Operator spin foam models

2011 Class. Quantum Grav. 28 105003

(http://iopscience.iop.org/0264-9381/28/10/105003)

Class. Quantum Grav. **28** (2011) 105003 (23pp) doi:10.1088/0264-9381/28/10/105003

Operator spin foam models

Benjamin Bahr[1,2], **Frank Hellmann**[3], **Wojciech Kamiński**[3], **Marcin Kisielowski**[3] and **Jerzy Lewandowski**[3]

[1] DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, UK
[2] Albert Einstein Institute, Muehlenberg 1, 14476 Golm, Germany
[3] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, 00-681 Warszawa (Warsaw), Polska. Poland

Received 11 November 2010, in final form 22 February 2011
Published 6 April 2011
Online at stacks.iop.org/CQG/28/105003

Abstract
The goal of this paper is to introduce a systematic approach to spin foams. We define *operator* spin foams, that is foams labelled by group representations and operators, as our main tool. A set of moves we define in the set of the operator spin foams (among other operations) allows us to split the faces and the edges of the foams. We assign to each operator spin foam a *contracted* operator, by using the contractions at the vertices and suitably adjusted face amplitudes. The emergence of the face amplitudes is the consequence of assuming the invariance of the contracted operator with respect to the moves. Next, we define spin foam *models* and consider the class of models assumed to be symmetric with respect to the moves we have introduced, and assuming their partition functions (state sums) are defined by the contracted operators. Briefly speaking, those operator spin foam models are invariant with respect to the cellular decomposition, and are sensitive only to the topology and colouring of the foam. Imposing an extra symmetry leads to a family we call *natural* operator spin foam models. This symmetry, combined with assumed invariance with respect to the edge splitting move, determines a complete characterization of a general natural model. It can be obtained by applying arbitrary (quantum) constraints on an arbitrary BF spin foam model. In particular, imposing suitable constraints on a spin(4) BF spin foam model is exactly the way we tend to view 4D quantum gravity, starting with the BC model and continuing with the Engle–Pereira–Rovelli–Livine (EPRL) or Freidel–Krasnov (FK) models. That makes our framework directly applicable to those models. Specifically, our operator spin foam framework can be translated into the language of spin foams and partition functions. Among our natural spin foam models there are the BF spin foam model, the BC model, and a model corresponding to the EPRL intertwiners. Our operator spin foam

framework can also be used for more general spin foam models which are not symmetric with respect to one or more moves we consider.

PACS number: 04.60.Pp

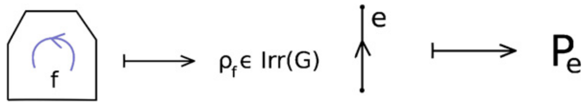
(Some figures in this article are in colour only in the electronic version)

1. Introduction

The successful application of the 3D BF spin foam theory to 3D quantum gravity (see [1, 2] and references therein) produced and still produces activity in the 4D spin foam approaches to the 4D quantum gravity [1–15]. After the decade of the Barrett–Crane model [3], a breakthrough has come with the new models: the Engle–Pereira–Rovelli–Livine (EPRL) model [4, 5] and the Freidel–Krasnov (FK) model [6]. For the first time, the existence of a relation between the 4D spin foam theory, on the one hand, and the kinematics of the (3+1) loop quantum gravity [16–20] has become plausible. The theory accommodates all the states of LQG labelled by graphs embedded in an underlying 3-manifold [7], although seems not to be sensitive on linking and knotting [8].

The spin networks and spin foams featuring in the spin foam models may be thought of as just combinatorial tools used to extract numbers. However, they also admit their own structure and natural operations that deserve understanding. The spin networks emerge in loop quantum gravity as invariant elements of the tensor products of representations. Consistently, the spin foams arise as cobordisms between the spin networks, and hence should be described in terms of operators mapping the invariants into invariants.

The goal of this paper is to introduce a systematic approach to spin foams. We define *operator* spin foams, that is foams labelled by group representations and operators, as our main tool. A set of moves we introduce in the set of the operator spin foams allows (among other operations) us to split the faces and the edges of the foams. The moves are used to introduce an equivalence relation. The equivalence relation is used in this paper as a symmetry of the structures we define. We do not consider equivalent operator spin foams to be the same operator spin foam (however such identification is possible). We assign to each operator spin foam a *contracted* operator, by using the contractions at the vertices and suitably adjusted face amplitudes. The emergence of the face amplitudes is the consequence of assuming the invariance of the contracted operator with respect to the moves. Next, we define spin foam *models* and consider the class of models assumed to be symmetric with respect to the moves we have introduced, and assuming that their partition functions (state sums) are defined by the contracted operators. Briefly speaking, those operator spin foam models are invariant with respect to the cellular decomposition, and are sensitive only to the topology and colouring of the foam. Imposing an extra symmetry leads to a family that we call *natural* operator spin foam models. This symmetry, combined with assumed invariance with respect to the edge splitting move, determines a complete characterization of a general natural model. It can be obtained by applying arbitrary (quantum) constraints on an arbitrary BF spin foam model. In particular, imposing suitable constraints on the spin(4) BF spin foam model is exactly the way we tend to view 4D quantum gravity, starting with the BC model and continuing with the EPRL or FK models. That makes our framework directly applicable to those models. Specifically, our operator spin foam framework can be translated into the language of spin foams and partition functions. Among our natural spin foam models there are the BF spin



(*a*) Colouring of faces (*b*) Colouring of edges

Figure 1. Operator form of spin foam.

foam model, the BC model and a model corresponding to the EPRL intertwiners. The result is that of [9], rather than the one defined in the original EPRL paper [4]. The choice of the EPRL intertwiners and the vertex amplitude is the same in both approaches. The ambiguity is in glueing the vertices. Of course we do not mean to insist that the proposal of [9], which also follows from the current paper, is better than the original EPRL one. We just find a set of natural properties that lead to the former proposal, and the bottom line is that the latter proposal is necessarily inconsistent with one of the conditions we spell out (this turns out to be a certain edge splitting condition).

Our operator spin foam framework can also be used for more general spin foam models which are not symmetric with respect to one or all the moves we consider.

2. Operator spin foam

2.1. Definition

Let κ be a locally linear, oriented 2-complex with boundary $\partial\kappa$ [1, 7] and let G be a compact Lie group. Denote by $\kappa^{(0)}$ the set of vertices (the 0-cells), by $\kappa^{(1)}$ the set of edges (1-cells) and by $\kappa^{(2)}$ the set of faces (2-cells) of the complex κ. For simplicity of the presentation, we will be assuming throughout this paper that every face of κ is topologically a disc[4]. Every edge $e \in \kappa^{(1)}$ is contained in at least one face. If e is contained in exactly one face, we call it boundary edge. Otherwise e is an internal edge. If a vertex $v \in \kappa^{(0)}$ is contained in a boundary edge, we call it boundary vertex. Otherwise v is internal. We will be denoting the set of internal edges/vertices by $\text{int}\kappa^{(1)}/\text{int}\kappa^{(0)}$.

The 1-complex set by the boundary edges and boundary vertices is denoted by $\partial\kappa$ and called the boundary of κ.

An operator spin foam that we define in this paper is a triple (κ, ρ, P), where ρ and P are colourings by representations and, respectively, operators defined below. The first one, ρ, is familiar with spin foam theories, namely

- ρ is a colouring of the faces with irreducible representations of G (figure 1(*a*)):

$$\rho : \kappa^{(2)} \to \text{Irr}(G), \tag{2.1}$$

$$f \mapsto \rho_f. \tag{2.2}$$

The colouring ρ can be used to assign Hilbert spaces to the faces and the edges of κ. To every face f, there is assigned a Hilbert space $\mathcal{H}_f$

$$f \mapsto \mathcal{H}_f \tag{2.3}$$

[4] That is, no point of a face is glued to another point of a same face; below we introduce an equivalence relation which allows us to split/glue faces and edges. It will be obvious how to use those moves to relax this assumption.

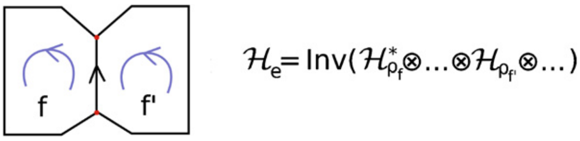


Figure 2. The edge Hilbert space $\mathcal{H}_e$.

