alpha, \dot\beta = \dot1, \dot2$, and $i, j = 1, \ldots, 4$. Here P represents translations, L Lorentz rotations, K special conformal transformations, D dilatations, and R the $SU(4)$-symmetry while Q and $\bar{Q}$ are the Poincaré supercharges and S and $\bar{S}$ their superconformal partners. We shall assume that the generators P, K, D, L are Hermitian while $R_i{}^j = -(R_j{}^i)^\dagger$, $Q^{i\alpha} = (\bar{Q}_i^{\dot\alpha})^\dagger$, and $S_i^\alpha = (\bar{S}^{i\dot\alpha})^\dagger$. Later on, we will also make use of the standard vector/bispinor index identification $\{a\} = \{\alpha\dot\beta\}$ as discussed in Appendix A.

In terms of these generators, the $\mathbb{Z}_4$-splitting (3.3) is then given as

$$\begin{aligned}
\mathfrak{h} &= \mathrm{span}\left\{\frac{1}{2}(P_a - K_a), L_{ab}, R_{(ij)}\right\},\\
\mathfrak{g}_{(1)} &= \mathrm{span}\left\{\frac{1}{2}(Q^{i\alpha} + C^{ij}S_j^\alpha), \frac{1}{2}(\bar{Q}_i^{\dot\alpha} + C_{ij}\bar{S}^{j\dot\alpha})\right\},\\
\mathfrak{g}_{(2)} &= \mathrm{span}\left\{\frac{1}{2}(P_a + K_a), D, R_{[ij]}\right\},\\
\mathfrak{g}_{(3)} &= \mathrm{span}\left\{\frac{-\mathrm{i}}{2}(Q^{i\alpha} - C^{ij}S_j^\alpha), \frac{\mathrm{i}}{2}(\bar{Q}_i^{\dot\alpha} - C_{ij}\bar{S}^{j\dot\alpha})\right\}.
\end{aligned} \tag{3.25}$$

Here, $R_{ij} := C_{ik}R_j{}^k$ and in $R_{(ij)}$ and $R_{[ij]}$ the parentheses mean normalized symmetrization and the square brackets mean normalized antisymmetrization. The constant matrix C_{ij} is an $Sp(4)$-metric and has the properties[18]

$$\begin{aligned}
&C_{ij} = -C_{ji} =: \frac{1}{2}\epsilon_{ijkl}C^{kl},\\
&C_{ij} = (C^{ij})^* \quad \text{and} \quad C_{ik}C^{jk} = \delta_i^j,
\end{aligned} \tag{3.26}$$

and it may be interpreted as a charge conjugation acting on $SU(4)$-spinors. We should stress that the particular choice of C_{ij} will not matter in the end, since physical quantities will not depend on it.

E. Poincaré parametrization of supercoset representative

Writing the current (3.4) in the basis (3.24), we get

$$\begin{aligned}
j = {} & j_{P_a}P_a + j_{L_{ab}}L_{ab} + j_{K_a}K_a + j_D D + j_{R_i{}^j}R_i{}^j\\
& + \mathrm{i}(j_{Q^{i\alpha}}Q^{i\alpha} - j_{\bar{Q}_i^{\dot\alpha}}\bar{Q}_i^{\dot\alpha} + j_{S_i^\alpha}S_i^\alpha - j_{\bar{S}^{i\dot\alpha}}\bar{S}^{i\dot\alpha}),
\end{aligned} \tag{3.27}$$

where the factor of i in front of the fermionic part was chosen to make j skew-Hermitian. Our aim now is to find the explicit form of these components in the parametrization of the supercoset corresponding to the Poincaré form of the $\mathrm{AdS}_5 \times S^5$ metric:

$$ds^2 = -\frac{1}{2}Y^2 dX_{\alpha\dot\beta}dX^{\dot\beta\alpha} + \frac{1}{4Y^2}dY_{ij}dY^{ij}. \tag{3.28}$$

Here, $(X, Y) = (X^{\dot\alpha\beta}, Y^{ij})$ represent the 10 independent bosonic coordinates (see also Appendix A)[19]

$$\begin{aligned}
&X_{\alpha\dot\beta} := \sigma^a_{\alpha\dot\beta}X_a = \epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta}X^{\dot\delta\gamma} = (X_{\dot\beta\alpha})^*,\\
&Y_{ij} = -Y_{ji} = \frac{1}{2}\epsilon_{ijkl}Y^{kl} = (Y^{ij})^*,\\
&Y^2 := \frac{1}{4}Y_{ij}Y^{ij}.
\end{aligned} \tag{3.29}$$

The coset representative $g \in G$ of $[g] \in G/H$ adapted to the metric (3.28) may be chosen as[20]

$$g(X, Y, \Theta) = B(X, Y)\mathrm{e}^{-F(\Theta)}, \tag{3.30a}$$

with

$$\begin{aligned}
B(X, Y) &= \mathrm{e}^{\mathrm{i}X^{\dot\alpha\beta}P_{\beta\dot\alpha}}\mathrm{e}^{\mathrm{i}\log(Y)D}\Lambda(Y) = \mathrm{e}^{\mathrm{i}X^{\dot\alpha\beta}P_{\beta\dot\alpha}}Y^{\mathrm{i}D}\Lambda(Y),\\
F(\Theta) &= \mathrm{i}[C_{ij}\epsilon_{\alpha\beta}(\theta_+^{i\alpha}Q^{j\beta} + \theta_-^{i\alpha}C^{jk}S_k^\beta)\\
&\quad - C^{ij}\epsilon_{\dot\alpha\dot\beta}(\bar\theta_{+i}^{\dot\alpha}\bar{Q}_j^{\dot\beta} + \bar\theta_{-i}^{\dot\alpha}C_{jk}\bar{S}^{k\dot\beta})],
\end{aligned} \tag{3.30b}$$

where[21]

$$\Lambda(Y) = (\Lambda^i{}_j) := \frac{1}{Y}(C^{ik}Y_{kj}). \tag{3.30c}$$

Here, $\Theta = (\theta_\pm^{i\alpha}, \bar\theta_{\pm i}^{\dot\alpha})$ represents the 32 independent fermionic coordinates satisfying the following reality condition:

$$\theta_\pm^{i\alpha} = (\bar\theta_{\pm i}^{\dot\alpha})^\dagger. \tag{3.31}$$

Then the current may be written as

$$\begin{aligned}
j = g^{-1}dg &= \mathrm{e}^F j_B \mathrm{e}^{-F} + \mathrm{e}^F d\mathrm{e}^{-F}\\
&= j_B(X, Y) + j_F(X, Y, \Theta),
\end{aligned} \tag{3.32a}$$

where

$$j_B := j(X, Y, \Theta = 0) = B^{-1}dB. \tag{3.32b}$$

A simple calculation shows that

$$j_B = \underbrace{\mathrm{i}YdX^{\dot\alpha\beta}}_{=:j_{B_{P_{\beta\dot\alpha}}}}P_{\beta\dot\alpha} + \underbrace{\frac{\mathrm{i}}{Y}dY}_{=:j_{B_D}}D + \underbrace{2\mathrm{i}(\Lambda^{-1})^i{}_k d\Lambda^k{}_j}_{=:j_{B_{R_i{}^j}}}R_i{}^j. \tag{3.33}$$

As reviewed in Appendix C, the fermionic part of the current can be expressed as [36]

[18] Here, $*$ denotes complex conjugation and ϵ_{ijkl} is totally antisymmetric with $\epsilon_{1234} = 1$.

[19] Here, $\epsilon_{12} = -\epsilon_{21} = \epsilon_{\dot1\dot2} = -\epsilon_{\dot2\dot1} = 1$.

[20] This corresponds to a particular gauge fixing of the local H-symmetry [cf. (3.25)]. A similar parametrization was used in [25].

[21] Notice that $\Lambda^{-1} = \Lambda^\dagger$ and $\det\Lambda = 1$ upon using Eqs. (3.29) and the reality condition.

$$j_F = -\frac{\sinh(\mathcal{M})}{\mathcal{M}}\nabla F - 2\left[F, \frac{\sinh^2(\mathcal{M}/2)}{\mathcal{M}^2}\nabla F\right], \quad (3.34)$$

where we have introduced the operators

$$\nabla\cdot := d\cdot + [j_B, \cdot] \quad \text{and} \quad \mathcal{M}^2\cdot := [F, [F, \cdot]]. \quad (3.35)$$

Note that in (3.34), the first term on the right-hand side is proportional to the fermionic generators of the superconformal algebra, while the second one is proportional to the bosonic ones.

F. κ-symmetry gauge fixing

Having fixed the local H-symmetry gauge in (3.30), let us now discuss a specific κ-symmetry gauge choice [24,25] that will simplify the structure of the string action and is natural in the present context. In the notation used in (3.30), this gauge amounts to setting

$$\theta^{i\alpha}_{-} = 0 = \bar\theta^{\dot\alpha}_{-i}, \quad (3.36)$$

so that the fermionic part e^{-F} of g determined by

$$F(\Theta) = \mathrm{i}[C_{ij}\epsilon_{\alpha\beta}\theta^{i\alpha}_{+}Q^{j\beta} - C^{ij}\epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}_{+i}\bar Q^{\dot\beta}_{j}], \quad (3.37)$$

does not contain terms with S generators (for that reason we shall follow [25] and refer to this gauge as "S-gauge").

One may then readily check that in this case $\mathcal{M}^2\nabla F = 0$. From Eq. (3.34), we deduce

$$\begin{aligned} j_F &= -\nabla F - \frac{1}{2}[F, \nabla F] \\ &= -\mathrm{i}C_{ij}\epsilon_{\alpha\beta}\nabla\theta^{i\alpha}_{+}Q^{j\beta} + \mathrm{i}C^{ij}\epsilon_{\dot\alpha\dot\beta}\nabla\bar\theta^{\dot\alpha}_{+i}\bar Q^{\dot\beta}_{j} \\ &\quad - \frac{1}{2}(\bar\theta^{\dot\alpha}_{+i}\nabla\theta^{i\beta}_{+} - \nabla\bar\theta^{\dot\alpha}_{+i}\theta^{i\beta}_{+})P_{\beta\dot\alpha}, \end{aligned} \quad (3.38a)$$

where

$$\begin{aligned} \nabla\theta^{i\alpha}_{+} &= d\theta^{i\alpha}_{+} + \omega^{i\alpha}{}_{j\beta}\theta^{j\beta}_{+}, \quad \text{with} \\ \omega^{i\alpha}{}_{j\beta} &:= \frac{\mathrm{i}}{2}\delta^{\alpha}{}_{\beta}(\delta^{i}{}_{j}j_{B_D} - C^{ki}C_{jl}j_{B_{R_l}{}^k}). \end{aligned} \quad (3.38b)$$

Note that $\nabla\theta^{i\alpha}_{+} = (\nabla\bar\theta^{\dot\alpha}_{+i})^{\dagger}$. Upon substituting the expressions of j_{B_D} and $j_{B_{R_i}{}^j}$ given in (3.33) into (3.38b), one may recast $\nabla\theta^{i\alpha}_{+}$ as

$$\nabla\theta^{i\alpha}_{+} = Y^{1/2}\Lambda^{i}{}_{k}d(Y^{-1/2}(\Lambda^{-1})^{k}{}_{j}\theta^{j\alpha}_{+}), \quad (3.39)$$

where $\Lambda^{i}{}_{j}$ was defined in (3.30c). This then suggests performing the following fermionic field redefinition:

$$\begin{aligned} &(\theta^{i\alpha}_{+}, \bar\theta^{\dot\alpha}_{+i}) \mapsto (\theta^{i\alpha}, \bar\theta^{\dot\alpha}_{i}), \\ &\theta^{i\alpha} := Y^{-1/2}(\Lambda^{-1})^{i}{}_{j}\theta^{j\alpha}_{+}, \quad \text{with} \quad \theta^{i\alpha} = (\bar\theta^{\dot\alpha}_{i})^{\dagger}. \end{aligned} \quad (3.40)$$

Then the current $j = j_B + j_F$ expressed in terms of the bosonic coordinates (X, Y) and the new 16 independent fermionic coordinates θ takes the following form

$$\begin{aligned} j &= \mathrm{i}Y\left[dX^{\dot\alpha\beta} + \frac{\mathrm{i}}{2}(\bar\theta^{\dot\alpha}_{i}d\theta^{i\beta} - d\bar\theta^{\dot\alpha}_{i}\theta^{i\beta})\right]P_{\beta\dot\alpha} + \frac{\mathrm{i}}{Y}dYD \\ &\quad + 2\mathrm{i}(\Lambda^{-1})^{i}{}_{k}d\Lambda^{k}{}_{j}R_{i}{}^{j} - \mathrm{i}C_{ij}\epsilon_{\alpha\beta}Y^{1/2}\Lambda^{i}{}_{k}d\theta^{k\alpha}Q^{j\beta} \\ &\quad + \mathrm{i}C^{ij}\epsilon_{\dot\alpha\dot\beta}Y^{1/2}(\Lambda^{-1})^{k}{}_{i}d\bar\theta^{\dot\alpha}_{k}\bar Q^{\dot\beta}_{j}. \end{aligned} \quad (3.41)$$

A particular choice of the coset representative and a particular κ-symmetry gauge fixing makes part of the global symmetries nonmanifest; the manifest symmetries left after our gauge choices are the Poincaré translations, Lorentz rotations, dilatations, $SU(4)$-rotations, and the Poincaré (or Q, $\bar Q$) supersymmetry.

