

LINEAR AND NONLINEAR TAILS I: GENERAL RESULTS AND PERTURBATION THEORY

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Received 13 Oct. 2007

Revised 2 Jul. 2008

Communicated by Lars Andersson

Abstract. For nonlinear wave equations with a potential term, we prove pointwise space-time decay estimates and develop a perturbation theory for small initial data. We show that the perturbation series has a positive convergence radius by a method which reduces the wave equation to an algebraic one. We demonstrate that already first and second perturbation orders, satisfying linear equations, can provide precise information about the decay of the full solution to the nonlinear wave equation.

Keywords: Nonlinear wave equations; decay estimates; late-time tails; asymptotics.

Mathematics Subject Classification 2000: 35L05, 35B40, 35B20, 35B35

1. Introduction

It is a well-known fact that the presence of a long-range potential term (power-law decay at spatial infinity) in the wave equation violates Huygens principle and gives rise to a late-time tail in the solution with a power-law decay (both powers are related) [3, 5]. It is not so well-known that nonlinear terms like u^p cause the same effect. We study equations where both these effects are present and give pointwise decay estimates on the solutions. Further, we develop a perturbation theory for these equations and, by its means, argue that presented estimates should give optimal decay rates at late times. A rigorous proof of this fact will appear in a following publication [6] (Part II).

We consider linear and nonlinear wave equations with a potential term of the general form

$$\square u + Vu = F(u) \tag{1.1}$$

in 3 spatial dimensions, i.e. $u : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \equiv \mathbb{R}_+^{1+3} \rightarrow \mathbb{R}$, and solve the initial value problem with

$$u(0, x) = f(x), \quad \partial_t u(0, x) = g(x). \tag{1.2}$$

First, we construct an iteration scheme and show its convergence in a weighted space-time L^∞ -norm what reproduces the decay estimate from [3, 5]

$$|u(t, x)| \leq \frac{C}{(1+t+|x|)(1+|t-|x||)^{q-1}} \quad \forall (t, x) \in \mathbb{R}_+^{1+3}$$

with $q := \min(m-1, k, p-1)$ provided the potential V and the initial data f, g satisfy pointwise bounds

$$|V(x)| \leq \frac{V_0}{(1+|x|)^k}, \quad k > 2$$

$$|f(x)| \leq \frac{f_0}{(1+|x|)^{m-1}}, \quad |\nabla f(x)| \leq \frac{f_1}{(1+|x|)^m}, \quad |g(x)| \leq \frac{g_0}{(1+|x|)^m}, \quad m > 3,$$

with small V_0, f_0, f_1, f_2 and the nonlinearity is analytic and satisfies for $p > 1 + \sqrt{2}$

$$|F(u)| \leq F_1|u|^p, \quad |F(u) - F(v)| \leq F_2|u - v| \max(|u|, |v|)^{p-1} \quad \text{for } |u|, |v| < 1.$$

Next, we construct a perturbation series representing the solution u and prove its convergence (with finite convergence radius) in the same weighted space-time L^∞ -norm. It implies pointwise convergence in \mathbb{R}_+^{1+3} what allows us to control the decay at every perturbation order and obtain estimate on the remainder of the perturbation series for any order. Finally, if we can show that at some perturbation level our decay estimate is optimal, i.e. we know the true asymptotics for late times (what is not very difficult because the perturbation equations are linear), then we immediately know the asymptotics of u . It is the same as that of the given perturbation order because all higher terms in the perturbation series, summed up, are too small to be able to modify the asymptotics. The issue of optimal decay estimates and precise asymptotics compared with numerical results will be addressed in a forthcoming publication [6] which will be focused on spherical symmetry where the perturbation equations can be solved almost exactly.

The proof of convergence of the perturbation series is essential for justifying the perturbation scheme as a rigorous approximation and being able to provide exact decay rates. We show it by relating the (inverted) wave equation

$$u = \square^{-1}F(u) - \square^{-1}(Vu) + \varepsilon I(f, g)$$

(where $\varepsilon I(f, g)$ stands for initial data contribution to the solution of the free wave equation $\square u = 0$) to an algebraic equation of a similar form

$$W = C\tilde{F}(W) + \delta W + \varepsilon D$$

(\tilde{F} is obtained from F by transformation of its Taylor series), which arises from comparison of the perturbation schemes for both problems. We make an interesting observation that the nonlinear wave equation has a solution $u(\varepsilon)$ analytic in ε , and hence representable by a convergent series in ε , if the same holds for the solution $W(\varepsilon)$ of the corresponding algebraic equation. The latter, however, is always true when $F(u)$ is analytic at $u = 0$ what we assume.

Regarding regularity, we can go a safe way and consider only the classical solutions, i.e. assume $(f, g) \in \mathcal{C}^3(\mathbb{R}^3) \times \mathcal{C}^2(\mathbb{R}^3)$, $V \in \mathcal{C}^2(\mathbb{R}^3)$ and $F \in \mathcal{C}^2(\mathbb{R})$ and obtain $u \in \mathcal{C}^2(\mathbb{R}_+^{1+3})$. However, all results remain true also for weak solutions where $(f, g) \in \mathcal{C}^1(\mathbb{R}^3) \times \mathcal{C}^0(\mathbb{R}^3)$, $V \in \mathcal{C}^0(\mathbb{R}^3)$ and $F \in \mathcal{C}^0(\mathbb{R})$ and we have $u \in \mathcal{C}^0(\mathbb{R}_+^{1+3})$, because Lemmas A.1–A.4, which constitute the main “engine” of all estimates, preserve the continuity (see [5] for a detailed discussion of the weak solutions).

This paper is organized as follows. It has three main sections addressing the linear wave equation with potential, nonlinear wave equation with and without potential, respectively. The idea is to develop tools for the simplest, linear problem and then to generalize them to the nonlinear situation. Every section has subsections presenting an iterative and a perturbative approach to the construction of solutions and a discussion of the optimal decay rates. Appendix collects some lemmas used in the proofs, cited from other works.

Notation

With the symbol $\langle x \rangle := 1 + |x|$, we define spatial and space-time weighted- L^∞ norms

$$\begin{aligned} \|f\|_{L_m^\infty} &:= \|\langle x \rangle^m f(x)\|_{L^\infty(\mathbb{R}^3)}, \\ \|u\|_{L_{q,p}^\infty} &:= \|\langle t + |x| \rangle^q \langle t - |x| \rangle^{p-q} u(t, x)\|_{L^\infty(\mathbb{R}_+^{1+3})}, \end{aligned}$$

of which we will most frequently use

$$\|u\|_{L_{1,p}^\infty} := \|\langle t + |x| \rangle \langle t - |x| \rangle^{p-1} u(t, x)\|_{L^\infty(\mathbb{R}_+^{1+3})}.$$

Its finiteness guarantees the decay of u like $1/t$ on the lightcone $t \sim |x|$ and like $1/t^p$ for fixed x as well as $1/|x|^p$ for fixed t . Note that functions with compact support in \mathbb{R}^3 belong to all spaces L_m^∞ with any $m > 0$.

We introduce the following notation for solutions of the wave equations. Let I_V be a linear map from the space of initial data to the space of solutions of the wave equations (1.1) and (1.2) with $F(u) = 0$, so that $u = I_V(f, g)$. For wave equations with a source term and null initial data

$$\square u + Vu = F, \quad u(0, x) = 0, \quad \partial_t u(0, x) = 0,$$

let us denote the solutions by $u = L_V(F)$, where L_V is a linear map from the space of source functions to the space of solutions to the above problem. Note that, due to

linearity, the solution u of a wave equation with source F and non-vanishing initial data f, g is a sum of these two contributions

$$u = L_V(F) + I_V(f, g).$$

Observe that if we put the potential term on the right-hand side, we obtain

$$\square u = -Vu + F$$

which, treated as a wave equation without potential (on the left-hand side), is formally solved by

$$u = -L_0(Vu) + L_0(F) + I_0(f, g).$$

Here the solution u appears on both sides what seems to make the formula useless, but it will allow us to formulate various iteration schemes, e.g.

$$u_{n+1} = -L_0(Vu_n) + L_0(F(u_n)) + I_0(f, g)$$

for which we will prove convergence in suitable $L_{1,q}^\infty$ norms.

Finally, we define constants which arise from estimates proved in [5], improved in [4]

$$C_m := \max\left(\frac{9}{2(m-2)}, 5\right),$$

$$C_{p,q} := 2 + \frac{8}{p-1} + \frac{2}{q-1}.$$

The latter will be referred to as a bound on the allowed strength V_0 of the potential. Our purpose is to emphasize that this bound, although not optimal, is finite and not arbitrarily small what is crucial when a potential with a given value V_0 is studied (like e.g. in the Regge–Wheeler equation describing waves on Schwarzschild geometry).

2. Linear Case with Potential

First, we consider a linear wave equation

$$\square u + \lambda V(x)u = 0, \tag{2.1}$$

where $\lambda > 0$ is a small parameter, bounded by some finite constant $C_V > 0$ (which will be defined later). We first show that a standard iteration scheme converges for all $\lambda < C_V$ to a solution in $L_{1,p}^\infty$, i.e. there exists a constant C such that

$$|u(t, x)| \leq \frac{C}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-1}} \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

with some $p > 2$ provided the potential V and the initial data $f, \nabla f, g$ are (at least) continuous and satisfy pointwise bounds

$$|V(x)| \leq \frac{1}{\langle x \rangle^k}, \quad k > 2 \tag{2.2}$$

and

$$|f(x)| \leq \frac{f_0}{\langle x \rangle^{m-1}}, \quad |\nabla f(x)| \leq \frac{f_1}{\langle x \rangle^m}, \quad |g(x)| \leq \frac{g_0}{\langle x \rangle^m}, \quad m > 3. \tag{2.3}$$

Then, we show that a perturbation scheme based on expansion in powers of λ is, due to linearity, equivalent to the iteration scheme and the perturbation series has convergence radius C_V .