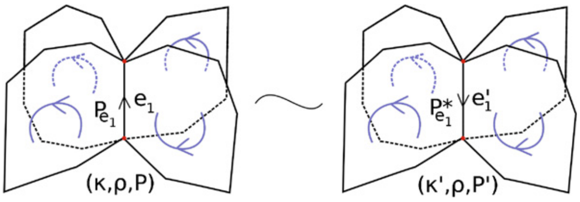


Figure 3. Invariance under the face subdivision.

on which the representation ρ_f acts. To every edge e there is assigned a Hilbert space H_e defined by the Hilbert spaces of the faces containing e:

$$\mathcal{H}_e = \bigotimes_{f \text{ incoming to } e} \mathcal{H}^*_f \otimes \bigotimes_{f' \text{ outgoing from } e} \mathcal{H}_{f'} \tag{2.4}$$

where a face is called incoming to (outgoing from) an edge e if its orientation agrees with (is opposite to) that of e, and by $\mathcal{H}^*_f$ we denote the algebraic dual (see figure 2). Given a representation $\mathcal{H}$ of G (irreducible), the subspace of invariant elements is denoted by Inv$\mathcal{H}$.

Having in mind those Hilbert spaces we introduce the operator labelling:

- P is a colouring of the internal edges with operators (figure 1(b))

$$\text{int}\kappa^{(1)} \ni e \mapsto P_e \tag{2.5}$$

$$P_e : \text{Inv}\mathcal{H}_e \to \text{Inv}\mathcal{H}_e. \tag{2.6}$$

2.2. The moves and the equivalence relation they define

In the space of operator spin foams we consider a set of moves and an equivalence relation they define. The moves allow us to subdivide edges and faces, change their orientation, use colourings with equivalent representations and add faces and edges. In the following paragraphs we describe that equivalence relation in detail. The moves correspond to analogous moves in the space of the spin networks except for the edge splitting move. Two equivalent operator spin foams are not literally identified in this paper. The equivalence relation is used as a symmetry of the structures we will define in this paper.

2.2.1. Edge reorientation. Given an operator spin foam (κ, ρ, P), let us switch the orientation of its edge e_1:

$$e'_1 = e_1^{-1}, \tag{2.7}$$

and leave all the other orientations unchanged (figure 3). Denote the resulting 2-complex by κ'. To define an operator spin foam (κ', ρ', P') which is equivalent to (κ, ρ, P), suppose first that the edge e_1 is internal and

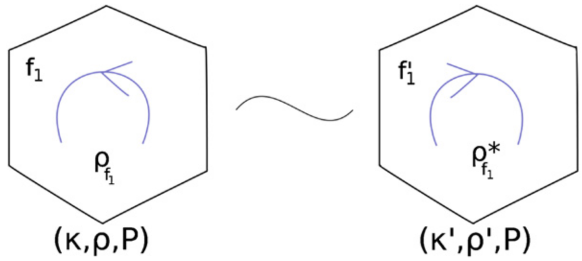


Figure 4. Invariance under the face subdivision.

- leave the labelling ρ, namely

$$\rho' = \rho. \tag{2.8}$$

Now, ρ' determines the Hilbert space $\mathcal{H}_{e'_1}$ to be

$$\mathcal{H}_{e'_1} = \mathcal{H}^*_{e_1} \tag{2.9}$$

where the algebraic dualization $*$ is applied to each factor on the right-hand side of (2.4). The natural choice for $P'_{e'_1}$ is

- for the reoriented edge $e'_1 = {e_1}^{-1}$

$$P'_{e'_1} = P^*_{e_1}, \tag{2.10}$$

- whereas for the remaining edges of κ' we leave

$$P'_e = P_e. \tag{2.11}$$

The operator spin foams (κ, ρ, P) and (κ', ρ, P') are equivalent:

$$(\kappa, \rho, P) \equiv (\kappa', \rho, P'). \tag{2.12}$$

The remaining case when the reoriented edge e_1 is boundary is yet simpler: both labellings ρ and P are defined on the faces/edges unaffected by the reorientation of e_1; we just leave them unchanged, that is we set $\rho' = \rho$ and $P' = P$.

2.2.2. Face reorientation. Given an operator spin foam (κ, ρ, P), let us switch the orientation of its face f_1 and denote the reoriented face f'_1. Denote the resulting 2-complex by κ' (figure 4). To define an operator spin foam (κ', ρ', P') equivalent to (κ, ρ, P), we modify the labelling ρ in the following way:

- for the reoriented face f'_1 we take the dual representation

$$\rho'_{f'_1} = \rho^*_{f_1}; \tag{2.13}$$

- for the remaining faces, the labelling ρ' coincides with ρ:

$$\rho'_f = \rho_f, \qquad \text{for} \quad f \neq f'_1. \tag{2.14}$$

At each edge e, the labelling ρ' defines the same Hilbert space $\mathcal{H}_e$ as ρ in (κ, ρ, P). Therefore, the following definition of P' is possible.

- For labelling P' the choice is

$$P' = P. \tag{2.15}$$

Again, we will consider (κ', ρ', P) and (κ, ρ, P) equivalent:

$$(\kappa, \rho, P) \equiv (\kappa', \rho', P). \tag{2.16}$$

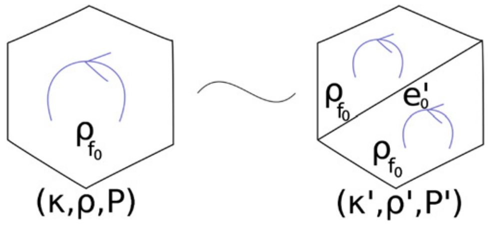


Figure 5. Invariance under face subdivision.

2.2.3. Face splitting. Consider an operator spin foam (κ, ρ, P). Split one of its faces, f_0 say, into f_1' and f_2' such that a resulting new edge e_0' (oriented arbitrarily) contained in f_1' and in f_2' connects two vertices belonging to $\kappa^{(0)}$. Choose an orientation of the new faces to be the one induced by f_0. The resulting new 2-cell complex κ' is obtained by replacing the face f_0 by the pair of faces f_1' and f_2' and by adding the edge e_0' (figure 5). Define a labelling ρ' on κ' in the following way:

- ρ' coincides with ρ on the unsplitted faces,

$$\rho'_{f'} = \rho_{f'}, \qquad \text{if} \quad f' \neq f_1', f_2', \tag{2.17}$$

- and ρ' agrees with ρ on the faces f_1', f_2' resulting from the splitting

$$\rho'_{f'} = \rho_{f_0}, \qquad \text{if} \quad f' = f_1', f_2'. \tag{2.18}$$

For the edge e_0', the corresponding Hilbert space is one dimensional by Schur's lemma:

$$\mathcal{H}_{e_0'} = \mathrm{Inv}(\mathcal{H}_{f_0} \otimes \mathcal{H}^*_{f_0}) \equiv \mathbb{C}. \tag{2.19}$$

Define a labelling P' of the edges of κ'

- to be the identity on the new edge e_0' resulting from the splitting

$$P'_{e'} = \mathrm{id}, \ \text{if } e' = e_0' \tag{2.20}$$

- and to coincide with P on the old edges

$$P'_{e'} = P_{e'}, \qquad \text{if} \quad e' \neq e_0'. \tag{2.21}$$

The resulting operator spin foam is equivalent to (κ, ρ, P):

$$(\kappa, \rho, P) \equiv (\kappa', \rho', P'). \tag{2.22}$$

2.2.4. Edge splitting. In an operator spin foam (κ, ρ, P) split an edge e_0 into e_1' and e_2'

$$e_0 = e_2' \circ e_1' \tag{2.23}$$

whose orientations are induced by e_0. Denote the resulting 2-complex by κ' (figure 6). An operator spin foam $(\kappa', \rho'.P')$ defined on κ' is equivalent to (κ, ρ, P),

$$(\kappa, \rho, P) \equiv (\kappa', \rho', P'), \tag{2.24}$$

whenever the following conditions are satisfied by ρ' and P':

- ρ is unchanged,

$$\rho' = \rho, \tag{2.25}$$

- P' coincides with P on the edges $e' \neq e_1', e_2'$,
- $P'_{e_1'}$ and $P'_{e_2'}$ satisfy the following constraint

$$P'_{e_2'} \circ P'_{e'1} = P_{e_0}, \tag{2.26}$$

provided the edge e_0 is internal.