In order to write down the action (3.6) in the S-gauge, we first need to extract the $j_{(m)}$ parts ($m = 1, 2, 3$) from the current (3.41) using Eqs. (3.25). We find (H.c. stands for Hermitian conjugation)

$$\begin{aligned} j_{(1)} &= -\frac{\mathrm{i}}{2}C_{ij}\epsilon_{\alpha\beta}Y^{1/2}\Lambda^{i}{}_{k}\theta^{k\alpha}(Q^{j\beta} + C^{jl}S^{\beta}_{l}) - \text{H.c.}, \\ j_{(2)} &= \frac{\mathrm{i}}{2}Y\Pi^{\dot\alpha\beta}(P_{\beta\dot\alpha} + K_{\beta\dot\alpha}) + \frac{\mathrm{i}}{Y}dYD \\ &\quad - 2\mathrm{i}C^{l[i}(\Lambda^{-1})^{j]}{}_{k}d\Lambda^{k}{}_{l}R_{[ij]}, \\ j_{(3)} &= -\frac{\mathrm{i}}{2}C_{ij}\epsilon_{\alpha\beta}Y^{1/2}\Lambda^{i}{}_{k}d\theta^{k\alpha}(Q^{j\beta} - C^{jl}S^{\beta}_{l}) - \text{H.c.}, \end{aligned} \quad (3.42)$$

where we defined

$$\Pi^{\dot\alpha\beta} := dX^{\dot\alpha\beta} + \frac{\mathrm{i}}{2}(\bar\theta^{\dot\alpha}_{i}d\theta^{i\beta} - d\bar\theta^{\dot\alpha}_{i}\theta^{i\beta}). \quad (3.43)$$

Upon inserting the above expressions into the action (3.6), we obtain [11]

$$\begin{aligned} S = -\frac{T}{2}\int_{\Sigma}\Big\{&-\frac{1}{2}Z^2\Pi_{\alpha\dot\beta}\wedge *\Pi^{\dot\beta\alpha} + \frac{1}{4Z^2}dZ_{ij}\wedge *dZ^{ij} \\ &+ \frac{1}{2}(\epsilon_{\alpha\beta}dZ_{ij}\wedge\theta^{i\alpha}d\theta^{j\beta} - \epsilon_{\dot\alpha\dot\beta}dZ^{ij}\wedge\bar\theta^{\dot\alpha}_{i}d\bar\theta^{\dot\beta}_{j})\Big\}. \end{aligned} \quad (3.44)$$

In deriving this form of the action, we have used the invariant form given in (B3) and performed the change of coordinates $Y_{ij} \mapsto Z_{ij}$, with

$$\begin{aligned} &Z_{ij} := YC_{kl}\Lambda^{k}{}_{i}\Lambda^{l}{}_{j}, \\ &Z_{ik}Z^{jk} = Y^2\delta_{i}{}^{j} \quad \text{and} \quad Z^2 = \frac{1}{4}Z_{ij}Z^{ij} = Y^2. \end{aligned} \quad (3.45)$$

As a result, the action does not depend on the choice of the constant matrix C_{ij}.

G. Gauge-fixed form of flat currents

Inserting the expressions (3.42) into (3.15), we immediately arrive at the κ-gauge-fixed version of the family of flat currents $\mathrm{j}(z)$. The construction of the other family of flat currents $\mathrm{J}(z)$ given in (3.17) requires more work. First, notice that we might rewrite (3.17) as

$$\mathrm{J}(z) = (z-1)J_{(1)} + \frac{1}{2}(z^2 + z^{-2} - 2)J_{(2)} + (z^{-1} - 1)J_{(3)} - \frac{1}{2}(z^2 - z^{-2}) * J_{(2)}, \tag{3.46a}$$

where we have defined

$$J_{(m)} := g j_{(m)} g^{-1}. \tag{3.46b}$$

Upon using successively the Baker-Campbell-Hausdorff formula, we arrive after some rather lengthy algebraic manipulations at

$$\begin{aligned} J_{(1)} = &-\frac{1}{2}(\bar\theta_i^{\dot\alpha} d\theta^{i\beta} - Z_{ij}X^{\dot\alpha\gamma} d\theta^i_\gamma \theta^{j\beta})P_{\beta\dot\alpha} + \frac{1}{3!}\epsilon_{\gamma\delta} d\theta^j_\alpha \bar\theta_{k\dot\beta}(\theta^{j\delta}\theta^{k(\alpha}P^{\dot\beta\gamma)} + \theta^{i(\alpha}\theta^{k\delta)}P^{\dot\beta\gamma}) + \frac{\mathrm{i}}{2} Z_{ij} d\theta^{i\alpha}\theta^{j\beta} L_{\alpha\beta} \\ &+ \frac{\mathrm{i}}{4} Z_{ij} d\theta^{i\alpha}\theta^j_\alpha D + \mathrm{i} Z_{ij} d\theta^{k\alpha}\theta^i_\alpha R_k{}^j + \frac{\mathrm{i}}{Z} Z_{ij} d\theta^i_\alpha Q^{j\alpha} - \frac{\mathrm{i}}{2Z^2} Z_{ij} Z_{kl}\epsilon_{\beta\gamma} d\theta^i_\alpha \left(\theta^{j(\alpha}\theta^{k\gamma)}Q^{l\beta} + \frac{1}{2}\theta^{j\gamma}\theta^{k\beta}Q^{l\alpha}\right) \\ &+ \frac{1}{2}\epsilon_{\dot\alpha\dot\beta} d\theta^i_\gamma \left(X^{\dot\alpha\gamma} - \frac{\mathrm{i}}{2}\bar\theta^{\dot\alpha}_j\theta^{j\gamma}\right)\bar Q_i^{\dot\beta} + \frac{\mathrm{i}}{2} Z d\theta^i_\alpha S_i^\alpha - \mathrm{H.c.}, \end{aligned} \tag{3.47a}$$

$$\begin{aligned} J_{(2)} = &\left\{\mathrm{i} Z\Pi^{\dot\delta\gamma} A^{\dot\alpha\beta}_{\gamma\dot\delta} + \frac{\mathrm{i}}{Z} dZ\left(X^{\dot\alpha\beta} + \frac{\mathrm{i}}{2}\bar\theta^{\dot\alpha}_i\theta^{i\beta}\right) - \frac{1}{2Z^2} Z^{ik} dZ_{kj}\bar\theta^{\dot\alpha}_i\theta^{j\beta}\right\}P_{\beta\dot\alpha} - \frac{\mathrm{i}}{2} Z^2 \Pi_{\alpha\dot\beta}\epsilon_{\gamma\delta}\left(X^{\dot\beta\delta} - \frac{\mathrm{i}}{2}\bar\theta^{\dot\beta}_i\theta^{i\delta}\right)L^{\alpha\gamma} \\ &- \frac{\mathrm{i}}{2} Z^2 \Pi_{\beta\dot\alpha}\epsilon_{\dot\gamma\dot\delta}\left(X^{\dot\delta\beta} + \frac{\mathrm{i}}{2}\bar\theta^{\dot\delta}_i\theta^{i\beta}\right)L^{\dot\alpha\dot\gamma} + \left(\frac{\mathrm{i}}{2} Z^2 \Pi_{\beta\dot\alpha}X^{\dot\alpha\beta} + \frac{\mathrm{i}}{Z} dZ\right)D + \frac{\mathrm{i}}{2} Z^2 \Pi^{\dot\alpha\beta} K_{\beta\dot\alpha} \\ &+ \left(\Pi_{\beta\dot\alpha} Z^{ik} Z_{jl}\bar\theta^{\dot\alpha}_k\theta^{l\beta} - \frac{\mathrm{i}}{Z^2} Z^{ik} dZ_{kj}\right)R_i{}^j + \mathrm{i}\left(Z\Pi_{\beta\dot\alpha}B^{\dot\alpha\beta}_{i\gamma} - \frac{1}{2Z} dZ_{ij}\theta^j_\gamma\right)Q^{i\gamma} \\ &- \mathrm{i}\left(Z\Pi_{\alpha\dot\beta}(B^{\dot\alpha\beta}_{i\gamma})^\dagger - \frac{1}{2Z} dZ^{ij}\bar\theta_{j\dot\gamma}\right)\bar Q_i^{\dot\gamma} - \frac{1}{2} Z Z^{ij}\Pi_{\alpha\dot\beta}\bar\theta^{\dot\beta}_j S_i^\alpha - \frac{1}{2} Z Z_{ij}\Pi_{\beta\dot\alpha}\theta^{j\beta}\bar S^{i\dot\alpha}, \end{aligned} \tag{3.47b}$$

$$J_{(3)} = \left\{\frac{\mathrm{i}}{Z} Z_{ij} d\theta^i_\alpha Q^{j\alpha} - \bar\theta^{\dot\alpha}_i d\theta^{i\beta} P_{\beta\dot\alpha} - \mathrm{H.c.}\right\} - J_{(1)}. \tag{3.47c}$$

Here, we have defined

$$\begin{aligned} A^{\dot\alpha\beta}_{\gamma\dot\delta} := &\frac{1}{2Z}(1 + X^2Z^2)\delta_{\dot\delta}{}^{\dot\alpha}\delta_\gamma{}^\beta + \frac{1}{2} Z X^{\dot\alpha\beta} X_{\gamma\dot\delta} \\ &- \frac{\mathrm{i}}{4} Z\Big(\epsilon^{\dot\alpha\dot\beta} X_{\alpha\dot\beta}\bar\theta_{i\dot\delta}\theta^{i(\alpha}\delta_\gamma{}^{\beta)} \\ &+ \frac{1}{3!Z^2} Z^{ij}\bar\theta_{i\dot\delta} B^{\dot\alpha\beta}_{j\gamma} - \mathrm{H.c.}\Big), \end{aligned} \tag{3.47d}$$

$$\begin{aligned} B^{\dot\alpha\beta}_{i\gamma} := -\frac{\mathrm{i}}{Z} Z_{ij}\Big(&\bar\theta^{\dot\alpha}_k\theta^k_\delta\theta^{j(\delta}\delta^{\beta)}_\gamma + 2\delta^\delta_\gamma\Big(\bar\theta^{\dot\alpha}_k\theta^k_\delta\theta^{j\beta} \\ &- \frac{1}{2}\theta^j_\delta\theta^{k\beta}\bar\theta^{\dot\alpha}_k\Big)\Big), \end{aligned} \tag{3.47e}$$

with $X^2 := -\frac{1}{2} X_{\alpha\dot\beta} X^{\dot\beta\alpha}$. Note that we have again performed the change of coordinates (3.45). Also note that in the final expression of the currents, the $Sp(4)$-metric C_{ij} does not appear as expected in view of $SU(4)$-invariance.

If we set $\theta^{i\alpha} = 0 = \bar\theta^{\dot\alpha}_i$, the fermionic parts $J_{(1)}$ and $J_{(3)}$ become identically zero, while the bosonic part $J_{(2)}$ reduces to

$$\begin{aligned} J_{(2)} = &\frac{\mathrm{i}}{2}\Big\{(1 + X^2Z^2) dX_{\alpha\dot\beta} + Z^2 X^{\dot\gamma\delta} dX_{\delta\dot\gamma} X_{\alpha\dot\beta} \\ &+ \frac{2}{Z} dZ X_{\alpha\dot\beta}\Big\}P^{\dot\beta\alpha} - \frac{\mathrm{i}}{2} Z^2 dX_{\alpha\dot\beta}\epsilon_{\gamma\delta} X^{\dot\beta\delta} L^{\alpha\gamma} \\ &- \frac{\mathrm{i}}{2} Z^2 dX_{\beta\dot\alpha}\epsilon_{\dot\gamma\dot\delta} X^{\dot\delta\beta} L^{\dot\alpha\dot\gamma} + \Big(\frac{\mathrm{i}}{2} Z^2 dX_{\alpha\dot\beta} X^{\dot\beta\alpha} \\ &+ \frac{\mathrm{i}}{Z} dZ\Big)D + \frac{\mathrm{i}}{2} Z^2 dX_{\alpha\dot\beta} K^{\dot\beta\alpha} - \frac{\mathrm{i}}{Z^2} Z^{ik} dZ_{kj} R_i{}^j. \end{aligned} \tag{3.48}$$

These are precisely the Noether currents for the bosonic sigma model on $\mathrm{AdS}_5 \times S^5$ in the metric (3.28), i.e. the Noether currents associated with the Killing vectors of (3.28) (see also [19]).

IV. DUALITY TRANSFORMATION ON AdS_5 COORDINATES (BOSONIC T-DUALITY)

Let us now turn to the discussion of the duality transformation of the superstring sigma model along the four isometry directions $X^{\dot\alpha\beta}$ of AdS_5 in the Poincaré coordinates following [11].[22] In particular, we will generalize the results of [19] and explain how to construct families of flat

[22]Note that we are performing the duality along the noncompact directions, i.e. as in [11], we are concerned here with a formal sigma model duality. We shall still refer to this as T-duality. Let us mention that T-duality transformations of type II superstrings were discussed also, e.g., in [37].

currents for the T-dual model, making its integrability manifest.