As next, we show that the lowest order u_0 has, in general, a different decay estimate than all higher orders, starting from u_1 . Finally, we prove that either u_0 or u_1 gives precise information about the decay rate of the full solution u .

2.1. Iteration

We define an iteration by

$$\begin{aligned} u_{-1} &:= 0 \\ u_n &:= I_0(f, g) - \lambda L_0(Vu_{n-1}), \quad n = 0, 1, 2, \dots \end{aligned}$$

Then we have the following

Theorem 2.1. *With f, g and V as above for any $m > 3$ and $k > 2$ the sequence u_n converges (in norm) in $L_{1,p}^\infty$ for $p = \min(k, m - 1)$ provided $\lambda < C_{p,k}^{-1}$. The limit $u := \lim_{n \rightarrow \infty} u_n$ satisfies*

$$|u(t, x)| \leq \frac{C}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-1}}, \quad \forall (t, x) \in \mathbb{R}_+^{1+3}$$

with some positive constant C depending only on f_0, f_1, g_0, λ and k, m .

This theorem was proved first for classical solutions in [3] and later generalized to weak solutions in [5] and stated in a more detailed form, which will be important here. We cite the essential part of the proof because some of the presented estimates will be used later.

Proof. For $g, \nabla f \in L_m^\infty$ and $f \in L_{m-1}^\infty$ with $m > 3$, from Lemma A.1, we get $u_0 = I_0(f, g) \in L_{1,m-1}^\infty$. Next, observe that if $u_n \in L_{1,p}^\infty$ with some $p > 1$, then

$$\|\langle x \rangle^k V u_n\|_{L_{1,p}^\infty} \leq \|\langle x \rangle^k V\|_{L^\infty} \|u_n\|_{L_{1,p}^\infty} = \|u_n\|_{L_{1,p}^\infty} < \infty$$

and from Lemma A.2 with $F \equiv V u_n$, we get $L_0(V u_n) \in L_{1,p}^\infty$ when $p \leq k$. Because $L_{1,p_1}^\infty \subset L_{1,p_2}^\infty$ when $p_1 \geq p_2$, we get $u_{n+1} \in L_{1,p}^\infty$ with $p \leq \min(m - 1, k)$.

By induction we obtain $u_n \in L_{1,p}^\infty$ for every $n = 0, 1, 2, \dots$ with the optimal value $p := \min(m - 1, k)$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|_{L_{1,p}^\infty} &= \lambda \|L_0(-V(u_n - u_{n-1}))\|_{L_{1,p}^\infty} \leq \lambda C_{p,k} \|\langle x \rangle^k V(u_n - u_{n-1})\|_{L_{1,p}^\infty} \\ &\leq \lambda C_{p,k} \|\langle x \rangle^k V\|_{L^\infty} \|u_n - u_{n-1}\|_{L_{1,p}^\infty} \leq \lambda C_{p,k} \|u_n - u_{n-1}\|_{L_{1,p}^\infty} \end{aligned}$$

again making use of Lemma A.2 with $\langle x \rangle^k F \equiv -\langle x \rangle^k V(u_n - u_{n-1}) \in L_{1,p}^\infty$. For $\delta := \lambda C_{p,k} < 1$, the iteration is a contraction in the normed space $L_{1,p}^\infty$. A simple argument shows that the sequence u_n is Cauchy. We have

$$\|u_{n+1} - u_n\|_{L_{1,p}^\infty} \leq \delta^{n+1} \|u_0 - u_{-1}\|_{L_{1,p}^\infty} = \delta^{n+1} \|I_0(f, g)\|_{L_{1,p}^\infty} \tag{2.4}$$

and for $n' > n$

$$\begin{aligned} \|u_{n'} - u_n\|_{L_{1,p}^\infty} &\leq \sum_{j=0}^{n'-n-1} \|u_{n'+j+1} - u_{n+j}\|_{L_{1,p}^\infty} \\ &\leq \sum_{j=0}^{n'-n-1} \delta^{j+n+1} \|I_0(f, g)\|_{L_{1,p}^\infty} \leq \frac{\delta^{n+1}}{1 - \delta} \|I_0(f, g)\|_{L_{1,p}^\infty}. \end{aligned} \tag{2.5}$$

This expression can be made arbitrarily small (smaller than any $\epsilon > 0$) for all $n, m > M(\epsilon)$. Hence, u_n is a Cauchy sequence in $L_{1,p}^\infty$ which is Banach and u_n has a limit $u \in L_{1,p}^\infty$ satisfying

$$u = I_0(f, g) - \lambda L_0(Vu). \tag{2.6}$$

This equation is equivalent to the wave equation (2.1) with the initial data (2.3). Finally, we find the $L_{1,p}^\infty$ -norm of u

$$\|u\|_{L_{1,p}^\infty} \leq \|I_0(f, g)\|_{L_{1,p}^\infty} + \lambda \|L_0(Vu)\|_{L_{1,p}^\infty} \leq C_m(f_0 + f_1 + g_0) + \lambda C_{p,k} \|u\|_{L_{1,p}^\infty},$$

thus

$$\|u\|_{L_{1,p}^\infty} \leq \frac{C_m(f_0 + f_1 + g_0)}{1 - \lambda C_{p,k}} \equiv C. \tag{2.7}$$

2.2. Perturbation series

Now, we define a perturbation series by

$$u = \sum_{n=0}^{\infty} \lambda^n v_n$$

and insert into the wave equation (2.1). It leads to the following perturbation scheme

$$\square v_0 = 0, \quad (v_0, \dot{v}_0)(0) = (f, g) \rightarrow v_0 = I_0(f, g), \tag{2.7}$$

$$\square v_{n+1} = -Vv_n, \quad (v_{n+1}, \dot{v}_{n+1})(0) = (0, 0) \rightarrow v_{n+1} = -L_0(Vv_n). \tag{2.8}$$

Due to linearity of (2.1), it turns out that the partial sums

$$\sum_{k=0}^n \lambda^k v_k = u_n$$

give the elements u_n obtained above by the iteration technique, so both methods (if they work) are equivalent. Theorem 2.1 implies convergence in $L_{1,p}^\infty$ for $\lambda < C_{p,k}^{-1}$ with $k > 2$, $m > 3$ and $p := \min(k, m - 1)$.

From (2.4) in the proof of Theorem 2.1, it follows that

$$\|v_n\|_{L_{1,p}^\infty} = \frac{\|u_n - u_{n-1}\|_{L_{1,p}^\infty}}{\lambda^n} \leq (C_{p,k})^n \|I_0(f, g)\|_{L_{1,p}^\infty},$$

hence $v_n \in L_{1,p}^\infty$ for all $n \geq 0$. Observe, however, that in the case when $m - 1 > k = p$, we have at the lowest order $v_0 = u_0$ a better decay estimate, namely $v_0 \in L_{1,m-1}^\infty$ (see first line of the proof). The reason that v_0 decays faster is that its decay comes only from initial data and is not influenced by the potential. At all higher orders, $v_n (n = 1, 2, \dots)$ contain the contribution from the scattering on the potential and are only in $L_{1,k}^\infty$. Since $u \in L_{1,p}^\infty = L_{1,k}^\infty$, we expect that all u_n starting from $u_1 \in L_{1,k}^\infty$ predict qualitatively correct asymptotic behavior of u while the lowest order $u_0 \in L_{1,m-1}^\infty$ fails in this. This becomes especially evident for initial data with compact support, for which $u_0 \in L_{1,q}^\infty$ with arbitrarily big q , but $u_1, u_2, \dots \in L_{1,k}^\infty \ni u$.

Knowing that the perturbation series converges for some λ , we can estimate the error of the n th perturbation's order relative to the exact solution by estimating the sum of all higher order terms. For the convergent sequence u_n we use the relation (2.5) which holds also in the limit $n' \rightarrow \infty$, $u_{n'} \rightarrow u$ and gives

$$\|u - u_n\|_{L_{1,p}^\infty} \leq \frac{\delta^{n+1}}{1 - \delta} \|I_0(f, g)\|_{L_{1,p}^\infty}.$$

It provides a pointwise bound on the error

$$|u(t, x) - u_n(t, x)| \leq \frac{(C_{p,k}\lambda)^{n+1}}{1 - C_{p,k}\lambda} \cdot \frac{C_m \cdot (f_0 + f_1 + g_0)}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-1}} \quad \forall (t, x) \in \mathbb{R}_+^{1+3}. \quad (2.9)$$

2.3. Optimal decay estimate

In this section, we sketch a proof how, under some conditions, the optimal decay estimate and precise asymptotic behavior of the solution u can be deduced from the behavior of low order perturbations. This will be studied in more detail in a forthcoming publication [6] dealing with spherical symmetry where the lowest perturbation orders can be calculated almost explicitly.