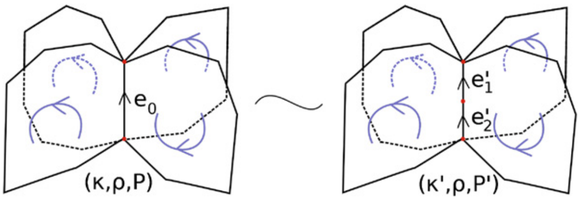


Figure 6. Invariance under the edge subdivision.

2.2.5. Rescaling of the operators. Every operator spin foam (κ, ρ, P) is equivalent to any operator spin foam (κ, ρ, P') defined by rescaling, for every internal edge e,

$$P'_e = a_e P_e, \qquad a_e \in \mathbb{C}, \tag{2.27}$$

such that

$$\prod_e a_e = 1. \tag{2.28}$$

2.2.6. Face relabelling with equivalent representations. Consider an operator spin foam (κ, ρ, P) and (κ, ρ', P'), where

- $\rho_f = \rho'_f$ for all but one face $f = f_0$, and for f_0, there exists an isomorphism $\mathcal{I} : \mathcal{H}_{f_0} \to \mathcal{H}'_{f_0}$ which intertwines the representations, namely $\mathcal{I} \circ \rho_{f_0} = \rho'_{f_0} \circ \mathcal{I}$;
- $P_e = P'_e$ for every edge e not contained in the face f_0;

$$P'_e = id \otimes \ldots \otimes \mathcal{I} \otimes id \otimes \ldots \otimes id \circ P_e \circ id \otimes \ldots \otimes \mathcal{I}^{-1} \otimes id \otimes \ldots \otimes id, \tag{2.29}$$

if the face f_0 is outgoing from the edge e;

$$P'_e = id \otimes \ldots \otimes \mathcal{I}^{*-1} \otimes id \otimes \ldots \otimes id \circ P_e \circ id \otimes \ldots \otimes \mathcal{I}^{*} \otimes id \otimes \ldots \otimes id, \tag{2.30}$$

if the face f_0 is incoming to the edge e.

The two spin foams are equivalent:

$$(\kappa, \rho, P) \equiv (\kappa, \rho', P'). \tag{2.31}$$

2.2.7. Adding a face labelled by the trivial representation. Our definition of the operator spin foams does not exclude the trivial representation from the set of labels assigned to the faces. Every spin foam (κ, ρ, P) will be considered equivalent to a spin foam (κ', ρ', P') obtained by adding a face f'_1 and labelling it by the trivial representation ρ_0, that is

$$\rho'(f') = \begin{cases} \rho(f'), & \text{if } f' \in \kappa^{(2)} \\ \rho_0, & \text{if } f' = f'_1 \end{cases} \tag{2.32}$$

provided every edge of κ the face f'_1 is glued to, is either internal, or a boundary edge of a face labelled by the trivial representation ρ_0. For every internal edge e' of κ', either: (i) it is also the internal edge of κ, the corresponding Hilbert spaces $\mathcal{H}_e$ coincide, and P' is defined to be,

$$P' = P, \tag{2.33}$$

or (ii) $P'_{e'} = 1 : \mathbb{C} \to \mathbb{C}$.

Note that conditions 2.2.3 and 2.2.4 relate colourings on different cellular decompositions of foams of the same topology and are analogous to one- and two-dimensional Pachner moves.

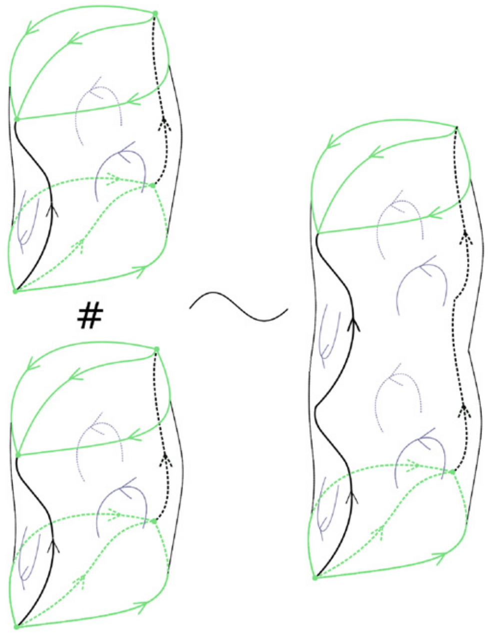


Figure 7. Glueing of the operator spin foams.

In fact if the 2-complex is finitely triangulable they exactly generate the Pachner moves, i.e. the move 2.2.4 generates the $1 \to 2$ (and $2 \to 1$) Pachner move, and the move 2.2.3 generates the $1 \to 3$ (and $3 \to 1$) and $2 \to 2$ Pachner moves. Thus all decompositions of foams with the same topology are related by finitely many such moves. By Schur's lemma all operator spin foams with non-isomorphic representations around a bivalent edge do not poses a non-trivial labelling.

2.3. Glueing the operator spin foams

In the space of the 2-complexes considered in this paper there is the obvious operation of glueing. It admits a natural extension to an operation of glueing the operator spin foams, which, for the sake of completeness, we describe in the following. Two oriented, locally linear 2-cell complexes κ and κ' can be glued along a connected component γ of the boundary $\partial\kappa$ and a connected component γ' of $\partial\kappa$, provided γ and γ' are isomorphic closed 1-cell complexes (unoriented graphs), and the orientations of the glued faces and, respectively, their sites match (figure 7). If $\phi : \gamma \to \gamma'$ is an isomorphism, then the glueing amounts to glueing along each link e of γ: a face f_e of κ containing e is glued with the face $f'_{\phi(e)}$ of κ' containing the link $\phi(e)$ of γ'. In what follows we will assume that the map

$$\gamma \ni e \mapsto f_e, \qquad \gamma' \ni e' \mapsto f'_{e'} \tag{2.34}$$

is 1-1 (each e has its own f_e). This can always be achieved by dividing the faces and edges. The resulting face $f_e \# f'_{\phi(e)}$ can be oriented either according to the orientation of f_e or according to the orientation of $f'_{\phi(e)}$; coinciding of the two orientations is the matching relation we have mentioned above. A similar matching condition applies to the oriented sides of the faces f_e and $f'_{\phi(e)}$. Repeating that glueing for every link e of γ, we complete the glueing of κ and κ' along γ. The result can be denoted by $\kappa\#\kappa'$ and it depends on the graphs γ, γ' and the isomorphism ϕ. If the 2-complexes above were endowed with the structures of the operator spin foams (κ, ρ, P), and respectively, (κ', ρ', P'), the operator spin foams can be glued into an operator spin foam $(\kappa\#\kappa', \rho\#\rho', P\#P')$ provided the representations agree on the boundary, and the glueing condition is

$$\rho'_{f'_{\phi(e)}} = \rho_{f_e} \tag{2.35}$$

for every pair e and $\phi(e)$ of the identified edges.

- For each of the boundary edges e, due to the glueing condition, we can set

$$(\rho\#\rho')_{f_e\#f'_{\phi(e)}} = \rho_{f_e} = \rho'_{\phi(e)}. \tag{2.36}$$

- For the remaining faces we use either ρ or, respectively, ρ'

$$(\rho\#\rho')_{f''} = \begin{cases} \rho_{f''}, & \text{if } f'' \in \kappa^{(2)}, \\ \rho'_{f''} & \text{if } f'' \in \kappa'^{(2)}, \end{cases}. \tag{2.37}$$

For the operator part $P\#P'$, the glueing consists in

- taking the composition of the operators for every pair $(\tilde{e}, \tilde{e'})$ of sides of the faces f_e, and respectively, $f'_{\phi(e)}$ that are glued into a side of the face $f_e\#f'_{\phi(e)}$, that is either

$$(P\#P')_{\tilde{e}\circ\tilde{e'}} = P_{\tilde{e}} \circ P_{\tilde{e'}} \tag{2.38}$$

or

$$(P\#P')_{\tilde{e'}\circ\tilde{e}} = P_{\tilde{e'}} \circ P_{\tilde{e}} \tag{2.39}$$

depending on the orientations.