A. T-duality transformation of the superstring action

To implement the duality along X, let us start with the first-order form of the action (3.44) (see also [8,9])

$$S = -\frac{T}{2}\int_\Sigma \Big\{ -\frac{1}{2}Z^2\Big(\mathrm{V}^{\dot\alpha\beta} + \frac{\mathrm{i}}{2}(\bar\theta_i^{\dot\alpha} d\theta^{i\beta} - d\bar\theta_i^{\dot\alpha}\theta^{i\beta})\Big) \wedge *\Big(\mathrm{V}_{\beta\dot\alpha} + \frac{\mathrm{i}}{2}(\theta^i_\beta d\bar\theta_{i\dot\alpha} - d\theta^i_\beta \bar\theta_{i\dot\alpha})\Big) + \tilde X_{\alpha\dot\beta} d\mathrm{V}^{\dot\beta\alpha} + \frac{1}{4Z^2} dZ_{ij}\wedge * dZ^{ij} + \frac{1}{2}(dZ_{ij}\wedge \theta^{i\alpha} d\theta^j_\alpha - dZ^{ij}\wedge \bar\theta_i^{\dot\alpha} d\bar\theta_{j\dot\alpha})\Big\}, \tag{4.1}$$

where V is an auxiliary one-form field and the field $\tilde X^{\alpha\dot\beta}$ (which will become the T-dual coordinate) plays the role of a Lagrange multiplier imposing the flatness of V, i.e. $d\mathrm{V} = 0 \Rightarrow \mathrm{V} = dX$. On the other hand, solving for V first yields

$$\mathrm{V}^{\dot\alpha\beta} + \frac{\mathrm{i}}{2}(\bar\theta_i^{\dot\alpha} d\theta^{i\beta} - d\bar\theta_i^{\dot\alpha}\theta^{i\beta}) = Z^{-2} * d\tilde X^{\dot\alpha\beta}, \tag{4.2}$$

and thus the T-dual action written in terms of $\tilde X^{\dot\beta\alpha}$ becomes [11]

$$S = -\frac{T}{2}\int_\Sigma \Big\{ -\frac{1}{2Z^2} d\tilde X_{\alpha\dot\beta}\wedge * d\tilde X^{\dot\beta\alpha} + \frac{1}{4Z^2} dZ_{ij}\wedge * dZ^{ij} + \frac{\mathrm{i}}{2} d\tilde X_{\beta\dot\alpha}\wedge(\bar\theta_i^{\dot\alpha} d\theta^{i\beta} - d\bar\theta_i^{\dot\alpha}\theta^{i\beta}) + \frac{1}{2}(dZ_{ij}\wedge\theta^{i\alpha} d\theta^j_\alpha - dZ^{ij}\wedge\bar\theta_i^{\dot\alpha} d\bar\theta_{j\dot\alpha})\Big\}. \tag{4.3}$$

One observes that the bosonic geometry is again $\mathrm{AdS}_5 \times S^5$ (to put the bosonic action into the exactly same form one needs to change coordinates Z_{ij} so that $Z \mapsto Z^{-1}$). Also, the dual action is quadratic in the fermions. Moreover, the fermionic part of the action is of WZ type and therefore does not depend on the world-sheet metric.[23]

Let us remark that the on-shell relation between the original and dual coordinates is

$$dX^{\dot\alpha\beta} + \frac{\mathrm{i}}{2}(\bar\theta_i^{\dot\alpha} d\theta^{i\beta} - d\bar\theta_i^{\dot\alpha}\theta^{i\beta}) = Z^{-2} * d\tilde X^{\dot\alpha\beta}. \tag{4.4}$$

[23]Note that when integrating out $\mathrm{V}_{\alpha\dot\beta}$ in the path integral, one picks up a factor $\Delta_B^{-1/2}$ involving the functional determinant $\Delta_B = \Pi_{\sigma\in\Sigma} Z^{16}(\sigma)$ (in units where $T = -2$) which needs to be regularized. Using heath kernel methods, this amounts to adding the term $-8\int_\Sigma \mathrm{dvol} R^{(2)} \log(Z)$ to the action (4.3) (cf. [8,38]), $R^{(2)}$ being the scalar curvature of Σ.

B. Flat currents for the T-dual model

In general, if the original model is classically integrable, the same applies to its dual counterpart: the flatness of the Lax connection gives first-order equations that "interpolate" between the original and dual model. Still, it is useful to find the explicit expression for the flat currents in terms of the dual coordinates as this may also help clarify the transformation of the conserved charges under the T-duality.

For our choice of the Poincaré coordinates and the κ-symmetry gauge, the current j depends on the original coordinate $X^{\dot\alpha\beta}$ only through its differential $dX^{\dot\alpha\beta}$ [see Eq. (3.41)]. The same then applies to the family of flat currents $\mathrm{j}(z)$ in (3.15), where the expressions for $j_{(1)}$, $j_{(2)}$, and $j_{(3)}$ are given in (3.42) and

$$A = j_{(0)} = \frac{\mathrm{i}}{2} Y \Pi^{\dot\alpha\beta}(P_{\beta\dot\alpha} - K_{\beta\dot\alpha}) - 2\mathrm{i} C^{l(i}(\Lambda^{-1})^{j)}{}_k d\Lambda^k{}_l R_{(ij)}. \tag{4.5}$$

Then it is straightforward to reexpress $\mathrm{j}(z)$ in terms of $\tilde X$ by using (4.4), i.e. by replacing $\Pi_{\alpha\dot\beta}$ with $Z^{-2} * d\tilde X_{\alpha\dot\beta}$. The resulting family of currents $\tilde{\mathrm{j}} := \mathrm{j}(X \mapsto \tilde X)$ is still flat since (4.4) holds on-shell. And vice versa, the flatness of $\tilde{\mathrm{j}}$ will imply the field equations of the T-dual model.

As already discussed above, the family $\mathrm{j}(z)$ is not H-gauge invariant, i.e. it depends on a choice of representative of G/H. In order to be able to discuss the physical conserved charges, it is therefore useful to repeat the same procedure of replacing X by $\tilde X$ for the other family of flat currents $\mathrm{J}(z)$ in (3.17) closely related to Noether charge. However, unlike $\mathrm{j}(z)$, the current $\mathrm{J}(z)$ which involves g explicitly depends on X and thus, if dualized directly, would nonlocally depend on $\tilde X$. One can bypass this problem and get a local expression for $\mathrm{J}(z)$ in terms of the dual coordinate $\tilde X$ by first performing a G-gauge transformation (3.22) [which preserves the flatness condition (3.20)] with the following parameter g:

$$g = \mathrm{e}^{\mathrm{i}X^{\dot\alpha\beta}P_{\beta\dot\alpha}}. \tag{4.6}$$

Then the gauge transformed current is

$$\mathrm{J}'(z) = (z-1)J'_{(1)} + \frac{1}{2}(z^2 + z^{-2} - 2)J'_{(2)} + (z^{-1} - 1)J'_{(3)} - \frac{1}{2}(z^2 - z^{-2}) * J'_{(2)} + \underbrace{\mathrm{i} dX^{\dot\alpha\beta} P_{\beta\dot\alpha}}_{=g^{-1}g}, \tag{4.7}$$

with

$$J'_{(m)} := g^{-1} J_{(m)} g = g' j_{(m)} g'^{-1}, \quad \text{with} \quad g' = g^{-1} g, \tag{4.8}$$

where g is given by (3.30) in the S-gauge (3.36).[24] Then

[24]Note that $\mathrm{J}'(z)$ is invariant under the H-gauge transformations (3.5b): under such transformations $g \mapsto g$ and $g' \mapsto g'h$ and thus $g' j_{(m)} g'^{-1} \mapsto g' h (h^{-1} j_{(m)} h) h^{-1} g'^{-1} = g' j_{(m)} g'^{-1}$ and $g^{-1} dg \mapsto g^{-1} dg$.

$g' = g(X = 0, Y, \theta_+, \theta_- = 0)$ and thus [cf. Eqs. (3.47)]

$$\begin{aligned} J'_{(1)} &= -\frac{1}{2}\bar{\theta}_i^{\dot\alpha} d\theta^{i\beta} P_{\beta\dot\alpha} + \frac{1}{3!}\epsilon_{\gamma\delta} d\theta^j_\alpha \bar\theta_{k\dot\beta}(\theta^{j\delta}\theta^{k(\alpha}P^{\dot\beta\gamma)} + \theta^{i(\alpha}\theta^{k\delta)}P^{\dot\beta\gamma}) + \frac{\mathrm{i}}{2} Z_{ij} d\theta^{i\alpha}\theta^{j\beta} L_{\alpha\beta} + \frac{\mathrm{i}}{4} Z_{ij} d\theta^{i\alpha}\theta^j_\alpha D \\ &\quad + \mathrm{i} Z_{ij} d\theta^{k\alpha}\theta^i_\alpha R_k{}^j + \frac{\mathrm{i}}{Z} Z_{ij} d\theta^i_\alpha Q^{j\alpha} - \frac{\mathrm{i}}{2Z^2} Z_{ij} Z_{kl}\epsilon_{\beta\gamma} d\theta^i_\alpha\left(\theta^{j(\alpha}\theta^{k\gamma)}Q^{l\beta} + \frac{1}{2}\theta^{j\gamma}\theta^{k\beta}Q^{l\alpha}\right) - \frac{\mathrm{i}}{4}\epsilon_{\dot\alpha\dot\beta} d\theta^i_\gamma \bar\theta^{\dot\alpha}_j \theta^{j\gamma}\bar Q_i^{\dot\beta} \\ &\quad + \frac{\mathrm{i}}{2} Z d\theta^i_\alpha S_i^\alpha - \text{H.c.}, \end{aligned} \tag{4.9a}$$

$$\begin{aligned} J'_{(2)} &= \left\{\mathrm{i} Z\Pi^{\dot\delta\gamma} A^{\dot\alpha\beta}_{\gamma\dot\delta} - \frac{1}{2Z} dZ\bar\theta_i^{\dot\alpha}\theta^{i\beta} - \frac{1}{2Z^2} Z^{ik} dZ_{kj}\bar\theta_i^{\dot\alpha}\theta^{j\beta}\right\} P_{\beta\dot\alpha} - \frac{1}{4} Z^2 \Pi_{\alpha\dot\beta}\epsilon_{\gamma\delta}\bar\theta_i^{\dot\beta}\theta^{i\delta} L^{\alpha\gamma} + \frac{1}{4} Z^2 \Pi_{\beta\dot\alpha}\epsilon_{\dot\gamma\dot\delta}\bar\theta_i^{\dot\delta}\theta^{i\beta} L^{\dot\alpha\dot\gamma} \\ &\quad + \frac{\mathrm{i}}{Z} dZD + \frac{\mathrm{i}}{2} Z^2 \Pi^{\dot\alpha\beta} K_{\beta\dot\alpha} + \left(\Pi_{\beta\dot\alpha} Z^{ik} Z_{jl}\bar\theta_k^{\dot\alpha}\theta^{l\beta} - \frac{\mathrm{i}}{Z^2} Z^{ik} dZ_{kj}\right) R_i{}^j + \mathrm{i}\left(Z\Pi_{\beta\dot\alpha} B^{\dot\alpha\beta}_{i\gamma} - \frac{1}{2Z} dZ_{ij}\theta^j_\gamma\right) Q^{i\gamma} \\ &\quad - \mathrm{i}\left(Z\Pi_{\alpha\dot\beta}(B^{\dot\alpha\beta}_{i\gamma})^\dagger - \frac{1}{2Z} dZ^{ij}\bar\theta_{j\dot\gamma}\right)\bar Q_i^{\dot\gamma} - \frac{1}{2} Z Z^{ij}\Pi_{\alpha\dot\beta}\bar\theta_j^{\dot\beta} S_i^\alpha - \frac{1}{2} Z Z_{ij}\Pi_{\beta\dot\alpha}\theta^{j\beta}\bar S^{i\dot\alpha}, \end{aligned} \tag{4.9b}$$

$$J'_{(3)} = \left\{\frac{\mathrm{i}}{Z} Z_{ij} d\theta^i_\alpha Q^{j\alpha} - \bar\theta_i^{\dot\alpha} d\theta^{i\beta} P_{\beta\dot\alpha} - \text{H.c.}\right\} - J'_{(1)}, \tag{4.9c}$$

where here

$$A^{\dot\alpha\beta}_{\gamma\dot\delta} := \frac{1}{2Z}\delta_{\dot\delta}{}^{\dot\alpha}\delta_\gamma{}^\beta - \frac{\mathrm{i}}{4!Z}(Z^{ij}\bar\theta_{i\dot\delta} B^{\dot\alpha\beta}_{j\gamma} - \text{H.c.}), \tag{4.9d}$$

$$B^{\dot\alpha\beta}_{i\gamma} := -\frac{\mathrm{i}}{Z} Z_{ij}\left(\bar\theta^{\dot\alpha}_k\theta^k_\delta\theta^{j(\delta}\delta_\gamma^{\beta)} + 2\delta^\delta_\gamma\left(\bar\theta^{\dot\alpha}_k\theta^k_\delta\theta^{j\beta} - \frac{1}{2}\theta^j_\delta\theta^{k\beta}\bar\theta^{\dot\alpha}_k\right)\right). \tag{4.9e}$$

Note that the bosonic truncation of the gauge transformed current $\mathrm{J}'(z)$ is given by

$$\begin{aligned} \mathrm{J}'(z) &= \frac{1}{2}(z^2 + z^{-2} - 2) J'_{(2)} + \frac{1}{2}(z^2 - z^{-2}) * J'_{(2)} \\ &\quad + \mathrm{i} dX^{\dot\alpha\beta} P_{\beta\dot\alpha}, \\ J'_{(2)} &= \frac{\mathrm{i}}{2}(1 + Z^2) dX_{\alpha\dot\beta} P_+^{\dot\beta\alpha} + \frac{\mathrm{i}}{2}(1 - Z^2) dX_{\alpha\dot\beta} P_-^{\dot\beta\alpha} \\ &\quad + \frac{\mathrm{i}}{Z} dZD - \frac{\mathrm{i}}{Z^2} Z^{ik} dZ_{kj} R_i{}^j, \end{aligned} \tag{4.10}$$

where $P_\pm^{\dot\alpha\beta} := \frac{1}{2}(P^{\dot\alpha\beta} \pm K^{\dot\alpha\beta})$. Up to a rotation by a constant matrix, these are the same gauge transformed currents as found directly in Ref. [19] without referring to the coset nature of the AdS space.