Consider first the case $m - 1 > k = p$, i.e. when the rate of decay of u is dominated by scattering on the potential (and not by decay of the initial data). We have $u_0 = v_0 \in L_{1,m-1}^\infty$ and $v_n \in L_{1,p}^\infty$ for $n \geq 1$. Below, we show that if the asymptotic behavior of v_1 is such as provided by its estimate (i.e. p in the norm $L_{1,p}^\infty$ is optimal), then Theorem 2.1 gives an optimal estimate for $u \in L_{1,p}^\infty$ with the same decay rate p . Here, we consider only the asymptotics in direction of timelike infinity (the case of spatial infinity can be treated similarly). Assume, we are able to show (by some explicit calculation, like in [6]) that $v_1(t, x) = L_0(Vv_0) \cong c_1(x)t^{-p} \neq 0$ for $t \gg 1$, where $c_1(x)$ is independent on λ . The approximation sign means that

for every small $\eta > 0$ and every $x \in \mathbb{R}^3$ there is a $T_0(x, \eta) > 0$ such that for all $t > T_0(x, \eta)$ the relative error is small, i.e.

$$\left| v_1(t, x) - \frac{c_1(x)}{t^p} \right| \leq \eta \frac{|c_1(x)|}{t^p}. \tag{2.10}$$

From (2.9) with $u_1 = v_0 + \lambda v_1$, we have

$$|u(t, x) - v_0(t, x) - \lambda v_1(t, x)| \leq \frac{(C_{p,k}\lambda)^2}{1 - C_{p,k}\lambda} \cdot \frac{C_m \cdot (f_0 + f_1 + g_0)}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-1}} =: \Delta_1(t, x)$$

for all $(t, x) \in \mathbb{R}_+^{1+3}$. A simple inequality^a

$$\frac{1}{(1 - \zeta)^\sigma} \leq \frac{1}{1 - \sigma\zeta} = 1 + \frac{\sigma\zeta}{1 - \sigma\zeta} \leq 2, \quad \forall \zeta \leq 1/(2\sigma), \quad \sigma > 1 \tag{2.11}$$

implies

$$\frac{1}{\langle t - |x| \rangle^q} = \frac{1}{(1 + t)^q \left(1 - \frac{|x|}{1+t}\right)^q} \leq \frac{2}{(1 + t)^q}$$

for $\zeta := |x|/(1 + t) \leq 1/(2q)$, hence is true for all $t \geq 2q|x|$. The error term can be estimated

$$\Delta_1(t, x) \leq 2(C_{p,k}\lambda)^2 \frac{2^2 C_m \cdot (f_0 + f_1 + g_0)}{(1 + t)^p} \equiv \tilde{C} \frac{\lambda^2}{(1 + t)^p},$$

where we have twice used (2.11) for $t \geq 2(p - 1)|x|$ and $\lambda \leq 1/(2C_{p,k})$. Further,

$$\tilde{C} \frac{\lambda^2}{(1 + t)^p} \leq \eta \lambda \frac{|c_1(x)|}{t^p}$$

provided λ is small enough such that $\lambda \leq \Lambda_\eta(x) := \eta c_1(x)/\tilde{C}$. Again from (2.11), we get

$$|v_0(t, x)| \leq \frac{c_0}{\langle t + |x| \rangle \langle t - |x| \rangle^{m-2}} \leq \frac{2c_0}{(1 + t)^{m-1}}$$

for all $t \geq 2(m - 2)|x|$. Then,

$$|v_0(t, x)| \leq \frac{2c_0}{(1 + t)^{m-1}} \leq \eta \lambda \frac{|c_1(x)|}{t^p}$$

provided $t > T_1(x, \eta, \lambda)$ is big enough, such that $t^{m-1-p} \geq 2c_0/(\eta \lambda |c_1(x)|)$.

Finally, we arrive at the statement that for every small $\eta > 0$ and every $x \in \mathbb{R}^3$, for sufficiently small $\lambda \leq \min[\Lambda_\eta(x), 1/(2C_{p,k})]$, and for sufficiently big $t >$

^aIt follows immediately from the Bernoulli's inequality.

$\max[T_0(x, \eta), T_1(x, \eta, \lambda), 2(m - 2)|x|]$, we have

$$\begin{aligned} \left| u(t, x) - \lambda \frac{c_1(x)}{t^p} \right| &\leq |u(t, x) - v_0(t, x) - \lambda v_1(t, x)| + |v_0(t, x)| + \lambda \left| v_1(t, x) - \frac{c_1(x)}{t^p} \right| \\ &\leq 3\eta\lambda \frac{|c_1(x)|}{t^p}, \end{aligned}$$

that is, for $p = k$,

$$u(t, x) \cong \lambda \frac{c_1(x)}{t^k}.$$

That gives a precise information about the time-decay of $u(t, x)$ and shows that the estimate in Theorem 2.1 is optimal (for $t \gg |x|$).

In the case $p = m - 1 \leq k$, the decay rate of u is determined by the decay of (long range) initial data and all $v_n \in L_{1,p}^\infty$. Analogously, if we can show that $v_0(t, x) \cong c_0(x)t^{-p} \neq 0$ for $t \gg 1$, then we can bound all higher perturbation orders for sufficiently small λ and big t by the same expression multiplied by an arbitrarily small η . To this aim we use again (2.9)

$$|u(t, x) - v_0(t, x)| \leq \frac{C_{p,k}\lambda}{1 - C_{p,k}\lambda} \cdot \frac{C_m \cdot (f_0 + f_1 + g_0)}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-1}} =: \Delta_0(t, x) \quad \forall (t, x) \in \mathbb{R}_+^{1+3},$$

and bound $\Delta_0(t, x)$ by $\eta|c_0(x)|t^{-p}$ as above. It leads to

$$\left| u(t, x) - \frac{c_0(x)}{t^p} \right| \leq |u(t, x) - v_0(t, x)| + \left| v_0(t, x) - \frac{c_0(x)}{t^p} \right| \leq 2\eta \frac{|c_0(x)|}{t^p},$$

what for $p = m - 1$ gives

$$u(t, x) \cong \frac{c_0(x)}{t^{m-1}}.$$

That again gives a precise information about the time-decay of $u(t, x)$ and shows that the estimate in Theorem 2.1 is optimal (for $t \gg |x|$).

3. Nonlinear Case without the Potential Term

Now, we consider a nonlinear wave equation of the form

$$\square u = F(u) \tag{3.1}$$

subject to initial data (f, g) satisfying (2.3) with $f_0, f_1, g_0 < \varepsilon$. The nonlinear term obeys $|F(u)| \leq F_1|u|^p$ for $|u| < 1$ and $|F(u) - F(v)| \leq F_2|u - v| \max(|u|, |v|)^{p-1}$. The second condition is satisfied e.g. for $F(u) = u^p$ with $F_2 = p$ or for $F \in \mathcal{C}^1$ such that $|F'(u)| \leq F_2|u|^{p-1}$ for $|u| < 1$.

3.1. Iteration

We define an iteration scheme

$$\begin{aligned} u_0 &:= 0, \\ u_{n+1} &:= I_0(f, g) + L_0(F(u_n)), \quad n \geq 0. \end{aligned} \tag{3.2}$$

For it we have the following

Theorem 3.1. *With f, g and $F(u)$ as above for any $m > 3$, $p > 1 + \sqrt{2}$ and sufficiently small ε the sequence u_n converges (in norm) in $L_{1,q}^\infty$ for $q = \min(p - 1, m - 1)$ to the solution u of Eq. (3.1). The limit $u := \lim_{n \rightarrow \infty} u_n$ satisfies*

$$|u(t, x)| \leq \frac{C}{\langle t + |x| \rangle \langle t - |x| \rangle^{q-1}}, \quad \forall (t, x) \in \mathbb{R}_+^{1+3}$$

with some positive constant C depending only on p, m and ε .

Proof. For $g, \nabla f \in L_m^\infty$ and $f \in L_{m-1}^\infty$ with $m > 3$ from Lemma A.1, we get $u_1 = I_0(f, g) \in L_{1,m-1}^\infty$. Next, if some $u_n \in L_{1,q}^\infty$ with some $q > 1$ then, since L_0 is a positive operator,^b we have $|L_0(F(u_n))| \leq F_1 L_0(|u_n|^p)$ and from Lemma A.3, we get

$$\|L_0(F(u_n))\|_{L_{1,q}^\infty} \leq F_1 \|L_0(|u_n|^p)\|_{L_{1,q}^\infty} \leq F_1 C \|u_n\|_{L_{1,q}^\infty}^p,$$

and hence $L_0(F(u_n)) \in L_{1,q}^\infty$ when $q \leq p - 1$. Then, $u_{n+1} \in L_{1,m-1}^\infty \cap L_{1,q}^\infty = L_{1,q}^\infty$ for $q := \min(m - 1, p - 1)$. Hence, by induction we obtain $u_n \in L_{1,q}^\infty$ for every $n = 0, 1, 2, \dots$ and

$$\|u_1\|_{L_{1,q}^\infty} \leq \|I_0(f, g)\|_{L_{1,q}^\infty} \leq C_m(f_0 + f_1 + g_0) \leq 3C_m\varepsilon$$

$$\|u_{n+1}\|_{L_{1,q}^\infty} \leq \|I_0(f, g)\|_{L_{1,q}^\infty} + \|L_0(F(u_n))\|_{L_{1,q}^\infty} \leq 3C_m\varepsilon + F_1 C \|u_n\|_{L_{1,q}^\infty}^p.$$