- For each of the remaining edges of $\kappa\#\kappa'$ we leave the corresponding operator of either κ or κ':

$$(P\#P')_{e''} = \begin{cases} P_{e''}, & \text{if } e'' \in \text{int}\kappa \\ P'_{e''}, & \text{if } e'' \in \text{int}\,\kappa'. \end{cases} \tag{2.40}$$

3. Spin foam operator

3.1. 2-edge contraction

Wherever two internal edges of a spin foam (κ, ρ, P) meet, the geometry of a spin foam defines a natural contraction between the corresponding operators. The easiest way to introduce it is to use the (abstract) index notation. It is as follows: given

$$w \in \text{Inv}\left(\bigotimes_{f \text{ incoming to } e} \mathcal{H}^*_f \otimes \bigotimes_{f' \text{ outgoing from } e} \mathcal{H}_{f'}\right) \tag{3.1}$$

we denote it in the index notation as

$$w = w_{A\ldots}{}^{A'\cdots} \tag{3.2}$$

where the lower/upper indices correspond to the spaces $\mathcal{H}^*_f/\mathcal{H}_{f'}$. The action of the operator P_e reads

$$(P_e w)_{A\ldots}{}^{A'\cdots} = P_e{}^{A'\ldots B\cdots}_{A\ldots B'\cdots} w_{B\ldots}{}^{B'\cdots}. \tag{3.3}$$

Moreover, the vector $w_{A\ldots}{}^{A'\cdots}$ is associated with the beginning of the given edge e, whereas the vector $(P_e w)_{A\ldots}{}^{A'\cdots}$ lives at the end of e. In this sense, the indices B, B' of $P_e{}^{A'\ldots B\cdots}_{A\ldots B'\cdots}$ are associated with the beginning point of e, whereas the indices A, A' of $P_e{}^{A'\ldots B\cdots}_{A\ldots B'\cdots}$ with the end

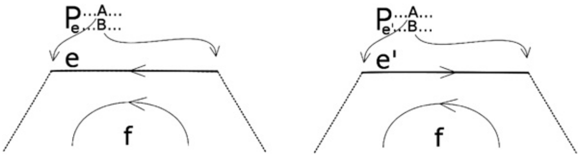


Figure 8. The rule of assigning an index of P_e to a corner v of a face f: given an edge e contained in a face f of an operator spin foam (κ, ρ, P), in the operator P_e, the indices corresponding to the Hilbert space $\mathcal{H}_f$ of the representation ρ_f are assigned to the end points of e such that the lower/upper index is assigned to the point that is the beginning/end point of e if the orientation of e is the same as that of f, and to the end/beginning point of e if the orientation of e is opposite. The oriented arc only marks the orientation of the polygonal face f.

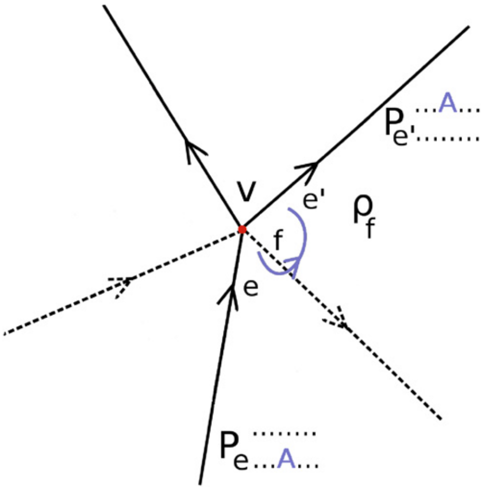


Figure 9. 2-edge contraction of indices. The edges e and e' are connected by the face f. Marked indices A of P_e and, respectively, $P_{e'}$ correspond to the Hilbert space $\mathcal{H}_f$ and get contracted by $\mathrm{Tr}_{v,f}$.

point of e. Therefore, for every edge e and for each face f containing e, there are two indices in the operator P_e: upper one and lower one. They correspond to the Hilbert space $\mathcal{H}_f$. The indices are associated with the ends of the edge e according to the rule introduced above and presented in figure 8.

Now, for every pair of edges e and e' which belong to the same face f and share a vertex v, if the index of P_e corresponding to f and v is upper/lower, then the index of $P_{e'}$ corresponding to f and v is lower/upper, respectively. In this way, the natural contraction $\mathrm{Tr}_{v,f}$ at v is defined (figure 9).

3.2. *Contracted operator spin foam*

The contraction at the vertices of the complex defines the contracted operator spin foam:

$$\mathrm{Tr}(\kappa, \rho, P) := \prod_{v,f} \mathrm{Tr}_{v,f}\left(\bigotimes_{e\in \mathrm{Int}\kappa^{(1)}} P_e\right). \tag{3.4}$$

Given an edge e, one of its ends v and a face f containing e, the corresponding index in P_e is contracted, provided there is another internal (that is contained in at least two different faces) edge e' contained in f and intersecting the point v. Otherwise, the index stays uncontracted. As a consequence, the contracted operator $\mathrm{Tr}(\kappa, \rho, P)$ is indeed an operator. Identifying each operator $P_e : \mathcal{H}_e \to \mathcal{H}_e$ with an element of $\mathcal{H}_e \otimes \mathcal{H}_e^*$, the contracted spin foam $\mathrm{Tr}(\kappa, \rho, P)$ is identified with an element of the Hilbert space

$$\mathcal{H}_{\partial\kappa} = \bigotimes_{e \text{ incoming to } \partial\kappa} \mathcal{H}_e \otimes \bigotimes_{e' \text{ outgoing from } \partial\kappa} \mathcal{H}_{e'}^*. \tag{3.5}$$

3.3. *Spin foam operator*

3.3.1. Contraction and the equivalence moves. Any splitting $\mathcal{H}_{\partial\kappa} = \mathcal{H}_{\mathrm{fin}} \otimes \mathcal{H}_{\mathrm{in}}^*$ makes the contracted operator spin foam $\mathrm{Tr}(\kappa, \rho, P)$ an operator $\mathcal{H}_{\mathrm{in}} \to \mathcal{H}_{\mathrm{fin}}$.

Expression (3.4) is not invariant with respect to the equivalence moves introduced in the previous subsection. Given an operator spin foam (κ, ρ, P), suppose that an operator spin foam (κ', ρ', P') is obtained from (κ, ρ, P) by one of the equivalence moves except for the face splitting move. Then

$$\mathrm{Tr}(\kappa', \rho', P') = \mathrm{Tr}(\kappa, \rho, P). \tag{3.6}$$

However, if an operator spin foam (κ', ρ', P') is obtained by splitting a face f_0 of (κ, ρ, P) and defining ρ' and P' as in section 2.2.3, then this move is not a symmetry of the trace. In that case, the Hilbert space

$$\mathrm{Inv}(\mathcal{H}_{f_1'} \otimes \mathcal{H}_{f_2'}^*) = \mathrm{Inv}(\mathcal{H}_{f_0} \otimes \mathcal{H}_{f_0}^*)$$

is spanned by the element, in the index notation, δ^a_b, and the operator $P'_{e_0'} = \mathrm{id}$ reads

$$P'^{\;ab'}_{e_0' a'b} = \frac{1}{d_{f_0}} \delta^a_b \delta^{b'}_{a'}. \tag{3.7}$$

It is easy to verify that

$$\mathrm{Tr}(\kappa', \rho', P) = \frac{1}{d_{f_0}} \mathrm{Tr}(\kappa, \rho, P) \tag{3.8}$$

where

$$d_{f_0} = \dim \mathcal{H}_{f_0}. \tag{3.9}$$

This shows that indeed the move is not a symmetry.