Finally, using the duality relation in Eq. (4.4), we find the expressions for the currents in terms of the T-dual coordinates:

$$\tilde{\mathrm{J}}(z; \tilde X, Z, \Theta) := \mathrm{J}'(z; X(\tilde X), Z, \Theta), \tag{4.11}$$

i.e.

$$\begin{aligned} \tilde{\mathrm{J}}(z) &= (z - 1)\tilde J_{(1)} + \frac{1}{2}(z^2 + z^{-2} - 2)\tilde J_{(2)} \\ &\quad + (z^{-1} - 1)\tilde J_{(3)} - \frac{1}{2}(z^2 - z^{-2}) * \tilde J_{(2)} \\ &\quad + \frac{\mathrm{i}}{Z^2}\left[* d\tilde X^{\dot\alpha\beta} - \frac{\mathrm{i}}{2} Z^2(\bar\theta_i^{\dot\alpha} d\theta^{i\beta} - d\bar\theta_i^{\dot\alpha}\theta^{i\beta})\right] P_{\beta\dot\alpha}, \end{aligned} \tag{4.12a}$$

with

$$\tilde J_{(m)} := J'_{(m)}(\Pi_{\alpha\dot\beta} \mapsto Z^{-2} * d\tilde X_{\alpha\dot\beta}). \tag{4.12b}$$

Note that $\tilde{\mathrm{J}}(z)$ is flat since the duality relation (4.4) holds on-shell. Having expressed the Lax connection in terms of the dual coordinates, one can in principle derive an infinite set of nonlocal charges in the T-dual model by using (3.21).

C. Bosonic duality as a symmetry of first-order system and Lax connection

With a motivation to eventually shed some light on how conserved charges of the original and T-dual models are related, let us go back to the purely bosonic sigma model on $\mathrm{AdS}_5 \cong SO(2,4)/SO(1,4)$.[25] We shall ignore the decoupled 5-sphere part here. As was shown in [19], the T-duality applied to the bosonic AdS_n model generically maps conserved local charges into nonlocal ones and vice versa. To make this more precise it is desirable to describe T-duality as a formal algebraic transformation on the phase space, i.e. on the components of the current subject to first-order equations.

Let us begin by recalling the $\mathbb{Z}_2$-automorphism of the conformal $\mathfrak{g} = \mathfrak{so}(2,4)$ algebra (cf. Appendix B)

[25] The subsequent discussions can of course be applied to the AdS_n case.

$$\Omega(P_{\alpha\dot\beta}) = -K_{\alpha\dot\beta}, \qquad \Omega(K_{\alpha\dot\beta}) = -P_{\alpha\dot\beta},$$
$$\Omega(L_{\alpha\beta}) = L_{\alpha\beta} \quad \text{and} \quad \Omega(D) = -D. \tag{4.13}$$

Then we may define the projectors

$$\mathcal{P}_{(0)} := \frac{1}{2}(1+\Omega) \quad \text{and} \quad \mathcal{P}_{(2)} := \frac{1}{2}(1-\Omega), \tag{4.14}$$

so that

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}_{(2)}, \quad \text{with} \quad \mathfrak{h} = \mathcal{P}_{(0)}(\mathfrak{g}) \quad \text{and} \quad \mathfrak{g}_{(2)} = \mathcal{P}_{(2)}(\mathfrak{g}). \tag{4.15}$$

Correspondingly, the current $j = g^{-1}dg$, for $g \in SO(2,4)$, decomposes as

$$j = A + j_{(2)}, \quad \text{with} \quad \mathcal{P}_{(0)}(j) = A \quad \text{and} \quad \mathcal{P}_{(2)}(j) = j_{(2)}. \tag{4.16}$$

The Maurer-Cartan equations and the equations of motion are the same as in (2.7) or found by setting $j_{(1)} = 0 = j_{(3)}$ in (3.10) and (3.12)

$$dA + A\wedge A + j_{(2)}\wedge j_{(2)} = 0,$$
$$\nabla j_{(2)} = 0 \quad \text{and} \quad \nabla *j_{(2)} = 0. \tag{4.17}$$

For the choice of the AdS_5 part of the coset representative in (3.30), i.e. $g = e^{iX^{\dot\alpha\beta}P_{\beta\dot\alpha}}Y^{iD}$, we have

$$j = j_P + j_D, \quad \text{with} \quad j_P = iYdX^{\dot\alpha\beta}P_{\beta\dot\alpha} \quad \text{and} \quad j_D = \frac{i}{Y}dYD. \tag{4.18}$$

In this parametrization, Eqs. (4.17) read explicitly as

$$\begin{aligned} dj_P + j_D\wedge j_P + j_P\wedge j_D &= 0,\\ dj_D &= 0,\\ d*j_P - j_D\wedge *j_P - *j_P\wedge j_D &= 0,\\ d*j_D - \frac{1}{2}j_P\wedge *\Omega(j_P) - \frac{1}{2}*\Omega(j_P)\wedge j_P &= 0. \end{aligned} \tag{4.19}$$

Here, $\Omega(j_P) = \Omega(j_{P^{\dot\alpha\beta}}P^{\dot\alpha\beta}) = j_{P^{\dot\alpha\beta}}\Omega(P^{\dot\alpha\beta}) = -j_{P^{\dot\alpha\beta}}K^{\dot\alpha\beta}$. The T-duality along the four isometry directions $X^{\dot\alpha\beta}$ corresponds to replacing $(X^{\dot\alpha\beta}, Y)$ by the dual fields $(\tilde X^{\dot\alpha\beta}, \tilde Y)$ according to[26]

$$d\tilde X^{\dot\alpha\beta} = Y^2 * dX^{\dot\alpha\beta} \quad \text{and} \quad \tilde Y = Y^{-1}. \tag{4.20}$$

Therefore, the components of the dual current $\tilde j = \tilde j_P + \tilde j_D$ are defined in terms of $(\tilde X^{\dot\alpha\beta}, \tilde Y)$ in exactly the same way that j in (3.32a) is defined in terms of $(X^{\dot\alpha\beta}, Y)$. It can then be expressed in terms of the original coordinates as follows:

$$j_P = iYdX^{\dot\alpha\beta}P_{\beta\dot\alpha} = i\tilde Y *d\tilde X^{\dot\alpha\beta}P_{\beta\dot\alpha} = *\tilde j_P,$$
$$j_D = \frac{i}{Y}dYD = -\frac{i}{\tilde Y}d\tilde Y D = -\tilde j_D. \tag{4.21}$$

The key point is that under this transformation, i.e.

$$j_P \mapsto \tilde j_P = *j_P \quad \text{and} \quad j_D \mapsto \tilde j_D = -j_D, \tag{4.22}$$

the set of first-order equations (4.19) is invariant; in particular, the Maurer-Cartan equation for j_P is interchanged with its equation of motion. Thus, we may forget about particular solutions for j in terms of X or $\tilde X$ and view the duality as a symmetry of the phase space equations (4.19).

The family of flat currents (3.15) here takes the form

$$\begin{aligned} \mathrm{j}(z) = {}& \frac{1}{4}(z+z^{-1})^2 j_P - \frac{1}{4}(z-z^{-1})^2\Omega(j_P)\\ & - \frac{1}{4}(z^2-z^{-2})*(j_P - \Omega(j_P)) + \frac{1}{2}(z^2+z^{-2})j_D\\ & - \frac{1}{2}(z^2-z^{-2})*j_D \end{aligned} \tag{4.23}$$

and its flatness condition implies the set of Eqs. (4.19). Given the fact that after the T-duality (combined with $Y \mapsto \tilde Y = Y^{-1}$) we obtain the very same AdS_5 sigma model, the corresponding expression for $\mathrm{j}(z)$ in the T-dual model should be the same as (4.23) with $(X^{\dot\alpha\beta}, Y) \mapsto (\tilde X^{\dot\alpha\beta}, \tilde Y)$. However, by applying the current duality transformation (4.22) to $\mathrm{j}(z)$, we find

$$\begin{aligned} \tilde{\mathrm{j}}(z) = {}& \frac{1}{4}(z+z^{-1})^2 *j_P - \frac{1}{4}(z-z^{-1})^2 *\Omega(j_P)\\ & - \frac{1}{4}(z^2-z^{-2})(j_P - \Omega(j_P)) - \frac{1}{2}(z^2+z^{-2})j_D\\ & + \frac{1}{2}(z^2-z^{-2})*j_D, \end{aligned} \tag{4.24}$$

which does not seem to be the same as (4.23) despite the fact that Eqs. (4.19) are invariant under (4.22). Superficially, that may seem to imply that there are two independent Lax connections with inequivalent monodromy matrices, Yangians, etc.

This, of course, is not the case: the Lax connections (4.23) and (4.24) are actually related by a (spectral parameter dependent) $\mathbb{Z}_2$-automorphism of the Lie algebra $\mathfrak{g}$ defined as follows ($T \in \mathfrak{g}$):

$$T \mapsto \mathcal{U}_z(T) := U_z\Omega(T)U_z^{-1}, \quad \text{with} \quad U_z := \left(\frac{z-z^{-1}}{z+z^{-1}}\right)^{iD}. \tag{4.25}$$

This implies the following action on the components of the current:

[26] Here, the field Y appears instead of Z when compared with (4.4) since we dropped the S^5 part and so the field redefinition (3.45) is not needed.

$$\mathcal{U}_z(j_P) = \frac{z - z^{-1}}{z + z^{-1}}\Omega(j_P),$$
$$\mathcal{U}_z(\Omega(j_P)) = \frac{z + z^{-1}}{z - z^{-1}} j_P \quad \text{and}$$
$$\mathcal{U}_z(j_D) = -j_D, \tag{4.26}$$

and it is easy to verify that this automorphism maps the two Lax connections into each other

$$\mathcal{U}_z(\mathrm{j}(z)) = \tilde{\mathrm{j}}(z). \tag{4.27}$$

Thus the T-duality for the bosonic sigma model can be abstractly understood as a symmetry of the Lax connection (integrable structure) induced by the automorphism of the conformal algebra $\mathfrak{so}(2,4)$. This symmetry then implies a certain map of conserved charges. We shall make few comments on conserved charges at the end of Sec. V C and in Appendix D. The present formulation makes the analysis done in [19] more transparent.

Finding an analogous automorphism once the fermions are included may not seem straightforward at first glance. One reason is that a particular κ-symmetry gauge choice makes some of the superisometries nonmanifest. For example, in the T-dual action (4.3) the original supersymmetry transformations reduced to fermionic shifts of $\theta^{i\alpha}$ and $\bar{\theta}_i^{\dot{\alpha}}$ (the T-dual bosonic coordinates $\tilde{X}^{\dot{\alpha}\beta}$, being related to supersymmetric invariants, were not transforming). Furthermore, the above construction of the automorphism (4.25) relied on the fact that after the T-duality we obtain the very same sigma model action.

In the next section, we will extend the above considerations by combining the bosonic duality transformation with a certain fermionic one [22]. This appears to require one to supplement the transformation (4.22) by a certain transformation (not involving the Hodge star) of the fermionic components of the current that should produce a symmetry of the full first-order system (3.10) and (3.12) written in the H-symmetry gauge (3.30).

It also appears necessary to consider a different real form of the complexified superconformal algebra. To repeat the above argument about the invariance of the Lax connection under the duality, we will then construct an extension of the $\mathbb{Z}_2$-automorphism (4.25) to a $\mathbb{Z}_4$-automorphism of the full superconformal algebra.

V. FERMIONIC DUALITY TRANSFORMATION AND SELF-DUALITY OF THE SUPERSTRING

The action (4.3) obtained from the gauge-fixed $\mathrm{AdS}_5 \times S^5$ superstring action (3.44) by the duality transformation applied to the four bosonic coordinates X has manifest conformal symmetry but not the full superconformal symmetry. Part of the supersymmetry became nonmanifest due to the κ-symmetry gauge choice[27] but part was made nonlocal (or trivial) as a result of the duality transformation. Since the duality is an equivalence transformation at the full $2d$ field theory level, the original global symmetry and the associated conserved charges should not actually disappear but they may become effectively nonlocal and thus hidden (and indeed not visible in the point-particle limit of the action).

One may ask if one may to recover the original global symmetry in a manifest way, i.e. also at the point-particle level, by combining the bosonic duality transformation with a similar one applied to fermions. This is indeed possible following the suggestion of [22]. As we will show below, starting with the action (4.3) obtained by the bosonic duality and applying a duality transformation to the fermionic coordinates $\theta^{i\alpha}$ (but not to their conjugates $\bar{\theta}_i^{\dot{\alpha}}$), one finds the action that can be interpreted as the original $\mathrm{AdS}_5 \times S^5$ superstring action written in a different κ-symmetry gauge. That means that the combination of the bosonic and the fermionic world-sheet duality transformations maps not only the bosonic $\mathrm{AdS}_5 \times S^5$ part, but the full superstring action into an equivalent action. As a result, we find the full global superconformal group now acting (modulo a compensating κ-symmetry transformation) on coordinates of the dual action.

The fact that the fermionic duality is performed along the complex (chiral) fermionic coordinates implies that the resulting action is not Hermitian. Indeed, to interpret it as a κ-symmetry gauge-fixed version of the $\mathrm{AdS}_5 \times S^5$ superstring action, we will need to formally complexify the action and choose a special κ-symmetry (previously considered in [26]).

We shall start with a discussion of the superstring action in this complex gauge and then show that this action becomes equivalent to the action (4.3) in the S-gauge upon application of a fermionic duality transformation. This combined action of bosonic and fermionic dualities thus maps the $\mathrm{AdS}_5 \times S^5$ sigma model into itself.[28]

We shall then explain the reason for the fermionic duality transformation by arguing that its combined action with the bosonic duality is eventually a symmetry of the first-order system and of the Lax connection of the superstring model (generalizing a similar symmetry of the bosonic model discussed in Sec. IV C).

[27] To recover it one needs to combine the symmetry transformation with a compensating κ-symmetry transformation as in, e.g., the usual light-cone gauge in flat space.

[28] This means, in particular, that this duality transformation induces a map on the space of solutions of the classical sigma model equations of motion. More precisely, we may interpret this T-duality as a dressing transformation acting on the space of solutions, like a Bäcklund transformation (see, e.g., [39]).