Choose $\varepsilon > 0$ such that $F_1 C (6C_m)^p \varepsilon^{p-1} < 3C_m \cdot \min(1, 2F_1/F_2)$. Then,

$$\|u_1\|_{L_{1,q}^\infty} \leq 6C_m\varepsilon$$

$$\|u_n\|_{L_{1,q}^\infty} \leq 6C_m\varepsilon \Rightarrow \|u_{n+1}\|_{L_{1,q}^\infty} \leq 3C_m\varepsilon + F_1 C (6C_m\varepsilon)^p \leq 6C_m\varepsilon,$$

hence $\|u_n\|_{L_{1,q}^\infty} \leq 6C_m\varepsilon$ for all $n \geq 1$. As next, we show convergence of the sequence u_n by demonstrating that it is Cauchy.

$$\begin{aligned} \|u_{n+1} - u_n\|_{L_{1,q}^\infty} &= \|L_0(F(u_n) - F(u_{n-1}))\|_{L_{1,q}^\infty} \\ &\leq F_2 \|L_0(|u_n - u_{n-1}| \max(|u_n|, |u_{n-1}|)^{p-1})\|_{L_{1,q}^\infty} \\ &\leq F_2 C \| |u_n - u_{n-1}| \max(|u_n|, |u_{n-1}|)^{p-1} \|_{L_{1,q}^\infty} \\ &\leq F_2 C (6C_m\varepsilon)^{p-1} \|u_n - u_{n-1}\|_{L_{1,q}^\infty} = \delta \|u_n - u_{n-1}\|_{L_{1,q}^\infty}, \end{aligned}$$

^bIn fact $L_0 = \square^{-1}$ is a measure on \mathbb{R}_+^{1+3} and therefore has a positive kernel. Then, $L_0(F) \geq 0$ if $F \geq 0$.

with $\delta := F_2 C(6C_m \varepsilon)^{p-1} < 1$, hence the iteration is a contraction in the normed space $L_{1,q}^\infty$ and u_n is a Cauchy sequence, because

$$\|u_{n+1} - u_n\|_{L_{1,q}^\infty} \leq \delta^n \|u_1 - u_0\|_{L_{1,q}^\infty} = \delta^n \|I_0(f, g)\|_{L_{1,q}^\infty} \leq \delta^n 3C_m \varepsilon \tag{3.3}$$

and for any $n' > n$

$$\begin{aligned} \|u_{n'} - u_n\|_{L_{1,q}^\infty} &\leq \sum_{j=0}^{n'-n-1} \|u_{n+j+1} - u_{n+j}\|_{L_{1,q}^\infty} \\ &\leq \sum_{j=0}^{n'-n-1} \delta^{j+n} 3C_m \varepsilon \leq \frac{\delta^n}{1-\delta} 3C_m \varepsilon. \end{aligned} \tag{3.4}$$

Since $L_{1,q}^\infty$ is Banach, u_n has a limit $u \in L_{1,q}^\infty$ satisfying

$$u = I_0(f, g) + L_0(F(u)) \tag{3.5}$$

and solving the wave equation (3.1) with the initial data (2.3). Its $L_{1,q}^\infty$ -norm satisfies

$$\|u\|_{L_{1,q}^\infty} \leq 6C_m \varepsilon. \tag{3.6}$$

□

From (3.4) it follows, in the limit $n' \rightarrow \infty$ an error bound

$$\|u - u_n\|_{L_{1,q}^\infty} \leq \frac{\delta^n}{1-\delta} 3C_m \varepsilon \leq C\varepsilon^{(p-1)n+1} \tag{3.7}$$

for small ε .

3.2. Perturbation series

In order to be able to construct a well-defined perturbation scheme to all orders, we have to assume additionally that $F(u)$ is analytic at $u = 0$, its Taylor series starts at power $p \geq 3$ and has convergence radius $R_F > 0$. Then, for small initial data

$$(u, \dot{u})(0) = (\varepsilon f, \varepsilon g), \tag{3.8}$$

we introduce a perturbation series for representing the solution of (3.1)

$$u = \sum_{n=1}^{\infty} \varepsilon^n v_n. \tag{3.9}$$

After inserting it into (3.1) and collecting terms according to powers of ε , we obtain the following perturbation scheme

$$\square v_1 = 0, \quad (v_1, \dot{v}_1)(0) = (f, g) \rightarrow v_1 = I_0(f, g) \tag{3.10}$$

$$\square v_{n+1} = F_n(v_1, \dots, v_n), \quad (v_{n+1}, \dot{v}_{n+1})(0) = (0, 0) \rightarrow v_{n+1} = L_0(F_n(v_1, \dots, v_n)), \tag{3.11}$$

for $n \geq 1$, where F_n result from collecting the nonlinear terms with the same powers of ε

$$F_n(v_1, \dots, v_n) = \sum_k a_k^n v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}, \tag{3.12}$$

where $\alpha_k^{n,m} \in \mathbb{N}$ satisfy $\sum_{m=1}^n m \alpha_k^{n,m} = n + 1$ and $\sum_{m=1}^n \alpha_k^{n,m} \geq p$ for every n, k .

We call this expansion a “zero background” case because the zero-order term v_0 is absent. If a v_0 term were present in the series above (i.e. the summation started at $n = 0$), we would have an additional equation $\square v_0 = F(v_0)$ which is truly nonlinear (opposite to the above system of linear wave equations with source terms). Its solution v_0 represents a “background” around which the perturbations v_n are calculated.

Below we show that the perturbation series converges to the solution u of the nonlinear wave equation (3.1) and has a positive convergence radius.

Theorem 3.2. *With f, g and $F(u)$ as above for any $m > 3$, $p > 1 + \sqrt{2}$ and sufficiently small ε the series defined in (3.9)–(3.11) converges (in norm) in $L_{1,q}^\infty$ for $q = \min(p - 1, m - 1)$ to the solution of Eq. (3.1) with initial data (3.8).*

Proof. For $g, \nabla f \in L_m^\infty$ and $f \in L_{m-1}^\infty$ with $m > 3$ from Lemma A.1, we get $v_1 = I_0(f, g) \in L_{1,m-1}^\infty$ with

$$\|v_1\|_{L_{1,m-1}^\infty} \leq C_m(f_0 + f_1 + g_0) =: D < \infty.$$

Next, we prove by induction $L_{1,q}^\infty$ bounds for all $n \geq 1$ with some $q > 1$. Assume that for a given $n \geq 1$ we have $v_m \in L_{1,q}^\infty$ for all $m \leq n$. Then, using (3.12), we get

$$\|v_{n+1}\|_{L_{1,q}^\infty} = \|L_0(F_n(v_1, \dots, v_n))\|_{L_{1,q}^\infty} \leq \sum_k |a_k^n| \cdot \|L_0(|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}|)\|_{L_{1,q}^\infty}.$$

Observe, that $|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}| = (\sqrt[p]{|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}|})^p$ and $\sqrt[p]{|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}|} \in L_{1,q}^\infty$, because of the following estimate for $b_1 + \dots + b_n = B \geq 1$ and $b_m \geq 0, m = 1, \dots, n$

$$\begin{aligned} \|w_1^{b_1} \dots w_n^{b_n}\|_{L_{1,q}^\infty} &= \|\langle t + |x| \rangle \langle t - |x| \rangle^{q-1} w_1^{b_1} \dots w_n^{b_n}\|_{L^\infty} \\ &\leq \|\langle t + |x| \rangle^{b_1/B} \langle t - |x| \rangle^{(q-1)b_1/B} w_1^{b_1}\|_{L^\infty} \dots \\ &\quad \times \|\langle t + |x| \rangle^{b_n/B} \langle t - |x| \rangle^{(q-1)b_n/B} w_n^{b_n}\|_{L^\infty} \\ &= \|\langle t + |x| \rangle^{1/B} \langle t - |x| \rangle^{(q-1)/B} w_1\|_{L^\infty}^{b_1} \dots \\ &\quad \times \|\langle t + |x| \rangle^{1/B} \langle t - |x| \rangle^{(q-1)/B} w_n\|_{L^\infty}^{b_n} \\ &\leq \|\langle t + |x| \rangle \langle t - |x| \rangle^{(q-1)} w_1\|_{L^\infty}^{b_1} \dots \|\langle t + |x| \rangle \langle t - |x| \rangle^{(q-1)} w_n\|_{L^\infty}^{b_n} \\ &= \|w_1\|_{L_{1,q}^\infty}^{b_1} \dots \|w_n\|_{L_{1,q}^\infty}^{b_n}, \end{aligned}$$

which used for $b_m := \alpha_k^{n,m}/p$ with $B := \alpha_k^{n,1}/p + \dots + \alpha_k^{n,n}/p \geq 1$ gives

$$\begin{aligned} \left\| \sqrt[p]{|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}|} \right\|_{L_{1,q}^\infty} &= \left\| |v_1|^{\alpha_k^{n,1}/p} \dots |v_n|^{\alpha_k^{n,n}/p} \right\|_{L_{1,q}^\infty} \\ &\leq \|v_1\|_{L_{1,q}^\infty}^{\alpha_k^{n,1}/p} \dots \|v_n\|_{L_{1,q}^\infty}^{\alpha_k^{n,n}/p} < \infty. \end{aligned}$$

Then, for $q \leq p - 1$, we can use Lemma A.3 with $u := \sqrt[p]{|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}|}$ to obtain

$$\begin{aligned} \|v_{n+1}\|_{L_{1,q}^\infty} &\leq C \sum_k |a_k^n| \cdot \left\| \sqrt[p]{|v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}|} \right\|_{L_{1,q}^\infty}^p = C \sum_k |a_k^n| \cdot \left\| v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}} \right\|_{L_{1,q}^\infty} \\ &\leq C \sum_k |a_k^n| \cdot \|v_1\|_{L_{1,q}^\infty}^{\alpha_k^{n,1}} \dots \|v_n\|_{L_{1,q}^\infty}^{\alpha_k^{n,n}}. \end{aligned}$$