3.3.2. Face amplitude restores the equivalence. Introducing suitable face amplitude makes the contraction Tr of operator spin foam exactly invariant with respect to all the moves. Consider a spin foam operator defined by a formula (tilde will be removed when we establish the final form of the operator)

$$\tilde{\mathcal{Z}}_{(\kappa,\rho,P)} = \left(\prod_{f \in \kappa^{(1)}} A_f \right) \mathrm{Tr}(\kappa, \rho, P) \tag{3.10}$$

where

$$f \mapsto A_f$$

is an unknown function, a face amplitude. Then, a unique solution for $f \mapsto A_f$ such that for every operator spin foam (κ, ρ, P) and every equivalent operator spin foam (κ', ρ', P')

$$\tilde{\mathcal{Z}}_{(\kappa,\rho,P)} = \tilde{\mathcal{Z}}_{(\kappa',\rho',P')}, \tag{3.11}$$

is

$$A_f = \dim \mathcal{H}_f. \tag{3.12}$$

3.3.3. Boundary amplitude restores the compatibility with the glueing. The introduction of the face amplitude destroys the compatibility with the glueing of the operator spin foams. Consider two operator spin foams (κ, ρ, P) and (κ', ρ', P'), and their composition $(\kappa, \rho, P)\#(\kappa', \rho', P')$ glued along a graph γ. The operator spin foam contraction induces the contraction of the operators $\tilde{\mathcal{Z}}(\kappa, \rho, P)$ and $\tilde{\mathcal{Z}}(\kappa', \rho', P')$; let us denote it by Tr_γ. The result is

$$\mathrm{Tr}_\gamma(\tilde{\mathcal{Z}}(\kappa, \rho, P) \otimes \tilde{\mathcal{Z}}(\kappa', \rho', P')) = \prod_{e\in\gamma} A(f_e)\tilde{\mathcal{Z}}(\kappa\#\kappa', \rho\#\rho', P\#P'). \tag{3.13}$$

To restore the compatibility of $\tilde{\mathcal{Z}}$ with glueing the operator spin foams we finally define the spin foam operator to be

$$\mathcal{Z}(\kappa, \rho, P) := \prod_{e\in(\partial\kappa)^{(1)}} \frac{1}{\sqrt{A_{f_e}}} \tilde{\mathcal{Z}}(\kappa, \rho, P), \tag{3.14}$$

where f_e is the face of κ containing e (and we are assuming that $e \neq e' \Rightarrow f_e \neq f_{e'}$ that can always be achieved by splitting faces and edges.). Now we have

$$\mathrm{Tr}_\gamma(\mathcal{Z}(\kappa, \rho, P) \otimes \mathcal{Z}(\kappa', \rho', P')) = \mathcal{Z}(\kappa\#\kappa', \rho\#\rho', P\#P'). \tag{3.15}$$

3.4. Relation with the spin foams and state sums

3.4.1. The spin foams. The operator spin foam formalism seem to differ from the usual formulation of spin foam amplitudes, in that there are projection operators assigned to edges instead of intertwiners. However, the projection operators P_e can be interpreted as the result of spin foam amplitudes where the sum over the intertwiners has already been carried out, i.e. we decompose each P_e,

$$P_e = \sum_{\iota_e\in\mathcal{B}_e} \sum_{\iota'_e\in\mathcal{B}^\dagger_{e'}} P^{\iota'_e}_{\iota_e} \iota_e \otimes \iota'_e, \tag{3.16}$$

in any basis,

$$\mathcal{B}_e \subset \mathcal{H}_e, \tag{3.17}$$

and the conjugate basis

$$\mathcal{B}^\dagger_e = \{\iota^\dagger_e : \iota_e \in \mathcal{B}_e\} \subset \mathcal{H}^*_e, \tag{3.18}$$

where $\mathcal{H} \ni v \mapsto v^\dagger \in \mathcal{H}^*$ is the canonical antilinear map (denoted by $|v\rangle \mapsto \langle v|$ in the Dirac notation).

After the substitution of the right-hand side of (3.16) for P_e, the tensor product $\bigotimes_e P_e$ becomes a linear combination of the tensor products

$$\bigotimes_e \iota_e \otimes \iota'_e, \tag{3.19}$$

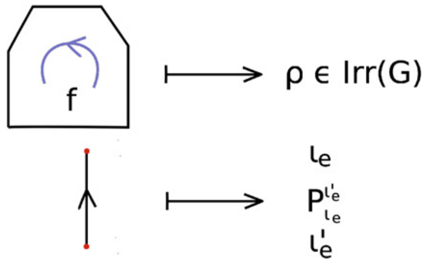


Figure 10. The operator approach is equivalent to the approach in which we assign an irreducible representation of group G to each face of the 2-complex and a pair of intertwiners ι_e,ι'_e together with the complex number $P^{\iota'_e}_{\iota_e}$ to each internal edge.

in which to each internal edge e there is assigned a (tensor product of a) pair of the intertwiners $\iota_e \otimes \iota'_e$, where $\iota_e \in \mathcal{B}_e$ and $\iota'_e \in \mathcal{B}^\dagger_e$ are independent of each other. In fact, from the point of view of the contractions we use, ι'_e is assigned to the beginning point of e whereas ι_e is assigned to the end point of e (figure 10). That is the generalized case of a spin foam that was derived in [9].

3.4.2. The vertex amplitude. Given a vertex v, the application of the constructions Tr_{vf} (see section 3.1) for all the faces f which intersect v, namely

$$\prod_{f\,:\,f\ni v} \mathrm{Tr}_{vf}\left(\bigotimes_e \iota_e \otimes \iota'_e\right), \tag{3.20}$$

produces a $\mathbb{C}$ number factor

$$A_v = \prod_{f\,:\,f\ni v} \mathrm{Tr}_{vf}\left(\bigotimes_{e\ \text{incoming}} \iota_e \otimes \bigotimes_{e'\ \text{outgoing}} \iota'_{e'}\right), \tag{3.21}$$

where e/e' ranges the set of edges that end/begin at v and each f connects a pair of the edges (either two unprimed, or two primed, or one primed and one unprimed). The factor A_v is known in the spin foam literature as the vertex amplitude.

3.4.3. The state sums. Finally, the substitution of the right hand side of (3.16) into the spin foam operator $\mathcal{Z}(\kappa, \rho, P)$ definition (3.4, 3.10, 3.12, section 3) gives the following sum with respect to all the labellings of the internal edges $e \in \mathrm{int}\kappa$,

$$\iota : e \mapsto \iota_e \otimes \iota'_e \in \mathcal{B}_e \otimes \mathcal{B}^\dagger_e, \tag{3.22}$$

namely

$$\mathcal{Z}(\kappa, \rho, P) = \sum_\iota \prod_e P^{\iota'_e}_{\iota_e} \prod_f d_f \prod_v A_v \prod_{\tilde{l}} \frac{1}{\sqrt{d_{f_{\tilde{l}}}}} \bigotimes_{\tilde{e}} \iota_{\tilde{e}} \otimes \bigotimes_{\tilde{e}'} \iota'_{\tilde{e}'} \tag{3.23}$$

where f runs through the set of faces and d_f is the dimension of ρ_f, v ranges the set of the internal vertices, l ranges the set of the boundary edges (links) and f_l is the face containing l, and $\tilde{e}/\tilde{e}'$ ranges the set of edges which intersect $\partial\kappa$ at the end/beginning point. Finally, the familiar partition function emerges from (3.23) after summing with respect to the labellings ι which induce a same labelling $\partial\iota$ of the nodes of the boundary graph. The result reads

$$\mathcal{Z}(\kappa, \rho, P) = \sum_{\partial\iota} Z(\kappa, \rho, \partial\iota) \bigotimes_{\tilde{e}} \iota_{\tilde{e}} \otimes \bigotimes_{\tilde{e}'} \iota'_{\tilde{e}'}, \tag{3.24}$$

4. Operator spin foam models

4.1. Definition, natural models

4.1.1. Definition. A G operator spin foam model, where G is a compact group, can be defined as an assignment of an operator spin foam (κ, ρ, P) to each locally linear 2-complex κ endowed with a labelling ρ of the faces of κ with the irreducible representations of G (see section 2.1):

$$(\kappa, \rho) \;\mapsto\; (\kappa, \rho, P). \tag{4.1}$$

We will be assuming throughout this paper that all the equivalence moves of section 2.2 are symmetries of the models we consider. That is, given an assignment (4.1), whenever (κ, ρ, P) emerges in (4.1) then so does any (κ', ρ', P') that can be obtained from (κ, ρ, P) by the equivalence moves. We will be also assuming that for every model, an operator defined by its partition function (state sum) assigned to (κ, ρ) is the spin foam operator $\mathcal{Z}(\kappa, \rho, P)$ constructed in section 3.