A. Superstring action in a complex κ-symmetry gauge

Let us go back to our choice of the coset representative (3.30). Instead of choosing the real S-gauge (3.36) where $\theta^{i\alpha}_{-}$ and its conjugate $\bar\theta^{\dot\alpha}_{-i}$ are set to zero, we may also consider the following gauge:

$$\theta^{i\alpha}_{-} = 0 = \bar\theta^{\dot\alpha}_{+i}. \tag{5.1}$$

More precisely, to be able to choose such a gauge requires a complexification of the $AdS_5 \times S^5$ action, i.e. a relaxation of the reality condition (3.31). A similar gauge appeared earlier in [26] where the authors considered the superstring action for a different (Wick rotation related) slice of the complexified version of the $AdS_5 \times S^5$ coset superspace (3.1). A need for such complexification or analytic continuation seems intimately related to the notion of dual superconformal symmetry (cf. [21]).[29]

In this gauge, the fermionic part of the coset representative $g = B(X, Y)e^{-F(\Theta)}$ in (3.30) becomes [cf. (3.37)]

$$F(\Theta) = i[C_{ij}\epsilon_{\alpha\beta}\theta^{i\alpha}_{+}Q^{j\beta} + \epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}_{-i}\bar S^{i\dot\beta})], \tag{5.2}$$

so that we get a mixture of Q and $\bar S$ generators while the $\bar Q$ and S parts are gauged away (we shall thus refer to this gauge as $\bar{Q}$S-gauge).

An interesting feature of this gauge (observed in [26]) is that here the superstring action becomes quadratic in the fermions even before T-duality in X as in [11].[30] Indeed, one may easily verify that here the current j in (3.32a) becomes simply [cf. (3.34)]

$$j = j_B - \nabla F. \tag{5.3}$$

Going through the same steps as in Sec. III F, we then find

$$\begin{aligned} j = {} & iY dX^{\dot\alpha\beta}P_{\beta\dot\alpha} + \frac{i}{Y}dYD + 2i(\Lambda^{-1})^i{}_k d\Lambda^k{}_j R_i{}^j \\ & - iY^{1/2}\Lambda^j{}_i(d\zeta_{j\alpha} + i dX_{\alpha\dot\beta}\bar\vartheta^{\dot\beta}_j)Q^{i\alpha} \\ & + i\epsilon_{\dot\alpha\dot\beta}Y^{-1/2}\Lambda^j{}_i d\bar\vartheta^{\dot\beta}_j\bar S^{i\dot\alpha}, \end{aligned} \tag{5.4a}$$

where we have defined

$$\begin{aligned} \zeta^\alpha_i &:= Y^{-1/2}C_{ik}(\Lambda^{-1})^k{}_j\theta^{j\alpha}_{+} \quad \text{and} \\ \bar\vartheta^{\dot\alpha}_i &:= Y^{1/2}(\Lambda^{-1})^j{}_i\bar\theta^{\dot\alpha}_{-j}. \end{aligned} \tag{5.4b}$$

Extracting the $j_{(m)}$ parts from the above current, the superstring action (3.6) is then found to take the following explicit form:

$$\begin{aligned} S = -\frac{T}{2}\int_\Sigma \Big[& -\frac{1}{2}W^2 dX_{\alpha\dot\beta}\wedge *dX^{\dot\beta\alpha} + \frac{1}{4W^2}dW_{ij}\wedge *dW^{ij} \\ & + \frac{1}{2}\epsilon^{\alpha\beta}W^{ij}(d\zeta_{i\alpha} + i dX_{\alpha\dot\gamma}\bar\vartheta^{\dot\gamma}_i)\wedge(d\zeta_{j\beta} + i dX_{\beta\dot\delta}\bar\vartheta^{\dot\delta}_j) \\ & - \frac{1}{2W^2}\epsilon_{\dot\alpha\dot\beta}W^{ij}d\bar\vartheta^{\dot\alpha}_i\wedge d\bar\vartheta^{\dot\beta}_j\Big]. \end{aligned} \tag{5.5}$$

In deriving this expression, we have used the invariant form (B3) and the identity $C_{ik}\Lambda^k{}_j = -C_{jk}\Lambda^k{}_i$ and replaced Y_{ij} by the $SU(4)$ "rotated" coordinates W_{ij} [similarly to $Y_{ij} \mapsto Z_{ij}$ in (3.45), recall that $\Lambda^i{}_j = Y^{-1}C^{ik}Y_{kj}$]

$$\begin{aligned} & Y_{ij} \mapsto W_{ij} := YC_{kl}(\Lambda^{-1})^k{}_i(\Lambda^{-1})^l{}_j \quad \text{and} \\ & W^2 = \frac{1}{4}W_{ij}W^{ij} = Y^2. \end{aligned} \tag{5.6}$$

B. Fermionic duality transformation

Let us now go back to the T-dual action (4.3) found after the bosonic duality transformation $X \mapsto \tilde X$ in the superstring action (3.44) in the S-gauge and show that after the $2d$ duality applied to the fermionic coordinates $\theta^{i\alpha}$ (but not to their conjugates $\bar\theta^{\dot\alpha}_i$) one finds precisely the non-Hermitian action (5.5) in the $\bar{Q}$S-gauge (5.1).

We begin with the following first-order form of the action (4.3):

$$\begin{aligned} S = -\frac{T}{2}\int_\Sigma \Big[& -\frac{1}{2Z^2}d\tilde X_{\alpha\dot\beta}\wedge *d\tilde X^{\dot\beta\alpha} + \frac{1}{4Z^2}dZ_{ij}\wedge *dZ^{ij} \\ & - i\tilde X_{\beta\dot\alpha}d\bar\theta^{\dot\alpha}_i\wedge\mathcal{V}^{i\beta} - \frac{1}{2}Z_{ij}\mathcal{V}^{i\alpha}\wedge\mathcal{V}^j_\alpha \\ & - \tilde\theta_{i\alpha}\wedge d\mathcal{V}^{i\alpha} + \frac{1}{2}Z^{ij}d\bar\theta^{\dot\alpha}_i\wedge d\bar\theta_{j\dot\alpha}\Big], \end{aligned} \tag{5.7}$$

where we observed that since (4.3) depends on $\theta^{i\alpha}$ only through its differential we can replace $d\theta^{i\alpha}$ by $\mathcal{V}^{i\alpha}$ adding the constraint $d\mathcal{V}^{i\alpha} = 0$ with the fermionic Lagrange multiplier $\tilde\theta_{i\alpha}$. The variation with respect to the gauge potential $\mathcal{V}^{i\alpha}$ yields

$$\mathcal{V}^{i\alpha} = -\frac{1}{Z^2}Z^{ij}\epsilon^{\alpha\beta}(d\tilde\theta_{j\beta} - i\tilde X_{\beta\dot\alpha}d\bar\theta^{\dot\alpha}_j). \tag{5.8}$$

Note that the on-shell relation

$$d\theta^{i\alpha} = -\frac{1}{Z^2}Z^{ij}\epsilon^{\alpha\beta}(d\tilde\theta_{j\beta} - i\tilde X_{\beta\dot\alpha}d\bar\theta^{\dot\alpha}_j) \tag{5.9}$$

here is different compared to the bosonic duality case (4.4) in that it does not involve the Hodge duality operation. This has to do with a peculiarity of the above GS action[31] where

[29] From the field theory point of view, this complexification seems to be related to the PCT self-conjugacy of the $\mathcal{N} = 4$ SYM multiplet which admits a holomorphic description in the on-shell superspace [21].

[30] The usual Hermitian $AdS_5 \times S^5$ action in a real κ-symmetry gauge can be at best made quartic in the fermions [24,25].

[31] For example, if we compare the two model Lagrangians $L = f d\theta \wedge d\theta$ and $\tilde L = f^{-1}d\tilde\theta \wedge d\tilde\theta$ where θ and $\tilde\theta$ may carry indices and f is a (symmetric) matrix depending on world-sheet coordinates then the dual equations of motion $df \wedge d\theta = 0$ and $df^{-1} \wedge d\tilde\theta = 0$ are first-order in fermions (as is standard for the GS string).

the fermions which we dualize appear only in the WZ term.[32]

Substituting $\mathcal{V}^{i\alpha}$ in (5.8) into (5.7), we end up with the fermionic dual of this action

$$S = -\frac{T}{2}\int_\Sigma\Big[-\frac{1}{2Z^2}d\tilde{X}_{\alpha\dot\beta}\wedge *d\tilde{X}^{\dot\beta\alpha} + \frac{1}{4Z^2}dZ_{ij}\wedge *dZ^{ij} - \frac{1}{2Z^2}Z^{ij}\epsilon^{\alpha\beta}(d\tilde\theta'_{i\alpha} + \mathrm{i}d\tilde{X}_{\alpha\dot\gamma}\bar\theta_i^{\dot\gamma})\wedge(d\tilde\theta'_{j\beta} + \mathrm{i}d\tilde{X}_{\beta\dot\delta}\bar\theta_j^{\dot\delta}) + \frac{1}{2}Z^{ij}d\bar\theta_i^{\dot\alpha}\wedge d\bar\theta_{j\dot\alpha}\Big], \tag{5.10}$$

where we have performed the following fermionic field redefinition:

$$\tilde\theta'_{i\alpha} := \tilde\theta_{i\alpha} - \mathrm{i}\tilde{X}_{\alpha\dot\beta}\bar\theta_i^{\dot\beta}. \tag{5.11}$$

Comparing now the actions (5.5) and (5.10), we conclude that they coincide provided we make the following field identifications:

$$X^{\dot\alpha\beta}\mapsto\tilde{X}^{\dot\alpha\beta},\qquad W^{ij}\mapsto Z^{-2}Z^{ij},\qquad W = Y\mapsto Z^{-1},\qquad \zeta_{i\alpha}\mapsto -\mathrm{i}\tilde\theta'_{i\alpha},\qquad \vartheta_i^{\dot\alpha}\mapsto\bar\theta_i^{\dot\alpha}. \tag{5.12}$$

We conclude that a combination of the bosonic duality [11] and the fermionic duality [22] transformations relates the $\mathrm{AdS}_5\times S^5$ superstring action in the supercoset parametrization (3.30) and in the κ-symmetry S-gauge (3.36) to the same action in the κ-symmetry $\bar{\mathrm{Q}}$S-gauge (5.1) (modulo the necessity of complexification in the transformation process).[33]

This implies that the original $\mathrm{AdS}_5\times S^5$ action after bosonic and fermionic dualities has an equivalent (in a complexified sense) superconformal $PSU(2,2|4)$ global symmetry group, modulo the fact that some of the supersymmetries are not manifest due to a special κ-symmetry gauge choice. In particular, as discussed in [19] and above, (part of[34]) the corresponding Noether charges of the dual model should have their origin in the hidden charges of the original model and vice versa.

In the remainder of this section, we shall explain the need for the fermionic duality transformation from a more general point of view: we will show that the combined action of the bosonic and fermionic dualities leaves the superstring first-order system of equations and Lax connection invariant, generalizing what we have done in the bosonic case in Sec. IV C

C. Combined bosonic/fermionic duality as a symmetry of the Lax connection

For the bosonic AdS_5 sigma model we have shown that the action of T-duality can be interpreted as a symmetry of first-order system of equations combined with a particular automorphism of the conformal group. In this section we show how to extend that symmetry to the full superstring by relating it to an automorphism of the superconformal algebra.

To start with, we need to extend the action of the operator Ω used in Sec. IV C to the full set of the superconformal generators

$$\begin{aligned}
&\Omega(P_{\alpha\dot\beta}) = -K_{\alpha\dot\beta}, && \Omega(K_{\alpha\dot\beta}) = -P_{\alpha\dot\beta},\\
&\Omega(D) = -D, && \Omega(L_{\alpha\beta}) = L_{\alpha\beta},\\
&\Omega(R_{[ij]}) = -R_{[ij]}, && \Omega(R_{(ij)}) = R_{(ij)},\\
&\Omega(Q^{i\alpha}) = \mathrm{i}C^{ij}S_j^\alpha, && \Omega(\bar{Q}_i^{\dot\alpha}) = \mathrm{i}C_{ij}\bar{S}^{j\dot\alpha},\\
&\Omega(S_i^\alpha) = -\mathrm{i}C_{ij}Q^{j\alpha}, && \Omega(\bar{S}^{i\dot\alpha}) = -\mathrm{i}C^{ij}\bar{Q}_j^{\dot\alpha}.
\end{aligned} \tag{5.13}$$

It is easy to verify that Ω is a $\mathbb{Z}_4$-automorphism of the $\mathfrak{psu}(2,2|4)$ algebra.

We can then introduce the following projectors

$$\begin{aligned}
\mathcal{P}_{(0)} &:= \frac{1}{4}(1+\Omega+\Omega^2+\Omega^3),\\
\mathcal{P}_{(1)} &:= \frac{1}{4}(1-\mathrm{i}\Omega-\Omega^2+\mathrm{i}\Omega^3),\\
\mathcal{P}_{(2)} &:= \frac{1}{4}(1-\Omega+\Omega^2-\Omega^3),\\
\mathcal{P}_{(3)} &:= \frac{1}{4}(1+\mathrm{i}\Omega-\Omega^2-\mathrm{i}\Omega^3),
\end{aligned} \tag{5.14}$$

which give the $\mathbb{Z}_4$-decomposition of the algebra

$$\mathfrak{g}_{(m)} = \mathcal{P}_{(m)}(\mathfrak{g}) \tag{5.15}$$

presented in (3.25). From the above analysis we know that the superstring action in the S-gauge is mapped under the combined action of the bosonic and fermionic dualities into the superstring action in the $\bar{\mathrm{Q}}$S complex gauge. To understand this relation from more general perspective, let us start by presenting the $\mathbb{Z}_4$-decomposition of the currents in the S-gauge

[32] Let us mention that since the fermionic duality can be performed via a Gaussian path integral, it can be promoted to a duality of the quantum sigma model. In particular, when performing this duality at the path integral level, one picks up a factor $\Delta_F^{1/2}$ involving the functional determinant $\Delta_F = \Pi_{\sigma\in\Sigma}Z^{16}(\sigma)$. Notice that this is the very same functional determinant which already appeared in the bosonic case (see footnote [23]). For the combination of bosonic and fermionic dualities to be promoted to a quantum symmetry of the GS string on $\mathrm{AdS}_5\times S^5$, the fermionic determinant should then be regularized the same way as the bosonic one, so that $\Delta_B^{-1/2}\Delta_F^{1/2} = 1$ at the end.