Unfortunately, we were not able to find an estimate for $\sum_k |a_k^n|$ being good enough to prove a geometric growth of $\|v_n\|_{L_{1,q}^\infty}$ and guaranteeing convergence of the series (3.9). If one tries so, e.g. assuming $\|v_m\|_{L_{1,q}^\infty} \leq D^m$ for all $m \leq n$, then

$$\begin{aligned} \|v_{n+1}\|_{L_{1,q}^\infty} &\leq C \sum_k |a_k^n| \cdot \|v_1\|_{L_{1,q}^\infty}^{\alpha_k^{n,1}} \dots \|v_n\|_{L_{1,q}^\infty}^{\alpha_k^{n,n}} \leq C \sum_k |a_k^n| \cdot D^{(1\alpha_k^{n,1} + \dots + n\alpha_k^{n,n})} \\ &= CD^{n+1} \sum_k |a_k^n|. \end{aligned}$$

The best estimate we were able to find is $\sum_k |a_k^n| \leq \tilde{C}n^p$ (imposing further assumptions on $F(u)$) which does not allow to close the induction argument. Therefore, we choose a different way and use some trick, relating the wave equation to an algebraic one.

To this goal, we need to relate the coefficients of the power series for $F(u)$

$$F(u) = \sum_{n=p}^\infty b_n u^n,$$

which converges for $|u| < R_F$, to the expansion coefficients a_k^n which result from a formal insertion of the series $u = \sum_{k=1}^\infty \varepsilon^k v_k$ into $F(u)$

$$\begin{aligned} F\left(\sum_{k=1}^\infty \varepsilon^k v_k\right) &\equiv \sum_{n=p-1}^\infty \varepsilon^{n+1} F_n(v_1, \dots, v_n) \\ &\equiv \sum_{n=p-1}^\infty \varepsilon^{n+1} \sum_k a_k^n v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}}. \end{aligned} \tag{3.13}$$

By some manipulation of sums, we obtain

$$a_k^n = b_{\alpha_k^{n,1} + \dots + \alpha_k^{n,n}} \left(\begin{matrix} \alpha_k^{n,1} + \dots + \alpha_k^{n,n} \\ \alpha_k^{n,1}, \dots, \alpha_k^{n,n} \end{matrix} \right),$$

where the symbol in delimiters represents the multinomial coefficient. Since there is an analogous relation between the absolute values of the coefficients

$$|a_k^n| = |b_{\alpha_k^{n,1} + \dots + \alpha_k^{n,n}}| \binom{\alpha_k^{n,1} + \dots + \alpha_k^{n,n}}{\alpha_k^{n,1}, \dots, \alpha_k^{n,n}}$$

we observe that the series (3.13) with a_k^n replaced by $|a_k^n|$ gives rise to a new function \tilde{F}

$$\sum_{n=p-1}^{\infty} \varepsilon^{n+1} \sum_k |a_k^n| v_1^{\alpha_k^{n,1}} \dots v_n^{\alpha_k^{n,n}} = \tilde{F} \left(\sum_{k=1}^{\infty} \varepsilon^k v_k \right)$$

such that

$$\tilde{F}(u) = \sum_{n=p}^{\infty} |b_n| u^n. \tag{3.14}$$

$\tilde{F}(u)$ is also analytic at $u = 0$ and the convergence radius is the same as that of $F(u)$, i.e. $R_{\tilde{F}} = R_F$ what follows from standard theory of analytic functions.

Now, instead of the system of estimates

$$\|v_1\|_{L_{1,q}^{\infty}} \leq D, \tag{3.15}$$

$$\|v_{n+1}\|_{L_{1,q}^{\infty}} \leq C \sum_k |a_k^n| \cdot \|v_1\|_{L_{1,q}^{\infty}}^{\alpha_k^{n,1}} \dots \|v_n\|_{L_{1,q}^{\infty}}^{\alpha_k^{n,n}} \tag{3.16}$$

with $q := \min(m - 1, p - 1)$, we consider a system of equations

$$w_1 = D, \tag{3.17}$$

$$w_{n+1} = C \sum_k |a_k^n| \cdot w_1^{\alpha_k^{n,1}} \dots w_n^{\alpha_k^{n,n}} \tag{3.18}$$

and it is easy to see (e.g. by induction) that $\|v_n\|_{L_{1,q}^{\infty}} \leq w_n$ for all $n \geq 1$. Now comes the trick. Using the above relations we can find that this system is equivalent to

$$\sum_{n=1}^{\infty} \varepsilon^n w_n = C\tilde{F} \left(\sum_{n=1}^{\infty} \varepsilon^n w_n \right) + D\varepsilon. \tag{3.19}$$

Introducing $W = \sum_{n=1}^{\infty} \varepsilon^n w_n$, we can write

$$W = C\tilde{F}(W) + D\varepsilon. \tag{3.20}$$

Since $\tilde{F}(W)$ is analytic at $W = 0$, so is $G(W)$

$$G(W) := \frac{W - C\tilde{F}}{D} = \varepsilon$$

and also its inverse $G^{-1}(\varepsilon)$ at $\varepsilon = 0$, because $G'(0) = 1/D > 0$ (see e.g. (real) analytic inverse function theorem in [2]), what follows from the fact that the Taylor series for \tilde{F} starts (as that for F) at the power at least $p > 2$. Then $G^{-1}(\varepsilon)$ has a

Taylor series with a positive convergence radius $R_{G^{-1}} > 0$. The solution $W(\varepsilon)$ of (3.20) can be then represented by a convergent series for $|\varepsilon| < R_{G^{-1}}$

$$W(\varepsilon) = G^{-1}(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n w_n. \tag{3.21}$$

In order to guarantee that this series can act as a good argument of \tilde{F} , we choose a possibly smaller radius $\tilde{R} \leq R_{G^{-1}}$ such that $|W(\varepsilon)| < R_F$ for all $|\varepsilon| < \tilde{R}$. Then $\tilde{F}(W(\varepsilon))$ can be represented by a convergent series (3.14) in $W(\varepsilon)$. Finally, this allows us to insert this series into (3.20) and obtain first (3.19) and then the system (3.17) and (3.18).

Essential for the trick is that the series in (3.21) converges for all $|\varepsilon| < \tilde{R}$. Now, since $\|v_n\|_{L_{1,q}^\infty} \leq w_n$ for all $n \geq 1$, we get from the comparison criterion that the series $\sum_{n=1}^{\infty} \varepsilon^n \|v_n\|_{L_{1,q}^\infty}$ converges as well for all $|\varepsilon| < \tilde{R}$. Thus, the series (3.9) converges in norm in $L_{1,q}^\infty$ for all $|\varepsilon| < \tilde{R}$ to some $\tilde{u} \in L_{1,q}^\infty$ which satisfies

$$\begin{aligned} \tilde{u} &:= \sum_{n=1}^{\infty} \varepsilon^n v_n = \varepsilon I_0(f, g) + \sum_{n=1}^{\infty} \varepsilon^{n+1} L_0(F_n(v_1, \dots, v_n)) \\ &= \varepsilon I_0(f, g) + L_0\left(F\left(\sum_{n=1}^{\infty} \varepsilon^n v_n\right)\right) \\ &= I_0(\varepsilon f, \varepsilon g) + L_0(F(\tilde{u})) \end{aligned}$$

what is equivalent to the wave equation (3.1) with initial data (3.8). Uniqueness of solutions follows easily from Theorem 3.1.

An important consequence of the convergence of $\sum_{n=1}^{\infty} \varepsilon^n \|v_n\|_{L_{1,q}^\infty}$ is that there exist constants $0 < M < \infty$ and $\tilde{R}^{-1} < \rho < \infty$ such that $\|v_n\|_{L_{1,q}^\infty} < M\rho^n$ for every $n \geq 1$. □

Since the introduction of the auxiliary parameter ε in the series expansion is only a way to generate the system of linear equations equivalent to the original nonlinear equation, we can now remove the parameter ε and replace the condition on the initial data by requiring $f_0, f_1, g_0 < \tilde{R}$. If we solve the system (3.10)–(3.11), then we obtain a solution of the nonlinear wave equation (3.1) by summing up the convergent series $\sum_{n=1}^{\infty} v_n = u$.

3.3. Optimal decay estimate

In the nonlinear case, the iteration sequence u_n is different than the perturbation sequence $\tilde{u}_n := \sum_{m=1}^n v_m$, therefore, the question whether information about the decay rate of u can be read-off from the low order terms must be studied separately for both cases. On the one hand, in the iterative scheme the form of the source

terms $F(u_n)$ is much simpler than that in the perturbative scheme, $F_n(v_1, \dots, v_n)$. On the other hand, in practise, it is much easier to calculate v_n 's than u_n 's. Below, we address both situations.

Analogously like for the linear equation, we will have two cases depending on whether m is smaller or bigger than p . In the first case, the initial data will dominate the late-time decay rate of u , in the second case the power p of the nonlinearity, through nonlinear scattering, will determine the the decay rate of u .