4.1.2. Natural operator spin foam models. Below we will consider a class of natural operator spin foam models, that is models such that, briefly speaking,

- the assignment $e \mapsto P_e$ depends only on the unordered sequence of labels ρ_f such that $e \subset f$ and is independent of the other parts of a given 2-complex κ—see below for a technical definition. We will also be assuming that the assignment P is self-adjoint, that is
- for every internal edge $e \in \mathrm{int}\kappa^{(1)}$

$$P_e^\dagger = P_e, \tag{4.2}$$

(of course, P_e is defined only for the internal edges).

Technically, the first assumption means that for every unordered sequence R of irreducible representations of the group G, we fix an operator

$$P_R \;:\; \mathrm{Inv} \bigotimes_{\rho\in R} \mathcal{H}_\rho \;\to\; \mathrm{Inv} \bigotimes_{\rho\in R} \mathcal{H}_\rho. \tag{4.3}$$

Next, given any (κ, ρ) on the left-hand side of (4.1), we can use the equivalence relation to reorient the faces f containing e, such that their orientations agree with that of e, and therefore an operator P_e should be a map

$$P_e \;:\; \bigotimes_{f\supset e} \mathcal{H}_f \to \bigotimes_{f\supset e} \mathcal{H}_f. \tag{4.4}$$

Finally set

$$P_e = P_{R_e} \tag{4.5}$$

where R_e is the unordered sequence of the representations ρ_f, such that f ranges the set of faces containing e.

4.1.3. A general solution for the conditions defining natural models. It is not hard to see, that the set of conditions defining the class of the natural operator spin foam models has a general solution. First, the assumed symmetry with respect to the face splitting move of section 2.2.3 implies that

$$P_R = \mathrm{id} \tag{4.6}$$

for every unordered sequence R given by the pair of elements ρ and ρ^*. Secondly, the consequence of the symmetry with respect to the edge splitting move of section 2.2.4 is that for every unordered sequence R of irreducible representations, the operator P_R (4.3) satisfies

$$P_R P_R = P_R. \tag{4.7}$$

Hence, each operator P_e is an orthogonal projection onto a subspace

$$\mathcal{H}^{\mathrm{s}}_R \subset \mathcal{H}_R. \tag{4.8}$$

The subspaces $\mathcal{H}^{\mathrm{s}}_R$ are subject to the isomorphisms following from (2.29), (2.30). They give rise to subspaces $\mathcal{H}^{\mathrm{s}}_e$ assigned to the internal edges e of the 2-complexes.

4.2. Examples

In the following, we will show how different choices of the operator labelling P, defining different operator spin foam models, reproduce different state-sum models. All the examples that we discuss below fall into the class of the natural operator spin foam models. Hence, by construction, each operator (2.5) is a projection. The freedom consists in fixing a subspace (4.8)

$$\mathcal{H}^{\mathrm{s}}_R \;\subset\; \mathcal{H}_R = \mathrm{Inv} \bigotimes_{\rho\in R} \mathcal{H}_\rho \tag{4.9}$$

for every unordered sequence R of the equivalence classes of irreducible representations of G (see conditions (2.29), (2.30)).

4.2.1. Surjective P: BF theory. The easiest nontrivial choice is, of course, choosing P_e to be the identity, for every edge e,

$$P_e = \mathrm{id} : \mathcal{H}_e \to \mathcal{H}_e, \tag{4.10}$$

that is, the fixed Hilbert subspace for each unordered sequence R of the irreducible representations is the full Hilbert space of invariants,:

$$\mathcal{H}^{\mathrm{s}}_R = \mathcal{H}_R. \tag{4.11}$$

Within this model, consider all the possible operator spin foams (κ, ρ, P) defined on a fixed 2-complex κ without boundary. Note that in the boundary-free case, the operator spin foam operator $\mathcal{Z}(\kappa, \rho, P)$ of (section 3) is a $\mathbb{C}$-number. It was shown in [21] that in this case[5] for any set of square-integrable functions

$$\{S_f : G \to \mathbb{C} \;:\; f \in \kappa^{(2)}\} \tag{4.12}$$

one has that

$$\int_{G^E} \left(\prod_e \mathrm{d}h_e\right) \prod_f S_f(g_f) = \sum_\rho \left(\prod_f \hat{S}_f(\rho_f)\right) \mathcal{Z}(\kappa, \rho, P) \tag{4.13}$$

[5] Strictly speaking, [21] only considered the 2-complex of a hypercubical lattice—however, the results can easily be generalized to the case of arbitrary 2-complexes.

where e ranges through the set of edges $\kappa^{(1)}$, $E = |\kappa^{(1)}|$, f runs through the set of faces $\kappa^{(2)}$,

$$g_f := \overrightarrow{\prod_{e\in\partial f}} h_e \tag{4.14}$$

is the holonomy around a face f and

$$\hat{S}_f(\rho) = \frac{1}{\dim\rho}\int_G \mathrm{d}g\, S_f(g)\chi_\rho(g) \tag{4.15}$$

is the Fourier coefficient of S_f as provided by the Peter–Weyl theorem. In the formal limit of all S_f approaching the delta function on G, one has $\hat{S}_f \equiv 1$, and the right-hand side of (4.13) approaches the (unregularized) discretized BF-theory amplitude, e.g. when $G = SU(2)$ and κ is dual to a triangulation of a 3D manifold one obtains the Ponzano–Regge amplitude. One therefore recovers BF theory as the most basic example for the spin foam operator formalism.

It should be noted that, due to (4.13), there are two dual ways of viewing the spin foam operator (which, in the case of a two-complex without boundary, is just a $\mathbb{C}$-number) as the vacuum-to-vacuum amplitude of a path integral for discretized BF-theory[6]. One of them focuses on the assignment of different amplitudes to the cells in κ and summation over all assigned representations $\{\rho_f\}$ (as well as over an orthonormal basis in the decomposition (3.16) of the P_e). This is close to the state-sum language, and is the usual way in which spin foam models are written down. The dual (in the sense of Fourier transform) way of describing the amplitude is given by the left-hand side of (4.13), and can be interpreted as path integral for a lattice theory, with some gauge-invariant action functional determined by the S_f [25, 27, 28], depending on finitely many holonomies. In this formulation there is a direct connection to an action functional which is determined by the S_f. The fact that different actions lead to different face amplitudes $\hat{S}_f$ has been used to define generalizations to BF theory [23].

The step to (four-dimensional) gravity is usually obtained by changing P_e to be a projector on a smaller subspace of $\mathrm{Inv}(\mathcal{H}^*_{\rho_f} \otimes \ldots \otimes \mathcal{H}_{\rho_{f'}})$, being interpreted as the solution space of (a discretization of) the simplicity constraint, which turns topological BF theory into general relativity. The correct choice for this subspace, motivated by correct semiclassical limit of the theory, has been the subject of extensive research, leading to the different generalizations of BF theory: Barrett–Crane model, EPRL model (whose limiting case is the Barrett–Crane model) and FK model [3–6]. The holonomy representation similar to (4.13) exists for the Barrett–Crane model [26] and it has been recently defined also for the EPRL model [29].[7] It should be noted, however, that for all current quantizations of the discretized simplicity constraint it is still an open question whether the degrees of freedom of general relativity are captured in the correct manner, and doubts have been spelled out both from the geometrical point of view [30], as well as with respect to the question whether diffeomorphism symmetry is implemented correctly [31].

All interpretational issues aside, in the following we will give two further examples of (Euclidean) spin foam models which correspond to operators P_e with non-surjective P_e, namely the Barret–Crane model and the EPRL model (we will also comment on the possible extensions to the Lorentzian case).

4.2.2. Rank 1 P_e: the Barrett–Crane model. The next model on the list of easy nontrivial examples is the case when for every edge e of each operator spin foam (κ, ρ, P) of a

[6] More generally, when κ has a boundary, the spin foam amplitudes are give by the matrix entries of the spin foam operator, describing the transition amplitudes between different in- and out-states.