[33] Let us note also that one may consider more general combinations of the bosonic and fermionic dualities. For example, one may first perform the fermionic duality and then the bosonic one; the resulting action will be different (and much more complicated). One may also consider combining these dualities with linear field redefinitions, getting an analog of the usual $O(d,d)$ duality group.

[34] The Lorentz and the R-symmetry $SO(6)$ symmetries are shared by the dual models.

$$\begin{aligned} j_{(0)} &= A = \frac{1}{2}(1+\Omega)(j_P + j_R), \\ j_{(1)} &= \frac{1}{2}(1-\mathrm{i}\Omega)(j_Q + j_{\bar{Q}}), \\ j_{(2)} &= \frac{1}{2}(1-\Omega)(j_P + j_R + j_D), \\ j_{(3)} &= \frac{1}{2}(1+\mathrm{i}\Omega)(j_Q + j_{\bar{Q}}). \end{aligned} \tag{5.16}$$

As follows from (4.4) and (5.9), the combined duality is equivalent to the following action on the superstring fields:

$$\begin{aligned} dX^{\dot\beta\alpha} + \frac{\mathrm{i}}{2}(\bar\theta_i^{\dot\beta} d\theta^{i\alpha} - d\bar\theta_i^{\dot\beta}\theta^{i\alpha}) &= Z^{-2} * d\tilde X^{\dot\beta\alpha}, \\ d\theta^{i\alpha} &= -\frac{1}{Z^2} Z^{ij}\epsilon^{\alpha\beta}(d\tilde\theta_{j\beta} - \mathrm{i}\tilde X_{\beta\dot\alpha} d\bar\theta_j^{\dot\alpha}). \end{aligned} \tag{5.17}$$

In order to be able to compare currents before and after the duality, let us also change coordinates according to

$$\begin{aligned} &Z_{ij} \mapsto \tilde Y^{-2} C_{ik} C_{jl} \tilde Y^{kl}, \quad \text{with} \quad \tilde Y^2 = \frac{1}{4}\tilde Y_{ij}\tilde Y^{ij} = Z^{-2}, \\ &\tilde\theta_{i\alpha} \mapsto -\mathrm{i}(\zeta_{i\alpha} + \mathrm{i}\tilde X_{\alpha\dot\beta}\bar\vartheta_i^{\dot\beta}) \quad \text{and} \quad \bar\theta_i^{\dot\alpha} \mapsto -\mathrm{i}\bar\vartheta_i^{\dot\alpha}. \end{aligned} \tag{5.18}$$

Upon applying the duality and the coordinate transformation as above, we can relate the components of the current in the S-gauge, $j = j_P + j_D + j_R + j_Q + j_{\bar Q}$, to the dual one in the $\bar{\mathrm{Q}}$S-gauge, $\tilde j = \tilde j_P + \tilde j_D + \tilde j_R + \tilde j_Q + \tilde j_{\bar S}$, as follows [see also (3.41) and (5.4a)]:

$$\begin{aligned} &j_P = \mathrm{i}Y\Pi^{\dot\alpha\beta}P_{\beta\dot\alpha} = \tilde Y * d\tilde X^{\dot\alpha\beta}P_{\beta\dot\alpha} = *\tilde j_P, \qquad j_D = \frac{\mathrm{i}}{Y}dYD = -\frac{\mathrm{i}}{\tilde Y}d\tilde Y D = -\tilde j_D, \\ &j_{R_a} = -2\mathrm{i}C^{l[i}(\Lambda^{-1}(Y))^{j]}{}_k d\Lambda(Y)^k{}_l R_{[ij]} = 2\mathrm{i}C^{l[i}(\Lambda^{-1}(\tilde Y))^{j]}{}_k d\Lambda(\tilde Y)^k{}_l R_{[ij]} = -\tilde j_{R_a}, \\ &j_{R_s} = -2\mathrm{i}C^{l(i}(\Lambda^{-1}(Y))^{j)}{}_k d\Lambda(Y)^k{}_l R_{(ij)} = -2\mathrm{i}C^{l(i}(\Lambda^{-1}(\tilde Y))^{j)}{}_k d\Lambda(\tilde Y)^k{}_l R_{(ij)} = \tilde j_{R_s}, \\ &j_Q = -\mathrm{i}C_{ij}\epsilon_{\alpha\beta}Y^{1/2}\Lambda(Y)^i{}_k d\theta^{k\alpha}Q^{j\beta} = \tilde Y^{1/2}\Lambda(\tilde Y)^j{}_i(d\zeta_{j\alpha} + \mathrm{i}d\tilde X_{\alpha\dot\beta}\bar\vartheta_j^{\dot\beta})Q^{i\alpha} = -\mathrm{i}\tilde j_Q, \\ &j_{\bar Q} = \mathrm{i}C^{ij}\epsilon_{\dot\alpha\dot\beta}Y^{1/2}(\Lambda^{-1}(Y))^k{}_i d\bar\theta_k^{\dot\alpha}\bar Q_j^{\dot\beta} = C^{ij}\epsilon_{\dot\alpha\dot\beta}\tilde Y^{-1/2}\Lambda(\tilde Y)^k{}_i d\bar\vartheta_k^{\dot\alpha}\bar Q_j^{\dot\beta} = -\Omega(\tilde j_{\bar S}). \end{aligned} \tag{5.19}$$

Here, R_a represents $R_{[ij]}$ while R_s represents $R_{(ij)}$. We can therefore formally summarize the action of the combined bosonic and fermionic dualities [including the coordinate transformation (5.18)] on the current as

$$\begin{aligned} &j_P \mapsto \tilde j_P = *j_P, \qquad j_D \mapsto \tilde j_D = -j_D, \\ &j_{R_a} \mapsto \tilde j_{R_a} = -j_{R_a}, \qquad j_{R_s} \mapsto \tilde j_{R_s} = j_{R_s}, \\ &j_Q \mapsto \tilde j_Q = \mathrm{i}j_Q, \qquad j_{\bar Q} \mapsto \tilde j_{\bar S} = \Omega(j_{\bar Q}). \end{aligned} \tag{5.20}$$

The family of flat currents or Lax connection in the S-gauge is

$$\begin{aligned} \mathrm{j}(z) = \mathrm{j}_B(z) &+ \frac{1}{2}(z+z^{-1})(j_Q + j_{\bar Q}) - \frac{\mathrm{i}}{2}(z-z^{-1})(\Omega(j_Q) \\ &+ \Omega(j_{\bar Q})), \end{aligned} \tag{5.21}$$

where $\mathrm{j}_B(z)$ is formally the current in (4.23), with j_P given in (3.41). Upon applying the duality transformation (5.20), we obtain the dual flat current family

$$\begin{aligned} \tilde{\mathrm{j}}(z) = \tilde{\mathrm{j}}_B(z) &+ \frac{\mathrm{i}}{2}(z+z^{-1})(j_Q - \mathrm{i}\Omega(j_{\bar Q})) \\ &+ \frac{1}{2}(z-z^{-1})(\Omega(j_Q) + \mathrm{i}j_{\bar Q}), \end{aligned} \tag{5.22}$$

where $\tilde{\mathrm{j}}_B(z)$ is the current in (4.24).

As in the bosonic case, we obtain two seemingly different Lax connections. However, one can show that the two Lax connections (5.21) and (5.22) are again related by a spectral parameter dependent automorphism of the superconformal algebra. Indeed, we can define the following $\mathbb{Z}_4$-automorphism:

$$\begin{aligned} &T \mapsto \mathcal{U}_z(T) := U_z\Omega(T)U_z^{-1}, \quad \text{with} \\ &\qquad U_z := \left(\frac{z-z^{-1}}{z+z^{-1}}\right)^{\mathrm{i}(\mathrm{B}+D)}, \end{aligned} \tag{5.23}$$

where T is a generic generator of the superconformal algebra and B generates a $U(1)$-automorphism, with nonvanishing (anti)commutators being

$$\begin{aligned} [\mathrm{B}, Q] &= \frac{\mathrm{i}}{2}Q, \qquad [\mathrm{B}, S] = -\frac{\mathrm{i}}{2}S, \\ [\mathrm{B}, \bar Q] &= -\frac{\mathrm{i}}{2}\bar Q, \qquad [\mathrm{B}, \bar S] = \frac{\mathrm{i}}{2}\bar S \end{aligned} \tag{5.24}$$

and $\Omega(\mathrm{B}) = -\mathrm{B}$.

If we define

$$f(z) := \frac{z-z^{-1}}{z+z^{-1}}, \tag{5.25}$$

we can represent the explicit action of the automorphism $\mathcal{U}_z$ on the generators as follows:

$$\begin{aligned}
&\mathcal{U}_z(P_{\alpha\dot\beta}) = f(z)\Omega(P_{\alpha\dot\beta}), && \mathcal{U}_z(K_{\alpha\dot\beta}) = f^{-1}(z)\Omega(K_{\alpha\dot\beta}),\\
&\mathcal{U}_z(D) = \Omega(D), && \mathcal{U}_z(R_i^j) = \Omega(R_i^j),\\
&\mathcal{U}_z(L_{\alpha\beta}) = \Omega(L_{\alpha\beta}), && \mathcal{U}_z(L_{\dot\alpha\dot\beta}) = \Omega(L_{\dot\alpha\dot\beta}),\\
&\mathcal{U}_z(Q^{i\alpha}) = f(z)\Omega(Q^{i\alpha}), && \mathcal{U}_z(S_i^\alpha) = f(z)^{-1}\Omega(S_i^\alpha),\\
&\mathcal{U}_z(\bar Q_i^{\dot\alpha}) = \Omega(\bar Q_i^{\dot\alpha}), && \mathcal{U}_z(\bar S^{i\dot\alpha}) = \Omega(\bar S^{i\dot\alpha}).
\end{aligned} \tag{5.26}$$

The action of this automorphism on the bosonic generators $\{P, L, K, D\}$ of the $\mathfrak{so}(2,4)$ algebra reduces to the action of the $\mathbb{Z}_2$-automorphism considered before in Sec. IV C. Altogether, we end up with

$$\tilde{\mathrm{j}}(z) = \mathcal{U}_z(\mathrm{j}(z)). \tag{5.27}$$

The conclusion is that we may interpret the combined action of the bosonic and the fermionic dualities as a symmetry of the first-order superstring system of equations induced by the above automorphism of the (complexified) $\mathfrak{psu}(2,2|4)$ algebra.

Furthermore, since the Noether charges may be derived from the flat current $\mathrm{j}(z)$ near $z = \pm 1$ [see Eqs. (3.17) and (3.21) and also Appendix D for more details], Eq. (5.23) suggests that the Noether charges associated with the generators of the superconformal algebra behave under the combined action of the bosonic and fermionic dualities as shown in Table I (modulo the issue of boundary conditions).

This is an immediate consequence of the fact that f goes to zero near $z = \pm 1$ while f^{-1} diverges as can be seen from the respective expansions around $z = \pm 1$

$$\begin{aligned}
&f(z) = \pm(z \mp 1) + \mathcal{O}((z \mp 1)^2) \sim 0 \quad \text{for } z \to \pm 1,\\
&f^{-1}(z) = \pm\frac{1}{z \mp 1} + \frac{1}{2} + \mathcal{O}((z \mp 1)) \sim \pm\frac{1}{z \mp 1} \quad \text{for } z \to \pm 1.
\end{aligned} \tag{5.28}$$

We can understand the behavior of $P_{\alpha\dot\beta}$ and $Q^{i\alpha}$ under T-duality also by observing that they do not act on the dual coordinates $\tilde X_{\alpha\dot\beta}$ and $\tilde\theta_{i\alpha}$ given in Eq. (5.17). This is what we mean by "trivial" in Table I. The resulting picture is in agreement with the conclusion announced in [22]. Remarkably, similar relations for the generators of the original and dual superconformal symmetry when acting on supergluon amplitudes appear also on the gauge theory side [21].[35]

Let us add that the automorphism (5.23) can, in principle, be used to obtain a map between the full set of conserved charges (local and nonlocal ones) before and after the duality.

It would be useful to give a more covariant version of the above analysis in which the κ-symmetry would not be fixed. This would make the global symmetries more manifest and would further clarify the mapping between the conserved charges in the two dual models. It would also be interesting to understand further the meaning of complexification of the superconformal algebra which was required in our string theory considerations and which apparently is also playing an important role on the dual gauge theory side [21] (being related to a possibility of having a chiral on-shell superspace description of the scattering amplitudes for the PCT self-conjugate $\mathcal{N} = 4$ SYM multiplet).

Needless to say, the major outstanding problem is to understand the precise relation between the superstring symmetries in the bulk and the symmetries of the supergluon scattering amplitudes in the boundary gauge theory. This would presumably require defining the IR-regularized amplitudes in terms of correlators of open-string vertex operators inserted on an IR D3-brane as in [12] (see also [40] for a review). For that, one would have to specify, in particular, the boundary conditions for the open strings stretching in the bulk of AdS_5 and ending on the IR brane. The presence of the IR regulator would break the (dual) superconformal symmetry, but in an anomalous, i.e. "controlled," way [21]: it will still lead to highly nontrivial constraints on the finite parts of the amplitudes.[36]

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APPENDIX A: SPINOR CONVENTIONS

1. Four-dimensional spinor conventions

We mostly follow the conventions of Wess and Bagger [41]. Consider 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ with metric $(\eta_{ab}) = \mathrm{diag}(-1, 1, 1, 1)$ and coordinates X^a, where $a, b, \ldots = 0, \ldots, 3$. We shall adopt the convention $(\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha}$ (we shall use † to denote Hermitian conju-

[35] Some generators act trivially and some do not act linearly as they are realized as second-order differential operators [21]. In this reference, the amplitudes are discussed in a chiral superspace. An equivalent choice would have been to consider an antichiral superspace. In the present discussion this change would amount to choosing a QS-gauge rather than a $\bar{\mathrm{Q}}$S-gauge and to performing the fermionic duality along the $\bar\theta$- instead of the θ-directions.