3.3.1. Iteration

In analogy to the linear case, basing on a decay information for some low order term in the iteration sequence and on its error bound we find the exact decay rate of u . From the error bound (3.7), it follows for large t

$$|u(t, x) - u_n(t, x)| \leq \frac{C\varepsilon^{(p-1)n+1}}{\langle t + |x| \rangle \langle t - |x| \rangle^{q-1}} \leq \frac{c_n(x)\varepsilon^{(p-1)n+1}}{(1+t)^q},$$

where $q := \min(p-1, m-1)$. If we are able to show that some $u_n(t, x) \cong \varepsilon d_n(x)t^{-q} \neq 0$ for large t (the asymptotic approximation is to be understood in the following sense:

$$\exists \eta_0 > 0 \forall 0 < \eta < \eta_0 \exists T < \infty \forall t > T \left| u_n(t, x) - \frac{\varepsilon d_n(x)}{t^q} \right| < \eta \frac{\varepsilon d_n(x)}{t^q}, \tag{3.22}$$

i.e. the relative error η becomes arbitrarily small for sufficiently big t , cf. (2.10)), then already u_n shows the correct decay rate, identical with this of u , because then, choosing $\eta := \varepsilon^{(p-1)n}$, we get

$$\left| u(t, x) - \varepsilon \frac{d_n(x)}{t^q} \right| \leq |u(t, x) - u_n(t, x)| + \left| u_n(t, x) - \varepsilon \frac{d_n(x)}{t^q} \right| \leq \eta \varepsilon \frac{c_n(x)}{t^q} + \eta \varepsilon \frac{d_n(x)}{t^q}$$

for sufficiently small ε . Hence, the decay rate of u at late times is exactly t^{-q} .

In case when $m > p = q + 1$, we have $u_1 = I_0(f, g) \in L_{1, m-1}^\infty$ and

$$|u_1(t, x)| \leq \frac{C_m \cdot (f_0 + f_1 + g_0)}{\langle t + |x| \rangle \langle t - |x| \rangle^{m-2}} \leq \frac{2C_m \cdot (f_0 + f_1 + g_0)}{(1+t)^{m-1}}$$

for $t \geq 2(m-2)|x|$, hence it decays faster than u and it cannot be shown that $u_1(t, x) \cong \varepsilon d_1(x)t^{-q}$. It is expected that it will be true for $u_2 \cong \varepsilon d_2(x)t^{-q}$, what means, that already u_2 would have the same rate of decay as u (see [6] for such results in spherical symmetry).

In case when $m \leq p$, we have $q := m - 1$. Then it should be possible to show $u_1(t, x) \cong c_1(x)t^{-(m-1)}$, what means, that already u_1 would have the same rate of decay as u .

3.3.2. Perturbation series

The perturbation scheme (3.10)–(3.11) can be written as

$$v_1 = I_0(f, g) \tag{3.23}$$

$$v_2 = v_3 = \dots = v_{p-1} = 0 \tag{3.24}$$

$$v_p = L_0(F_{p-1}(v_1, \dots, v_{p-1})) = a_0 L_0((v_1)^p) \tag{3.25}$$

$$v_{n+1} = L_0(F_n(v_1, \dots, v_n)), \quad n \geq p. \tag{3.26}$$

Assume we are in the more interesting case $m > p$ where the tail results from the nonlinear scattering. Then, $v_1 = I_0(f, g) \in L_{1, m-1}^\infty$ and $v_n \in L_{1, p-1}^\infty$ for $n \geq 2$. If we can show that $a_0 L_0((I(f, g))^p) \cong d_p(x)t^{-(p-1)}$, then already v_p has the correct decay rate, identical with this of u . To prove it, we need to show that $\varepsilon I_0(f, g)$ and $\varepsilon^{n+1} L_0(F_n(v_1, \dots, v_n))$ for $n \geq p$ are small relative to $\varepsilon^p d_p(x)t^{-(p-1)}$.

Again, for $v_1 = I_0(f, g) \in L_{1, m-1}^\infty$ the situation is obvious, $\varepsilon|v_1| = \varepsilon|I_0(f, g)(t, x)| \leq c_1(x)\varepsilon(1+t)^{-(m-1)}$ and it is much smaller than $\varepsilon^p d_p(x)t^{-(p-1)}$ for sufficiently large t .

From the convergence proof for the perturbation series, we know that there exist constants $M, \rho > 0$ such that $\|v_n\|_{L_{1, p-1}^\infty} \leq M\rho^n$ for all $n \geq 1$. Hence, we can estimate the remainder of the perturbation series

$$\left\| \sum_{m=p+1}^\infty \varepsilon^m v_m \right\|_{L_{1, p-1}^\infty} \leq M \sum_{m=p+1}^\infty \varepsilon^m \rho^m \leq \frac{M\varepsilon^{p+1}\rho^{p+1}}{1-\varepsilon\rho} \leq C\varepsilon^{p+1},$$

for sufficiently small $\varepsilon < 1/\rho$. It means that

$$\left| \sum_{m=p+1}^\infty \varepsilon^m v_m(t, x) \right| \leq \frac{C\varepsilon^{p+1}}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-2}} \leq \frac{c(x)\varepsilon^{p+1}}{(1+t)^{p-1}}$$

for big t (and fixed x). Then

$$|u(t, x) - \varepsilon^p v_p(t, x)| \leq |\varepsilon v_1(t, x)| + \left| \sum_{m=p+1}^\infty \varepsilon^m v_m \right| \leq \frac{c_1(x)\varepsilon}{t^{m-1}} + \frac{c(x)\varepsilon^{p+1}}{t^{p-1}},$$

hence with $v_p \cong d_p(x)t^{-(p-1)}$ and the relative error $\eta := \varepsilon$ (cf. (3.22) for the definition of “ \cong ”)

$$\left| u(t, x) - \frac{d_p(x)\varepsilon^p}{t^{p-1}} \right| \leq \frac{c_1(x)\varepsilon}{t^{m-1}} + \frac{[c(x) + d_p(x)]\varepsilon^{p+1}}{t^{p-1}}.$$

For small ε and big t such that $t^{m-p} \geq \frac{c_1(x)}{c(x)}\varepsilon^{-p}$, it follows

$$u(t, x) \cong \frac{d_p(x)\varepsilon^p}{t^{p-1}}.$$

Thus, v_p dominates the perturbation series for large times and small ε and has the same decay rate as the full solution of the nonlinear wave equation u .

4. Nonlinear Case with the Potential Term

Finally, let us consider a nonlinear wave equation with potential

$$\square u + Vu = F(u) \tag{4.1}$$

subject to initial data (f, g) satisfying (2.3) with $f_0, f_1, g_0 < \varepsilon$. The nonlinear term $F(u)$ is like in the previous section.

4.1. Iteration

4.1.1. Perturbative treatment of V

As in the previous sections, we define an iteration

$$\begin{aligned} u_0 &:= 0 \\ u_{n+1} &:= I_0(f, g) - \lambda L_0(Vu_n) + L_0(F(u_n)), \quad n \geq 0. \end{aligned}$$

We have the following

Theorem 4.1. *With f, g, V and $F(u)$ as above for any $m > 3, k > 2, p > 1 + \sqrt{2}, \lambda < C_{q,k}^{-1}$ and sufficiently small ε the sequence u_n converges (in norm) in $L_{1,q}^\infty$ for $q = \min(p - 1, k, m - 1)$ to the solution u of Eq. (4.1). The limit $u := \lim_{n \rightarrow \infty} u_n$ satisfies*

$$|u(t, x)| \leq \frac{C}{\langle t + |x| \rangle \langle t - |x| \rangle^{q-1}}, \quad \forall (t, x) \in \mathbb{R}_+^{1+3}$$

with some positive constant C depending only on p, k, m, λ and ε .

The proof is a combination of proofs of Theorems 3.1 and 3.2, therefore, we concentrate only on the points that differ.

Proof. For $g, \nabla f \in L_m^\infty$ and $f \in L_{m-1}^\infty$ with $m > 3$ from Lemma A.1, we get $u_1 = I_0(f, g) \in L_{1,m-1}^\infty$. Next, for $\delta := \lambda C_{q,k} < 1$, there exists $M > 3/(1 - \delta) > 0$ and if $u_n \in L_{1,q}^\infty$ with $\|u_n\|_{L_{1,q}^\infty} \leq MC_m \varepsilon$ for some $n \geq 1$ and $q > 1$, then from Lemmas A.1–A.3, we get

$$\begin{aligned} \|u_{n+1}\|_{L_{1,q}^\infty} &\leq \|I_0(f, g)\|_{L_{1,q}^\infty} + \lambda \|Vu_n\|_{L_{1,q}^\infty} + \|L_0(F(u_n))\|_{L_{1,q}^\infty} \\ &\leq C_m(f_0 + f_1 + g_0) + \lambda C_{q,k} \|u_n\|_{L_{1,q}^\infty} + F_1 C \|u_n\|_{L_{1,q}^\infty}^p \\ &\leq 3C_m \varepsilon + \delta MC_m \varepsilon + F_1 C (MC_m)^p \varepsilon^p < \infty \end{aligned}$$

and hence $u_{n+1} \in L_{1,q}^\infty$ if $q := \min(m - 1, k, p - 1)$. By induction we obtain $u_n \in L_{1,q}^\infty$ for every $n \geq 1$. For $\varepsilon > 0$ such that $F_1 C (MC_m)^p \varepsilon^{p-1} \leq \min[(M(1 - \delta) - 3)C_m, M\delta(1 - \delta)C_m F_1/F_2]$, we have