[7] Though in the latter case $G = SU(2) \times SU(2)$, in [29] the properties of the EPRL amplitude are used to write the left-hand side of (4.13) as integrals over only one copy of $SU(2)$, in order to make the connection to the loop quantum gravity Hilbert space more apparent.

model, the rank of the projection operator P_e is either 0 or 1. In fact, an example of a model of this type has been introduced by Barrett–Crane. In terms of our framework it is a $G = \mathrm{Spin}(4) \sim SU(2) \times SU(2)$ operator spin foam model. The representations associated with the faces of (κ, ρ, P) are therefore

$$\rho_f = (\rho_{j_f^+}, \rho_{j_f^-}),$$

where $j_f^{\pm}$ are half-integers labelling the $SU(2)$ representations, which—in the picture of Euclidean $4D$ gravity—constitute the self-dual and anti-self-dual parts of the Spin(4)-connection. The projector P_e assigned to each edge e is zero,

$$P_e = 0, \tag{4.16}$$

unless every representation associated with a face f hinging on the edge e is *balanced*, i.e. satisfies

$$j_f^+ = j_f^- \equiv j_f.$$

In the latter case, there is defined a unique element $\iota_{\mathrm{BC}} \in \mathcal{H}_e$, called the 'Barrett–Crane intertwiner', and P_e is set to be

$$P_e \; = \; \iota_{e\mathrm{BC}} \otimes \iota_{e\mathrm{BC}}^{\dagger}. \tag{4.17}$$

In the balanced case (below $\mathrm{Inv}_{\mathrm{SU}(2)} \ldots$ stands for the subspace of the SU(2) invariants; the subscript appears because we are also dealing with the spin(4) group),

$$\mathcal{H}_e = \mathrm{Inv}_{\mathrm{SU}(2)}\left(\bigotimes_{f:e\subset f} \mathcal{H}_{j_f}\right) \otimes \mathrm{Inv}_{\mathrm{SU}(2)}\left(\bigotimes_{f:e\subset f} \mathcal{H}_{j_f}\right) \tag{4.18}$$

where $\mathcal{H}_{j_f}$ is the carrier Hilbert space of the corresponding $SU(2)$ representation. The Barrett–Crane intertwiner is the bilinear form defined in the Hilbert space $\mathrm{Inv}_{\mathrm{SU}(2)}\big(\bigotimes_{f:e\subset f} \mathcal{H}_{j_f}\big)^*$ by the restriction of the canonical invariant bilinear form defined in $\bigotimes_{f:e\subset f} \mathcal{H}_{j_f}^*$.

It can be constructed as follows: denote by $\epsilon_j \in \mathcal{H}_j \otimes \mathcal{H}_j$ the unique up to rescaling SU(2) invariant. Furthermore, denote by

$$\pi \, : \, \bigotimes_{f:e\subset f} \mathcal{H}_f \; \to \; \mathcal{H}_e$$

the orthogonal projector. The Barrett–Crane intertwiner is then given by

$$\iota_{\mathrm{BC}} = c\, \pi\left(\bigotimes_{f:e\in f} \epsilon_{j_f}\right) \tag{4.19}$$

where c is a constant chosen such that ι_{BC} is normalized.

In a slightly different language the edge and face splitting condition in the case of the Barrett–Crane model was also discussed in [24].

4.2.3. Lessons from the previous two examples. The previous two examples give us an interpretation of the natural operator spin foam models. Each natural G operator spin foam model can be thought of as the G BF theory with constraints. Given an operator spin foam (κ, ρ, P) of a given model, elements of the Hilbert subspaces $\mathcal{H}_e^{\mathrm{s}}$ (4.8) assigned to the edges are quantum solutions to the constraints. In the case of the Barrett–Crane model, the constraint is intertwining the operators defined in $\bigotimes_f \mathcal{H}_{j_f^+}$, and, respectively, in $\bigotimes_f \mathcal{H}_{j_f^-}$, and the Barrett–Crane solution is the identity map, provided the representations are balanced.

4.2.4. The natural operator spin foam model for the EPRL intertwiners. The EPRL model [4] was developed to overcome some of the difficulties one was encountering with the attempt to interpret the Barrett–Crane model as a state-sum model for 4D Euclidean gravity. The fact that the operator labelling for the Barrett–Crane model assigns to the edges of the foams (at most) rank 1 operators lead to the argument that the theory does not capture enough degrees of freedom (and in particular is not compatible with an LQG boundary Hilbert space) [11].

In the Euclidean EPRL model, again $G = SU(2) \times SU(2)$.[8] Similarly, the projector P_e, for every edge e of an operator spin foam (κ, ρ, P), is defined by specifying its image, that is the corresponding subspace $\mathcal{H}^s_R$ of (4.8). The Euclidean EPRL model relies on the so-called Barbero–Immirzi parameter γ, which needs to be a rational number $\gamma \neq 0, \pm 1$. The EPRL model subspace $\mathcal{H}^s_e$ denoted here by $\mathcal{H}^{s,\text{EPRL}}_e$ is nonempty only if, for every face, there is a half-integer k_f such that

$$j^{\pm}_f = \tfrac{1}{2}|1 \pm \gamma| k_f \tag{4.20}$$

are also half-integers. The elements of this space $\mathcal{H}^{s,\text{EPRL}}_e$ are called 'EPRL intertwiners'. In [4] the EPRL map

$$\iota^{\text{EPRL}}_\gamma : \text{Inv}_{SU(2)}(\rho_{k_1} \otimes \ldots \otimes \rho_{k_n}) \longrightarrow \text{Inv}(\rho_{(j^+_1, j^-_1)} \otimes \ldots \otimes \rho_{(j^+_n, j^-_n)}) \tag{4.21}$$

is defined for any unordered sequences of admissible half-integers

$$\tilde{R} = (k_1, \ldots, k_n), \qquad R = \left((j^-_1, j^+_1), \ldots, (j^-_n, j^+_n)\right)$$

which maps $SU(2)$-intertwiners η to EPRL intertwiners $\iota^{\text{EPRL}}_\gamma(\eta)$. The space $\mathcal{H}^{s,\text{EPRL}}_R$ of the EPRL intertwiners is therefore the image of the map $\iota^{\text{EPRL}}_\gamma$, which can be shown to be one-to-one [7], but not an isometry, i.e. it does not preserve the Hilbert space inner product [9]. Using this map, one maps a (typically orthonormal) basis

$$\tilde{\mathcal{B}} \subset \text{Inv}_{SU(2)}(\rho_{k_1} \otimes \ldots \otimes \rho_{k_n})$$

into a basis

$$\mathcal{B}^{\text{EPRL}} \subset \mathcal{H}^{s,\text{EPRL}}_R$$

(typically not orthonormal). In this way, for every edge e, the corresponding subspace $\mathcal{H}^{s,\text{EPRL}}_e \subset \mathcal{H}_e$ is equipped with a basis $\mathcal{B}^{\text{EPRL}}_e \subset \mathcal{H}^{s,\text{EPRL}}_e$, elements of which are $\iota^{\text{EPRL}}_e(\eta_e)$, where η_e ranges through a basis $\tilde{B}_e$ of the corresponding space (via (4.20)) $\mathcal{H}^{\text{SU}(2)}_e$ of the $SU(2)$ intertwiners. We can expand the operator P_e in the basis $\mathcal{B}^{\text{EPRL}}_e$:

$$P_e = \sum_{\eta_e, \eta'_e} P^{\eta_e}_{\eta'_e} \iota^{\text{EPRL}}_\gamma(\eta_e) \otimes \left(\iota^{\text{EPRL}}_\gamma(\eta'_e)\right)^\dagger, \tag{4.22}$$

where the coefficients $P^{\eta_e}_{\eta'_e}$ are defined by the Hilbert product $(\cdot|\cdot)_e$ in $\mathcal{H}_e$, namely

$$\sum_{\eta'_e} P^{\eta_e}_{\eta'_e} \left(\iota^{\text{EPRL}}_\gamma(\eta'_e)|\iota^{\text{EPRL}}_\gamma(\eta''_e)\right)_e = \delta^{\eta_e}_{\eta''_e}. \tag{4.23}$$

As a result, given an operator spin foam (κ, ρ, P), instead of assigning an operator P_e to each edge e, one considers a set of assignments η of two $SU(2)$ intertwiners $\eta_e, \eta'_e \in \mathcal{H}^{\text{SU}(2)}_e$, to the end and, respectively, the beginning point of each edge e (figure 11). Following the

[8] There is—as well as for the Barrett–Crane model—a Lorentzian version available [22, 34], which uses different symmetry groups, but which are not discussed in this paper.