[36] In [21], the superconformal algebra is extended by a central charge which is suggested to be related to the helicity of the particles participating in the scattering process. To recover this central extension in the present approach, one has, presumably, to consider the action of the superconformal algebra on the (super)gluon vertex operators which define the scattering amplitude on the string theory side.

gation on Grassmann algebra elements), where $\alpha, \beta, \ldots = 1, 2$ and $\dot\alpha, \dot\beta, \ldots = \dot 1, \dot 2$.

Let $(\sigma^a) := (\sigma^a_{\alpha\dot\beta}) := (-\mathbb{1}_2, \vec\sigma)$. Here, $\vec\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. Then we define $\bar\sigma^{a\dot\alpha\beta} := \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}\sigma^a_{\delta\dot\gamma}$, with $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha{}^\beta$, $\epsilon_{\dot\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$ and $\epsilon_{12} = -\epsilon_{21} = \epsilon_{\dot 1\dot 2} = -\epsilon_{\dot 2\dot 1} = 1$. Next we introduce

$$X_{\alpha\dot\beta} := \sigma^a_{\alpha\dot\beta}X_a \Leftrightarrow X^a = -\frac{1}{2}\bar\sigma^{a\dot\alpha\beta}X_{\beta\dot\alpha}, \tag{A1}$$

where $X^a = \eta^{ab}X_b$. Explicitly, this reads as

$$(X^{\dot\alpha\beta}) = \begin{pmatrix} X^0 - X^3 & -X^1 + \mathrm{i}X^2 \\ -X^1 - \mathrm{i}X^2 & X^0 + X^3 \end{pmatrix}. \tag{A2}$$

The Minkowski space line element is then given by

$$\begin{aligned} ds^2 &= -\det(dX^{\dot\alpha\beta}) = -\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot\gamma\dot\delta}dX^{\dot\gamma\alpha}dX^{\dot\delta\beta} \\ &= -\frac{1}{2}dX_{\alpha\dot\beta}dX^{\dot\beta\alpha}. \end{aligned} \tag{A3}$$

From the Hermiticity of σ^a and $\bar\sigma^a$, i.e. $\sigma^a = (\sigma^a)^\dagger$ and $\bar\sigma^a = (\bar\sigma^a)^\dagger$ it follows that

$$X_{\alpha\dot\beta} = (X_{\beta\dot\alpha})^* \quad \text{and} \quad X^{\dot\alpha\beta} = (X^{\beta\dot\alpha})^*. \tag{A4}$$

2. Six-dimensional spinor conventions

Consider 6-dimensional Euclidean space $\mathbb{R}^6$ with metric δ_{rs} and coordinates Y^r, where $r, s, \ldots = 1, \ldots, 6$. Then we take the σ^r and $\bar\sigma^r$ matrices as antisymmetric 4×4 matrices,

$$\sigma^r = (\sigma^r_{ij}), \qquad \sigma^r_{ij} = -\sigma^r_{ji}, \qquad \bar\sigma^{rij} := \frac{1}{2}\epsilon^{ijkl}\sigma^r_{kl}, \tag{A5}$$

for $i, j, \ldots = 1, \ldots, 4$ and ϵ^{ijkl} is totally antisymmetric in its indices with $\epsilon^{1234} = 1$.

Next we introduce

$$Y_{ij} := \sigma^r_{ij}Y_r \Leftrightarrow Y^r = \frac{1}{4}\bar\sigma^{rij}Y_{ij}. \tag{A6}$$

Furthermore, by virtue of (A5) and $\bar\sigma^{rij} = (\sigma^r_{ij})^*$ we have

$$Y^{ij} = \frac{1}{2}\epsilon^{ijkl}Y_{kl} \quad \text{and} \quad Y^{ij} = (Y_{ij})^*. \tag{A7}$$

The line element of $\mathbb{R}^6$ in these coordinates is given by

$$ds^2 = \sqrt{\det(dY^{ij})} = \frac{1}{8}\epsilon_{ijkl}dY^{ij}dY^{kl} = \frac{1}{4}dY_{ij}dY^{ij}. \tag{A8}$$

APPENDIX B: SUPERCONFORMAL ALGEBRA

1. Commutation relations

The nontrivial (anti)commutation relations among the generators of the superconformal algebra $\mathfrak{psu}(2,2|4)$ are

$$\begin{aligned}
&\{Q^{i\alpha}, \bar Q^{\dot\beta}_j\} = -\delta^i_j P^{\dot\beta\alpha}, \qquad \{S^\alpha_i, \bar S^{j\dot\beta}\} = -\delta^j_i K^{\dot\beta\alpha}, \qquad \{Q^{i\alpha}, S^\beta_j\} = -\mathrm{i}\delta^i_j\Big(L^{\alpha\beta} + \frac{1}{2}\epsilon^{\alpha\beta}D\Big) + 2\mathrm{i}\epsilon^{\alpha\beta}R_j{}^i, \\
&[R_i{}^j, S^\alpha_k] = -\frac{\mathrm{i}}{2}(\delta^j_k S^\alpha_i - \frac{1}{4}\delta^i_j S^\alpha_k), \qquad [L^{\alpha\beta}, S^\gamma_i] = \mathrm{i}\epsilon^{\gamma(\alpha}S^{\beta)}_i, \qquad [P^{\dot\alpha\beta}, S^\gamma_i] = \epsilon^{\beta\gamma}\bar Q^{\dot\alpha}_i, \qquad [D, S^\alpha_i] = -\frac{\mathrm{i}}{2}S^\alpha_i, \\
&[R_i{}^j, Q^{k\alpha}] = \frac{\mathrm{i}}{2}(\delta^k_i Q^{j\alpha} - \frac{1}{4}\delta^j_i Q^{k\alpha}), \qquad [L^{\alpha\beta}, Q^{i\gamma}] = \mathrm{i}\epsilon^{\gamma(\alpha}Q^{i\beta)}, \qquad [K^{\dot\alpha\beta}, Q^{i\gamma}] = \epsilon^{\beta\gamma}\bar S^{i\dot\alpha}, \qquad [D, Q^{i\alpha}] = \frac{\mathrm{i}}{2}Q^{i\alpha}, \\
&[R_i{}^j, R_k{}^l] = \frac{\mathrm{i}}{2}(\delta^l_i R_k{}^j - \delta^j_k R_i{}^l), \qquad [D, P^{\dot\alpha\beta}] = \mathrm{i}P^{\dot\alpha\beta}, \qquad [D, K^{\dot\alpha\beta}] = -\mathrm{i}K^{\dot\alpha\beta}, \qquad [L^{\alpha\beta}, P^{\gamma\dot\delta}] = \mathrm{i}\epsilon^{\delta(\alpha}P^{\gamma\dot\beta)}, \\
&[L^{\alpha\beta}, K^{\dot\gamma\delta}] = \mathrm{i}\epsilon^{\delta(\alpha}K^{\dot\gamma\beta)}, \qquad [L_{\alpha\beta}, L^{\gamma\delta}] = -2\mathrm{i}\delta^{(\gamma}_{(\alpha}L^{\delta)}_{\beta)}, \qquad [P^{\dot\alpha\beta}, K^{\dot\gamma\delta}] = -\mathrm{i}(\epsilon^{\dot\alpha\dot\gamma}L^{\beta\delta} + \epsilon^{\beta\delta}L^{\dot\alpha\dot\gamma} + \epsilon^{\beta\delta}\epsilon^{\dot\alpha\dot\gamma}D).
\end{aligned} \tag{B1}$$

In writing these expressions, we have made use of the $4d$ vector index identification $\{a\} = \{\alpha\dot\beta\}$. In particular, this implies that the rotation generators decompose into the self-dual and anti-self-dual parts

$$\begin{aligned} &L_{ab} \leftrightarrow L_{\alpha\dot\beta\gamma\dot\delta} = -\frac{1}{2}(\epsilon_{\dot\beta\dot\delta}L_{\alpha\gamma} + \epsilon_{\alpha\gamma}L_{\dot\beta\dot\delta}), \\ &L_{\alpha\beta} = L_{\beta\alpha}, \qquad L_{\dot\alpha\dot\beta} = (L_{\alpha\beta})^\dagger. \end{aligned} \tag{B2}$$

2. Invariant form

The nonvanishing components of the invariant form of $\mathfrak{psu}(2,2|4)$ compatible with the above choice of the basis of the algebra are

$$\begin{aligned} \mathrm{str}(P_{\alpha\dot\beta}K_{\gamma\dot\delta}) &= \epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta}, \\ \mathrm{str}(DD) &= -1, \\ \mathrm{str}(L_{\alpha\beta}L_{\gamma\delta}) &= -\epsilon_{\alpha(\gamma}\epsilon_{\delta)\beta}, \\ \mathrm{str}(R_i{}^jR_k{}^l) &= \frac{1}{4}\Big(\delta^l_i\delta^j_k - \frac{1}{4}\delta^i_j\delta^l_k\Big), \\ \mathrm{str}(Q^{i\alpha}S^\beta_j) &= \delta^i_j\epsilon^{\alpha\beta}. \end{aligned} \tag{B3}$$

3. $\mathbb{Z}_4$-grading of the algebra

With the above choice of the generators, the $\mathbb{Z}_4$-decomposition (3.3) is not manifest. To find a manifest realization of the grading, let us start from the bosonic part

of the algebra, in particular, from $\mathfrak{so}(6) \cong \mathfrak{su}(4) = \mathrm{span}\{R_i{}^j\}$. Since $S^5 \cong SO(6)/SO(5) \cong SU(4)/Sp(4)$, we may pick some $Sp(4)$-metric C_{ij} with

$$C_{ij} = -C_{ji} =: \frac{1}{2}\epsilon_{ijkl}C^{kl}, \qquad C_{ij} = (C^{ij})^* \quad \text{and} \quad C_{ik}C^{jk} = \delta_i^j. \tag{B4}$$

Without loss of generality, $C = (C_{ij})$ may be chosen as

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{B5}$$

A particular choice of C_{ij} induces an isomorphism (non-canonical) between the 15-dimensional bivector representation $\mathbf{6} \wedge \mathbf{6}$ of $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ and the sum of the 5-dimensional vector representation $\mathbf{5}$ of $\mathfrak{so}(5) \cong \mathfrak{sp}(4) \subset \mathfrak{so}(6)$ and the 10-dimensional bivector representation $\mathbf{5} \wedge \mathbf{5}$, i.e. $\mathbf{6} \wedge \mathbf{6} \cong \mathbf{5} \oplus \mathbf{5} \wedge \mathbf{5}$. Explicitly, we then have

$$\underbrace{R_{ij} := C_{ik}R_j{}^k}_{\hat{=}\mathbf{6}\wedge\mathbf{6}} = \underbrace{C_{[ik}R_{j]}{}^k}_{\hat{=}\mathbf{5}} + \underbrace{C_{(ik}R_{j)}{}^k}_{\hat{=}\mathbf{5}\wedge\mathbf{5}}. \tag{B6}$$

Note that $C_{[ik}R_{j]}{}^k$ represents the 5 because of $R_i{}^i = 0$. We shall use the notation

$$R_{(ij)} := C_{(ik}R_{j)}{}^k \quad \text{and} \quad R_{[ij]} := C_{[ik}R_{j]}{}^k, \tag{B7}$$

where parentheses mean normalized symmetrization and square brackets mean normalized antisymmetrization. Then $R_{(ij)} \in \mathfrak{h}$ and $R_{[ij]} \in \mathfrak{g}_{(2)}$.

Next, consider $\mathfrak{so}(2,4) \cong \mathfrak{su}(2,2)$ which is generated by P_a, L_{ab}, K_a, D. Then $\frac{1}{2}(P_a - K_a)$ and L_{ab} are the remaining generators of $\mathfrak{h}$ while $\frac{1}{2}(P_a + K_a)$ and D are the remaining generators of the bosonic coset part $\mathfrak{g}_{(2)}$, respectively. One may proceed similarly with the fermionic generators. Eventually, one finds that the $\mathbb{Z}_4$-splitting is given by (3.25).

APPENDIX C: FERMIONIC CURRENT

Here, we shall briefly review the derivation of Eq. (3.34). The bosonic current j_B was already given. To get a handle on the fermionic one j_F, let us consider the one-parameter family ($t \in \mathbb{R}$)

$$j(t) := \mathrm{e}^{tF} j_B \mathrm{e}^{-tF} + \mathrm{e}^{tF} d \mathrm{e}^{-Ft}, \quad \text{with} \quad j(t=0) = j_B. \tag{C1}$$

This then implies

$$\partial_t j(t) = \mathrm{e}^{Ft}(-\nabla F)\mathrm{e}^{-tF}, \quad \text{with} \quad \nabla\cdot = d\cdot + [j_B, \cdot] \tag{C2}$$

and so ($j(t=1) = j$)

$$j(t) = j_B + \int_0^t dt' \mathrm{e}^{t'F}(-\nabla F)\mathrm{e}^{-t'F} \Rightarrow j_F = \int_0^1 dt' \mathrm{e}^{t'F}(-\nabla F)\mathrm{e}^{-t'F}. \tag{C3}$$

Upon recalling the formula $\mathrm{e}^{tA}B\mathrm{e}^{-tA} = \sum_n \frac{t^n}{n!}[A,B]^{(n)}$, where $[A,B]^{(n)} := [A,[A,B]^{(n-1)}]$ with $[A,B]^{(0)} := B$, we find

$$\begin{aligned} j_F &= \sum_n \frac{1}{(n+1)!}[F, -\nabla F]^{(n)} \\ &= \sum_n \frac{1}{(2n+1)!}[F, -\nabla F]^{(2n)} + \sum_n \frac{1}{(2n+2)!} \\ &\quad \times [F, -\nabla F]^{(2n+1)}. \end{aligned} \tag{C4}$$

Using the definition (3.35), then a short calculation reveals that

$$\sum_n \frac{1}{(2n+1)!}[F, -\nabla F]^{(2n)} = -\frac{\sinh(\mathcal{M})}{\mathcal{M}}\nabla F, \tag{C5a}$$

$$\sum_n \frac{1}{(2n+2)!}[F, -\nabla F]^{(2n+1)} = -2\left[\mathrm{i}f, \frac{\sinh^2(\mathcal{M}/2)}{\mathcal{M}^2}\nabla F\right]. \tag{C5b}$$

Altogether, we then obtain (3.34).