$$\begin{aligned} \|u_1\|_{L_{1,q}^\infty} &\leq 3C_m \varepsilon \leq MC_m \varepsilon \\ \|u_{n+1}\|_{L_{1,q}^\infty} &\leq 3C_m \varepsilon + \delta MC_m \varepsilon + (M(1 - \delta) - 3)C_m \varepsilon = MC_m \varepsilon, \end{aligned}$$

hence $\|u_n\|_{L_{1,q}^\infty} \leq MC_m\varepsilon$ for all $n \geq 1$. Analogously like in the previous proofs, we arrive at

$$\begin{aligned} \|u_{n+1} - u_n\|_{L_{1,q}^\infty} &\leq \|L_0(V(u_n - u_{n-1}))\|_{L_{1,q}^\infty} + \|L_0(F(u_n) - F(u_{n-1}))\|_{L_{1,q}^\infty} \\ &\leq \lambda C_{q,k} \|u_n - u_{n-1}\|_{L_{1,q}^\infty} + F_2 C(MC_m\varepsilon)^{p-1} \|u_n - u_{n-1}\|_{L_{1,q}^\infty} \\ &\leq \delta \|u_n - u_{n-1}\|_{L_{1,q}^\infty} + \delta(1 - \delta) \|u_n - u_{n-1}\|_{L_{1,q}^\infty} \\ &= \delta' \|u_n - u_{n-1}\|_{L_{1,q}^\infty}, \end{aligned}$$

where $\delta' := 2\delta - \delta^2 < 1$. It follows that u_n is a Cauchy sequence (see the above proofs) in the Banach space $L_{1,q}^\infty$ and hence u_n has a limit $u \in L_{1,q}^\infty$ satisfying

$$u = I_0(f, g) + L_0(Vu_n) + L_0(F(u)) \tag{4.2}$$

and solving the wave equation (4.1) with the initial data (2.3). Its $L_{1,q}^\infty$ -norm satisfies

$$\|u\|_{L_{1,q}^\infty} \leq MC_m\varepsilon \tag{4.3}$$

with some (finite) constant $M > 3/(1 - \delta) > 0$. □

Moreover, by analogous considerations like in the proof of Theorem 3.1, we find for $n' > n$

$$\|u_{n'} - u_n\|_{L_{1,q}^\infty} \leq \frac{\delta'^{n'}}{1 - \delta'} 3C_m\varepsilon,$$

and in the limit $n' \rightarrow \infty$

$$\|u - u_n\|_{L_{1,q}^\infty} \leq \frac{\delta'^n}{1 - \delta'} 3C_m\varepsilon. \tag{4.4}$$

4.1.2. Non-perturbative treatment of V

Building on the above results we can also define an alternative iteration scheme

$$\begin{aligned} u_0 &:= 0 \\ u_{n+1} &:= I_V(f, g) + L_V(F(u_n)), \quad n \geq 0 \end{aligned}$$

which is based on inversion of the operator $\square + \lambda V$. According to the discussion in the introduction, it is equivalent to

$$u_{n+1} = I_0(f, g) + L_0(F(u_n)) - \lambda L_0(Vu_{n+1}), \quad n \geq 0.$$

It converges under the same conditions as in Theorem 4.1. The proof has the only difference that now we have

$$\begin{aligned} \|u_{n+1}\|_{L_{1,q}^\infty} &\leq \|I_0(f, g)\|_{L_{1,q}^\infty} + \|L_0(F(u_n))\|_{L_{1,q}^\infty} + \lambda \|Vu_{n+1}\|_{L_{1,q}^\infty} \\ &\leq C_m(f_0 + f_1 + g_0) + F_1 C \|u_n\|_{L_{1,q}^\infty}^p + \lambda C_{q,k} \|u_{n+1}\|_{L_{1,q}^\infty} \\ &\leq (1 - \delta) MC_m\varepsilon + \delta \|u_{n+1}\|_{L_{1,q}^\infty} \end{aligned}$$

what gives

$$\|u_{n+1}\|_{L_{1,q}^\infty} \leq \frac{(1-\delta)MC_m\varepsilon}{1-\delta} = MC_m\varepsilon.$$

4.2. Perturbation series

Definig a perturbation scheme for the nonlinear wave equation with potential (4.1)

$$u = \sum_{n=1}^\infty \varepsilon^n v_n, \tag{4.5}$$

one encounters the problem of two scales which are introduced by parameters λ measuring the strength of the potential and ε measuring the strength of the initial data. Therefore, we propose two ways of looking at the problem: in first, we treat the potential non-perturbatively, in second, we assign to λ a scale of some power of ε .

4.2.1. Non-perturbative treatment of V ($\lambda \sim \varepsilon^0$)

In this perturbation scheme, we invert the operator $\square + \lambda V$, thus treating V in a non-perturbative way. For the sequence v_n defined by

$$v_1 := I_V(f, g) = I_0(f, g) - L_0(Vv_1) \tag{4.6}$$

$$v_{n+1} := L_V(F_n(v_1, \dots, v_n)) = L_0(F_n(v_1, \dots, v_n)) - L_0(Vv_{n+1}), \quad n \geq 1 \tag{4.7}$$

we have the following

Theorem 4.2. *With f, g, V and $F(u)$ as above for any $m > 3, k > 2, p > 1 + \sqrt{2}, \lambda < C_{q,k}^{-1}$ and sufficiently small ε the series defined in (4.5)–(4.7) converges (in norm) in $L_{1,q}^\infty$ for $q = \min(p - 1, k, m - 1)$ to the solution of the Eq. (4.1) with initial data (3.8).*

Proof. The proof is essentially identical with this of Theorem 3.2 with the following differences. For $q := \min(m - 1, k, p - 1)$, we obtain

$$\|v_1\|_{L_{1,m-1}^\infty} \leq D + \delta \|v_1\|_{L_{1,m-1}^\infty},$$

where $\delta := \lambda C_{q,k}$ and hence

$$\|v_1\|_{L_{1,q}^\infty} \leq \|v_1\|_{L_{1,m-1}^\infty} \leq \frac{D}{1-\delta} < \infty.$$

The same modification regards all other inequalities

$$\|v_{n+1}\|_{L_{1,q}^\infty} \leq C \sum_k |a_k^n| \cdot \|v_1\|_{L_{1,q}^\infty}^{\alpha_k^{n,1}} \cdots \|v_n\|_{L_{1,q}^\infty}^{\alpha_k^{n,n}} + \delta \|v_{n+1}\|_{L_{1,q}^\infty}$$

which leads to

$$\|v_{n+1}\|_{L_{1,q}^\infty} \leq \frac{C}{1-\delta} \sum_k |a_k^n| \cdot \|v_1\|_{L_{1,q}^\infty}^{\alpha_k^{n,1}} \cdots \|v_n\|_{L_{1,q}^\infty}^{\alpha_k^{n,n}}.$$

Repeating the trick used in the proof of Theorem 3.2, we can relate this problem to the algebraic equation, which now becomes

$$W = \frac{C}{1-\delta} \tilde{F}(W) + \frac{D}{1-\delta} \varepsilon. \tag{4.8}$$

Since $G(W)$ given by

$$G(W) := \frac{(1-\delta)W - C\tilde{F}}{D} = \varepsilon$$

is again analytic, so is $W(\varepsilon) = G^{-1}(\varepsilon)$, because $G'(0) = (1-\delta)/D > 0$. Repeating the reasoning, we arrive at the conclusion that $\sum_{n=1}^{\infty} \varepsilon^n \|v_n\|_{L_{1,q}^{\infty}}$ has a positive radius of convergence. It follows that the series (4.5) converges in norm in $L_{1,q}^{\infty}$ for all $\varepsilon < \tilde{R}$ to the solution of (4.1) with initial data (3.8). Uniqueness follows easily from Theorem 4.1. □

We can, again, remove the auxiliary parameter ε and replace the condition on the initial data by $f_0, f_1, g_0 < \tilde{R}$. Then, the series $\sum_{n=1}^{\infty} v_n$ defined by (4.6)–(4.7) converges to the solution of the nonlinear wave equation (4.1).

4.2.2. Perturbative treatment of V ($\lambda \sim \varepsilon^a$)

If we assume that the small scale of the potential’s strength λ is related to the small scale of the initial data, say $\lambda = \varepsilon^a \tilde{\lambda}$ with $a \in \mathbb{N}_+$, then the power series Ansatz

$$u = \sum_{n=1}^{\infty} \varepsilon^n v_n \tag{4.9}$$

inserted into the wave equation (4.1) gives

$$v_{-n} := 0, \quad n \geq 0 \tag{4.10}$$

$$v_1 := I_0(f, g) \tag{4.11}$$

$$v_{n+1} := -\tilde{\lambda} L_0(V v_{n+1-a}) + L_0(F_n(v_1, \dots, v_n)), \quad n \geq 1. \tag{4.12}$$

This system is much more appropriate for numerical techniques, because the equation on v_{n+1} is explicit, in contrast to the previous scheme, which includes implicit equations for v_{n+1} (i.e. appearing on both sides). Moreover, if we choose $a := p - 1$, then the lowest nontrivial order, v_p (all lower orders satisfy $v_n = 0$ for $1 < n < p$), contains both contributions from V and F and can be used as a good approximation to u (assuming the series converges), what will be discussed in the next section.

In this case, we also have a convergence result

Theorem 4.3. *With f, g, V and $F(u)$ as above for any $m > 3, k > 2, p > 1 + \sqrt{2}$, $\lambda < C_{q,k}^{-1}$ and sufficiently small ε the series defined in (4.9)–(4.12) converges (in norm) in $L_{1,q}^{\infty}$ for $q = \min(p - 1, k, m - 1)$ to the solution of Eq. (4.1) with initial data (3.8).*

Proof. The proof is again analogous to that of Theorems 3.2 and 4.2 with the following differences. We have for $q := \min(m - 1, k, p - 1)$

$$\|v_1\|_{L_{1,q}^\infty} \leq \|v_1\|_{L_{1,m-1}^\infty} \leq C_m(f_0 + f_1 + g_0) =: D < \infty$$

and

$$\begin{aligned} \|v_{n+1}\|_{L_{1,q}^\infty} &\leq \|L_0(F_n(v_1, \dots, v_n))\|_{L_{1,q}^\infty} + \tilde{\lambda} \|L_0(Vv_{n+1-a})\|_{L_{1,q}^\infty} \\ &\leq C \sum_k |a_k^n| \cdot \|v_1\|_{L_{1,q}^\infty}^{\alpha_k^{n,1}} \cdots \|v_n\|_{L_{1,q}^\infty}^{\alpha_k^{n,n}} + \tilde{\delta} \|v_{n+1-a}\|_{L_{1,q}^\infty} \end{aligned}$$

with $\tilde{\delta} := \tilde{\lambda}C_{q,k}$. The corresponding algebraic equation now becomes

$$W = C \tilde{F}(W) + \tilde{\delta}\varepsilon^a W + D\varepsilon. \tag{4.13}$$

It cannot be rewritten, like before, as $G(W) = \varepsilon$, but it can be written as

$$G(W, \varepsilon) := W - C \tilde{F}(W) - \tilde{\delta}\varepsilon^a W - D\varepsilon = 0.$$

Since a is a positive integer number, $G(W, \varepsilon)$ is analytic in both variables around the point $G(0, 0) = 0$. Moreover, $\partial G(W, 0)/\partial W|_{W=0} = 1 - \tilde{\delta}\varepsilon^a = 1 - \lambda C_{q,k} > 0$. Then, by (real) analytic implicit function thorem (see e.g. [2]), there exists a unique function $W(\varepsilon)$ such that $G(W(\varepsilon), \varepsilon) = 0$. Then, $W(\varepsilon)$ has a Taylor series representation with positive radius of convergence. Repeating the reasoning of the previous proofs, we arrive at the conclusion that $\sum_{n=1}^\infty \varepsilon^n \|v_n\|_{L_{1,q}^\infty}$ has a positive radius of convergence $\tilde{R} > 0$. It follows that the series (4.9)–(4.12) converges in norm in $L_{1,q}^\infty$ for all $0 < \varepsilon < \tilde{R}$ to the solution of (4.1) with initial data (3.8). Uniqueness follows again from Theorem 4.1. □

4.3. Optimal decay estimate

4.3.1. Iteration with perturbative treatment of V

From (4.4), we have

$$|u(t, x) - u_n(t, x)| \leq \frac{\delta^n(2 - \delta)^n}{(1 - \delta)^2} \cdot \frac{3C_m\varepsilon}{\langle t + |x| \rangle \langle t - |x| \rangle^{q-1}} \leq \frac{c_n(x)\lambda^n\varepsilon}{(1 + t)^q}$$

for sufficiently small ε and λ (such that $\lambda C_{q,k} = \delta < \delta_0 < 1$). If we are able to show that some $u_n(t, x) \cong \varepsilon d_n(x)t^{-q} \neq 0$ for large t (the asymptotic approximation “ \cong ” is to be understood in the sense defined in (3.22), with the relative error η), then already u_n shows the correct decay rate, identical with this of u , because then, choosing $\eta := \lambda^n$, we get

$$\left| u(t, x) - \frac{d_n(x)\varepsilon}{t^q} \right| \leq \frac{[d_n(x) + c_n(x)]\lambda^n\varepsilon}{t^q}.$$

For small λ , it follows

$$|u(t, x)| \cong \frac{d_n(x)\varepsilon}{t^q}.$$

Again, in case when $m > p$, we have $u_1 = I_0(f, g) \in L_{1,m-1}^\infty$ and

$$|u_1(t, x)| \leq \frac{C_m \cdot (f_0 + f_1 + g_0)}{\langle t + |x| \rangle \langle t - |x| \rangle^{m-2}} \leq \frac{2C_m \cdot (f_0 + f_1 + g_0)}{(1 + t)^{m-1}}$$

for $t \geq 2(m - 2)|x|$, hence, it decays faster than u and it cannot be shown that $u_1(t, x) \cong \varepsilon d_1(x)t^{-q}$. It is expected that it will be true for $u_2 \cong \varepsilon d_2(x)t^{-q}$, what means, that already u_2 would have the same rate of decay as u (see [6] for such results in spherical symmetry).

In case when $m \leq p$, we have $q := m - 1$. Then, it should be possible to show $u_1(t, x) \cong d_1(x)t^{-(m-1)}$, what means, that already u_1 would have the same rate of decay as u .

4.3.2. Perturbation series with perturbative treatment of V

Consider the system (4.10)–(4.12) and choose the constant $a := p - 1$ so that $v_2, \dots, v_{p-1} = 0$ and at the order v_p both effects, the nonlinear and linear (potential) scattering, appear simultaneously

$$v_{-n} := 0, \quad n \geq 0 \tag{4.14}$$

$$v_1 = I_0(f, g) \tag{4.15}$$

$$v_2 = v_3 = \dots = v_{p-1} = 0 \tag{4.16}$$

$$\begin{aligned} v_p &= -\tilde{\lambda}L_0(Vv_1) + L_0(F_{p-1}(v_1, \dots, v_{p-1})) \\ &= -\tilde{\lambda}L_0(Vv_1) + a_0L_0((v_1)^p) \end{aligned} \tag{4.17}$$

$$v_{n+1} = -\tilde{\lambda}L_0(Vv_{n-p+2}) + L_0(F_n(v_1, \dots, v_n)), \quad n \geq p. \tag{4.18}$$

Consider only the more interesting case $m - 1 > \min(p - 1, k) =: q$. If we can show that $v_p \cong d_p(x)t^{-q}$, then already v_p has the correct decay rate, identical with this of u . To prove it, we can repeat the reasoning from the section when we treated nonlinear wave equation without the potential term, because the only fact, which we use is that the perturbation series $\sum_{n=1} \varepsilon^n v_n$ has a positive radius of convergence and this is here guaranteed by Theorem 4.3. Analogously, we obtain

$$u(t, x) \cong \frac{d_p(x)\varepsilon^p}{t^q}$$

for all $t > T$ and sufficiently big $T = T(\varepsilon)$, so v_p dominates the perturbation series for large times and small ε and has the same decay rate as the full solution of the nonlinear wave equation u .

This is the simplest setting for applications. Here, we only need to solve (approximately) two linear wave equations, (4.15) and (4.17), in order to determine the decay rate for solutions of (4.1). This is the starting point of [6] where we solve the two equations under spherical symmetry.

Appendix A. Some Useful Estimates

The first two lemmas we cite from [5].

Lemma A.1. *Let the data $(f, g) \in L_{m-1}^\infty \times L_m^\infty$ with $m > 3$ satisfy*

$$f_0 := \|f\|_{L_{m-1}^\infty} < \infty, \quad f_1 := \|\nabla f\|_{L_m^\infty} < \infty, \quad g_0 := \|g\|_{L_m^\infty} < \infty.$$

Then there exists a unique weak solution $v(t, x) = I_0(f, g)$ of the free wave equation

$$\square v = 0, \quad v(0, x) = f(x), \quad \partial_t v(0, x) = g(x)$$

which satisfies

$$\|v\|_{L_{1,m-1}^\infty} \leq C(f, g) := C_m \cdot (g_0 + f_1 + f_0).$$

Lemma A.2. *Let the source F satisfy for some $q > 2$ and $1 < p \leq q$*

$$F_0 := \|\langle x \rangle^q F\|_{L_{1,p}^\infty} < \infty.$$

Then there exists a weak solution $v(t, x) = L_0(F)$ of the free wave equation with source

$$\square v = F,$$

and null initial data $v(0, x) = 0, \partial_t v(0, x) = 0$. Moreover, it satisfies

$$\|v\|_{L_{1,p}^\infty} \leq C_{p,q} F_0.$$

Next lemma we cite after Asakura [1, Corollory 2.4 and Eq. (2.33)] and state in our notation.

Lemma A.3. *Let $u \in C^2(\mathbb{R}_+^{1+3}) \cap L_{1,q}^\infty$ for some $q > 1$. Then for any $p > 1 + \sqrt{2}$*

$$\|L_0(|u|^p)\|_{L_{1,q}^\infty} \leq C \|u\|_{L_{1,q}^\infty}^p$$

with some $C > 0$ provided $q \leq p - 1$.

Note that it is a consequence of Lemma A.2, but only when $p > 3$, while for $1 + \sqrt{2} < p \leq 3$, it requires a more general proof. It can be easily deduced, also for weak solutions $u \in C^0(\mathbb{R}_+^{1+3})$, from a more general estimate [4].

Lemma A.4. *If*

$$|F(t, x)| \leq \frac{A}{\langle t + |x| \rangle^p \langle t - |x| \rangle^q}$$

with $p > 2, q > 1$, then

$$|L_0(F)(t, x)| \leq \frac{C}{\langle t + |x| \rangle \langle t - |x| \rangle^{p-2}}$$

with some positive constant C .

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