APPENDIX D: COMMENTS ON CONSERVED CHARGES

Let us make some comments on the construction of conserved charges for the bosonic sigma model discussed in Sec. IV C While having in mind the relation to scattering amplitudes in the dual SYM theory, it would be natural to discuss the T-duality acting on the open strings as in [12]. Here we shall formally assume that the string coordinates are periodic in the spatial world-sheet direction σ as would be the case in the closed string sector of the theory.

We have seen in Sec. III C that conserved charges follow directly from the current $\mathrm{J}(z)$. An alternative route to find them is to consider the parallel transport of the Lax connection $\mathrm{j}(z)$

$$M(z;\sigma,\tau;\sigma_0,\tau_0) := P\exp\left(\int_{\sigma_0,\tau_0}^{\sigma,\tau} \mathrm{j}(z)\right). \tag{D1}$$

A candidate conserved charge is given by the following composition of parallel transports

$$Q(z) := M(z;\sigma_0,\tau_0;\sigma+2\pi,\tau)\frac{\partial M}{\partial z}(z;\sigma+2\pi,\tau;\sigma,\tau) \times M(z;\sigma,\tau;\sigma_0,\tau_0). \tag{D2}$$

Assuming that the Lax connection is periodic, $\mathrm{j}(z;\sigma+2\pi,\tau) = \mathrm{j}(z;\sigma,\tau)$, the charge obeys the following differential equation

$$\frac{d}{d\tau}Q(z) = M(z;\sigma_0,\tau_0;\sigma+2\pi,\tau) \times \left[\frac{\partial \mathrm{j}}{\partial z}(z;\sigma,\tau), M(z;\sigma+2\pi,\tau;\sigma,\tau)\right] \times M(z;\sigma,\tau;\sigma_0,\tau_0). \tag{D3}$$

In other words, the charge is conserved if the commutator on the right-hand side vanishes. The charges $Q(z)$ and the dual charges $\tilde{Q}(z)$ are related through (4.27), though not in an obvious way.

Generically, the commutator can vanish only at specific values of z, in particular, at $z = \pm 1, \pm \mathrm{i}$ (see [34]). The Lax connection at $z = \pm 1$ can be easily integrated

$$M(\pm 1;\sigma,\tau;\sigma_0,\tau_0) = P\exp\left(\int_{\sigma_0,\tau_0}^{\sigma,\tau} j\right) = g(\sigma,\tau)^{-1}g(\sigma_0,\tau_0). \tag{D4}$$

In particular, due to the assumed periodicity of g, one finds

$$M(\pm 1;\sigma+2\pi,\tau;\sigma,\tau) = g(\sigma+2\pi,\tau)^{-1}g(\sigma,\tau) = 1, \tag{D5}$$

and therefore the charge at $z = \pm 1$ is manifestly conserved, $\frac{d}{d\tau}Q(\pm 1) = 0$. This charge is the standard Noether charge for the $\mathfrak{so}(2,4)$ symmetry [see also Eqs. (3.18) and (3.19)]

$$Q(\pm 1) = \mp g(\sigma_0,\tau_0)^{-1}\left(\oint *J_N\right)g(\sigma_0,\tau_0), \quad \text{with} \quad J_N = g j_{(2)} g^{-1}, \tag{D6}$$

i.e. J_N is the bosonic Noether current (3.48).

Consider now the dual charge $\tilde{Q}(z)$ at $z = \pm 1$. By similar arguments it is conserved if $\tilde{M}(\pm 1;\sigma+2\pi,\tau;\sigma,\tau)$ commutes with $\partial_z \tilde{\mathrm{j}}(\pm 1;\sigma,\tau)$. However, this crucially depends on the periodicity of the dual coordinates $\tilde{X}^{\dot{\alpha}\beta}$

$$\tilde{M}(\pm 1;\sigma+2\pi,\tau;\sigma,\tau) = \tilde{g}(\sigma+2\pi,\tau)^{-1}\tilde{g}(\sigma,\tau) = \exp[\mathrm{i}\tilde{Y}(\sigma)^{-1}(\tilde{X}^{\dot{\alpha}\beta}(\sigma+2\pi) - \tilde{X}^{\dot{\alpha}\beta}(\sigma))P_{\beta\dot{\alpha}}]. \tag{D7}$$

In terms of the original coordinates this expression reads

$$\tilde{M}(\pm 1;\sigma+2\pi,\tau;\sigma,\tau) = \exp\left(-2Y\oint *\Omega(J_{N,K})\right), \tag{D8}$$

where $J_{N,K}$ is the projection of the Noether current (3.48) along the K generator and Ω is the $\mathbb{Z}_2$-automorphism defined in Eq. (4.13). Thus the dual conformal symmetry acting on $\tilde{X}^{\dot{\alpha}\beta}$ coordinates is manifest only if the Noether charge of the original model satisfies

$$\oint *\Omega(J_{N,K}) = 0, \tag{D9}$$

i.e. the total momentum $\oint d\sigma Y^2 \partial_\tau X_{\alpha\dot{\beta}}$ vanishes.

To understand the meaning of this conclusion, let us recall that in the standard discussions of T-duality one usually assumes the compactness of the isometry direction along which the duality is performed. Provided the original X and the dual $\tilde{X}$ coordinates are periodic with radii a and $\tilde{a} = \frac{\alpha'}{a}$, the T-duality is then a symmetry of the spectrum of underlying conformal field theory: it interchanges the Kaluza-Klein momenta with the winding mode numbers. Viewing the noncompact isometry case as a limit of the compact one means that to preserve this symmetry one may assume that the dual coordinate is compactified on a circle of vanishing radius, $\tilde{a} \to 0$: then all finite-mass momentum modes are mapped into finite-mass winding modes (see Ref. [9]). A possible alternative is to restrict consideration to a subsector of states that do not carry momentum in the noncompact isometric X direction; then their duals are not required to have a winding in $\tilde{X}$ and thus $\tilde{X}$ may also be assumed to be noncompact. Indeed, Eq. (D9) may be interpreted as such zero X momentum or zero $\tilde{X}$ winding condition.

Since the T-duality along all the four translational isometries of the AdS_5 space acts also on the time direction, it is not clear, even assuming the above zero-momentum condition, if this duality may have some useful implications for the closed string spectrum of the superstring theory. Given a close relation between the T-self-duality of the $\mathrm{AdS}_5 \times S^5$ sigma model and its integrability that we uncovered above, one may still expect some connection between the duality and the closed string spectrum, but that probably requires a certain complexification of the set of charges that label string states (in addition to a constraint on their values implied by the above discussion).

[1] M. Lüscher and K. Pohlmeyer, Nucl. Phys. **B137**, 46 (1978).
[2] G. Mandal, N. V. Suryanarayana, and S. R. Wadia, Phys. Lett. B **543**, 81 (2002).
[3] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B533**, 109 (1998).
[4] I. Bena, J. Polchinski, and R. Roiban, Phys. Rev. D **69**, 046002 (2004).

[5] V. E. Zakharov and A. V. Mikhailov, Sov. Phys. JETP **47**, 1017 (1978).

[6] C. R. Nappi, Phys. Rev. D **21**, 418 (1980).

[7] B. E. Fridling and A. Jevicki, Phys. Lett. **134B**, 70 (1984); E. S. Fradkin and A. A. Tseytlin, Ann. Phys. (N.Y.) **162**, 31 (1985).

[8] T. H. Buscher, Phys. Lett. B **201**, 466 (1988).

[9] M. Roček and E. P. Verlinde, Nucl. Phys. **B373**, 630 (1992).

[10] E. Álvarez, L. Álvarez-Gaumé, and I. Bakas, Nucl. Phys. **B457**, 3 (1995); K. Sfetsos, Nucl. Phys. **B463**, 33 (1996); T. Curtright, T. Uematsu, and C. K. Zachos, Nucl. Phys. **B469**, 488 (1996).

[11] R. Kallosh and A. A. Tseytlin, J. High Energy Phys. 10 (1998) 016.

[12] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 06 (2007) 064; L. F. Alday and J. M. Maldacena, J. High Energy Phys. 11 (2007) 068.

[13] M. Kruczenski, R. Roiban, A. Tirziu, and A. A. Tseytlin, Nucl. Phys. **B791**, 93 (2008).

[14] J. M. Drummond, J. Henn, V. A. Smirnov, and E. Sokatchev, J. High Energy Phys. 01 (2007) 064; J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 385 (2008); J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev,Nucl. Phys. **B795**, 52 (2008); arXiv:0712.1223; arXiv:0803.1466.

[15] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, Phys. Rev. D **75**, 085010 (2007); Z. Bern, J. J. M. Carrasco, H. Johansson, and D. A. Kosower, Phys. Rev. D **76**, 125020 (2007); Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu, and A. Volovich, Phys. Rev. D **78**, 045007 (2008).

[16] Z. Komargodski, J. High Energy Phys. 05 (2008) 019.

[17] A. M. Polyakov, Int. J. Mod. Phys. A **14**, 645 (1999); Phys. At. Nucl. **64**, 540 (2001).

[18] A. Brandhuber, P. Heslop, and G. Travaglini, Nucl. Phys. **B794**, 231 (2008); J. McGreevy and A. Sever, J. High Energy Phys. 08 (2008) 078.

[19] R. Ricci, A. A. Tseytlin, and M. Wolf, J. High Energy Phys. 12 (2007) 082.

[20] M. Hatsuda and S. Mizoguchi, J. High Energy Phys. 07 (2006) 029.

[21] E. Sokatchev, in *Proceedings of the Conference on "Wonders of Gauge Theory and Supergravity, Paris, June 2008* (unpublished); G. Korchemsky, in *Proceedings of the Conference on "Wonders of Gauge Theory and Supergravity, Paris, June 2008* (unpublished); J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, arXiv:0807.1095.

[22] N. Berkovits, in *Proceedings of the Conference on "Wonders of Gauge Theory and Supergravity, Paris, June 2008* (unpublished), based on work in preparation with J. M. Maldacena.

[23] A. Giveon, M. Porrati, and E. Rabinovici, Phys. Rep. **244**, 77 (1994); E. Álvarez, L. Álvarez-Gaumé, and Y. Lozano, Nucl. Phys. B, Proc. Suppl. **41**, 1 (1995).

[24] R. Kallosh and J. Rahmfeld, Phys. Lett. B **443**, 143 (1998); I. Pesando, J. High Energy Phys. 11 (1998) 002.

[25] R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D **63**, 046002 (2001); R. R. Metsaev, C. B. Thorn, and A. A. Tseytlin, Nucl. Phys. **B596**, 151 (2001).

[26] R. Roiban and W. Siegel, J. High Energy Phys. 11 (2000) 024.

[27] T. Curtright and C. K. Zachos, Phys. Rev. D **49**, 5408 (1994); M. Sarisaman, arXiv:0704.3605.

[28] C.-H. Xiong, arXiv:0705.3546.

[29] A. Mikhailov, arXiv:hep-th/0511069.

[30] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov, and B. Zwiebach, Nucl. Phys. **B567**, 61 (2000).

[31] A. M. Polyakov, Mod. Phys. Lett. A **19**, 1649 (2004).

[32] L. F. Alday, G. Arutyunov, and A. A. Tseytlin, J. High Energy Phys. 07 (2005) 002; G. Arutyunov and S. Frolov, J. High Energy Phys. 02 (2005) 059; M. Grigoriev and A. A. Tseytlin, Nucl. Phys. **B800**, 450 (2008); A. Mikhailov and S. Schäfer-Nameki, J. High Energy Phys. 05 (2008) 075.

[33] C. A. S. Young, Phys. Lett. B **632**, 559 (2006).

[34] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, Commun. Math. Phys. **263**, 659 (2006).

[35] L. Dolan, Phys. Rep. **109**, 1 (1984).

[36] R. Kallosh, J. Rahmfeld, and A. Rajaraman, J. High Energy Phys. 09 (1998) 002; P. Claus and R. Kallosh, J. High Energy Phys. 03 (1999) 014.

[37] M. Cvetič, H. Lu, C. N. Pope, and K. S. Stelle, Nucl. Phys. **B573**, 149 (2000); B. Kulik and R. Roiban, J. High Energy Phys. 09 (2002) 007; S. F. Hassan, Nucl. Phys. B, Proc. Suppl. **102**, 77 (2001); L. F. Alday, G. Arutyunov, and S. Frolov, J. High Energy Phys. 06 (2006) 018.

[38] A. S. Schwarz and A. A. Tseytlin, Nucl. Phys. **B399**, 691 (1993).

[39] G. Arutyunov and M. Zamaklar, J. High Energy Phys. 07 (2005) 026.

[40] L. F. Alday and R. Roiban, arXiv:0807.1889.

[41] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University, Princeton, NJ, 1992), p. 259.

[42] N. Berkovits and J. M. Maldacena, J. High Energy Phys. 09 (2008) 062.