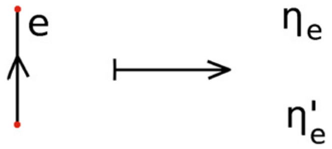


Figure 11. Two $SU(2)$ intertwiners η_e, η'_e are assigned to the end and, respectively, the beginning point of each edge e.

derivation of the amplitude form of the partition function done in [9] we obtain for the case of an oriented 2-complex with boundary:

$$\mathcal{Z}(\kappa, \rho, P) = \sum_{\eta} \prod_{e} P^{\eta_e}_{\eta'_e} \prod_{f} (2j^+_f + 1)(2j^-_f + 1) \prod_{v} A_v \\ \prod_{\tilde{l}} \frac{1}{\sqrt{(2j^+_{f_{\tilde{l}}} + 1)(2j^-_{f_{\tilde{l}}} + 1)}} \bigotimes_{\tilde{e}} \iota^{\mathrm{EPRL}}_{\gamma}(\eta_e) \otimes \bigotimes_{\tilde{e}'} \left(\iota^{\mathrm{EPRL}}_{\gamma}(\eta'_{e'})\right)^{\dagger} \tag{4.24}$$

where f runs through the set of faces, v ranges the set of the internal vertices, l ranges the set of the boundary edges (links) and f_l is the (unique) face containing l, and $\tilde{e}/\tilde{e}'$ ranges the set of edges which intersect $\partial\kappa$ at the end/beginning point and A_v is the vertex amplitude (3.21).

Note that the $P^{\eta_e}_{\eta'_e}$ matrix is not appearing in the original definition of the EPRL state sum in [4]. It has to be included if P_e is supposed to be an orthogonal projection, since the EPRL map $\iota^{\mathrm{EPRL}}_{\gamma}$ is not an isometry. The $P^{\eta_e\eta_{(w,e)}}$ can be interpreted as a measure factor appearing when summing over intertwiners. If the $P^{\eta_e}_{\eta'_e}$ factors are not included in the partition function, then the EPRL intertwiners are summed over with a different measure, and lead to P_e not being an orthogonal projection—in particular, the operator $\mathcal{Z}(\kappa, \rho, P)$ is no longer invariant under trivially subdividing an edge.

The case of the Lorentzian EPRL model is more subtle and complicated. It deserves further investigation. Let us mention here only important difficulties one encounters trying to translate our formalism into the $SL(2, \mathbb{C})$ case.

- First of all, $SL(2, \mathbb{C})$ invariants are not anymore normalizable vectors in the tensor product of irreducible representations. Procedures like group averaging are necessary to define Hilbert space structure on the space of invariants.
- Contractions of the invariants in vertices need additional regularisation. Such a regularization is well defined for 3-edge connected vertices [10]. However, separate treatment is needed for vertices that are not 3-edge connected.
- Having Hilbert space structure and the EPRL map, we can check whether this map is injective or unitary. It is plausible that as in the case of EPRL, the $SO(4)$ case injectivity holds but unitarity fails. Then we can use $SU(2)$ invariants to parametrize $SL(2, \mathbb{C})$ ones.
- The problem with non-3-edge-connected vertices leads to an ambiguity in the face splitting moves due to the fact that these vertices necessary appear in such a move; however, edge splitting move still leads to the conclusion of projectivity of the edge operator.

4.3. Further examples

4.3.1. Natural models for monoidal categories. Instead of considering operators and vector spaces one can more generally consider morphisms in a monoidal category. The models defined by Oeckl in [23] are then the most natural spin foam models. For the case of symmetric monoidal categories Oeckl defines these models by considering cables and wires that correspond to edges and faces of our foam. The cable or edges carry morphisms, the wires carry labels and determine the combinatorics of contraction, just as the faces do in our discussion. For semi-simple symmetric monoidal categories Oeckl defines an operator *T*[9] depending only on an unordered set of objects. In the case of the category of representations of a group *G* the operator *T* defined by Oeckl coincides with the projection on the invariant subspace of the tensor product of representations that we considered for BF theory. Oeckl generalizes this not by changing the operator *T*, as in the EPRL model, but by changing the face amplitudes.

It should also be noted that our edge and face splitting conditions are special cases of the fusion moves of Oeckl.

4.3.2. Simplicial group field theory. Group field theories generate spin foam amplitudes in their Feynman expansion. If the GFT is simplicial in the sense of [32], its expansion naturally leads to operator spin foam models with the edge operator given by the propagator of the theory. This is discussed in detail in [33]. The amplitudes defined there in terms of the 'EPRL/FK propagator'[10] are of the form of an operator spin foam with all faces labelled by $L^2(G, \mathrm{d}h)$. As this is an infinite dimensional Hilbert space the trace will not in general be finite. Note that from the perspective of GFTs the edge splitting condition, which forces the edge operators to be projectors, is not desirable, as it implies propagators with a trivial spectrum in the quantum field theory.

5. Summary

The operator spin foams that we have introduced are linear combinations of the usual spin foams; therefore, they should be robust in any spin foam context. In our paper we list the moves naturally defined in the space of the operator spin foams. The moves are used in our paper to define an equivalence relation—we would like to emphasize that 'equivalent' operator spin foams are not identical. Whenever we construct a model, the question we address is, whether or not all those moves are symmetries of our model. In the paper we considered the class of models which do admit all those symmetries. And we derived the consequences of that assumption. A spin foam model may not have the symmetries defined by our equivalence relation. One may restrict the set of spin foams that are allowed by a given model. In particular, the vertices obtained by splitting an internal edge can be just forbidden. However, we seem to agree that we would like to be able to identify an operator spin foam with a refined operator spin foam. So the allowed spin foams should admit at least sufficiently refined spin foams. Given an operator spin foam, there is the naturally defined operator denoted by Z in our paper. Nonetheless, an operator Z′ constructed within a given model from an operator spin foam may be different from the operator Z. For example one can introduce some extra structure at the vertices and use it to define Z′. The first three 'moves' defining the equivalence relation that we have constructed: reorientation of faces, edges, and splitting a face, are the consequence of analogous moves and equivalence of the spin networks. The equivalence upon splitting an edge and the suitable relation between the operators is a choice natural for the consistency between combining the operator spin foams and combining the corresponding operators. Together with splitting a face it also ensures independence from the cellular decomposition chosen for the 2-complex. As a result the natural spin foams depend only on the topology of the 2-complex

[9] Proposition 2.12 of [23].
[10] Defined in equation (20) of [33].

and a labelling of maximal faces. Also the contraction as well as the operator spin foam *operator* are naturally defined operations that exist independently on our believes and can be used as tools of any spin foam theory. The family of natural spin foam models we derived from assumed symmetry took appearance of constrained BF spin foam models. Each of the models is defined by the restriction of a proper spin foam model to a subspace in the space of intertwiners. Since gravity is often viewed in that way, one of the natural spin(4) operator spin foam model characterized by suitable subspace of solutions to the simplicity constraints could be the proper quantum gravity model. The most important example is given by the EPRL subspace of the spin(4) intertwiners. In that case, the corresponding natural operator spin foam model coincides with the proposal of [9], whereas it is different from the EPRL proposal [4]. That difference was already emphasised in [9]. The new conclusion coming from the current work is the set of rules governing operator spin foams that is satisfied in one case and is not satisfied by the other one. If experiment shows that nature favours the less natural model, we should still understand better its operator structure.

Acknowledgments

We have benefited a lot from stimulating discussions with Carlo Rovelli, Abhay Ashtekar, Jonathan Engle, Claudio Perini and Elena Magliaro. WK, MK and JL were partially supported by (i) the grants N N202 104838, N N202 287538 and 182/N-QGG/2008/0 (PMN) of Polish Ministerstwo Nauki i Szkolnictwa Wyzszego; (ii) the grant Mistrz of the Foundation for the Polish Science. JL also acknowledges the US National Science Foundation (NSF) grant PHY-0456913. MK also acknowledges financial support from the project 'International PhD Studies in Fundamental Problems of Quantum Gravity and Quantum Field Theory' of Foundation for Polish Science, cofinanced from the programme IE OP 2007-2013 within European Regional Development Fund. All the authors thank the ESF network Quantum Geometry and Quantum Gravity for short visit grants to collaborate.

References

[1] Baez J 2000 An introduction to spinfoam models of BF theory and quantum gravity *Lecture Notes Phys.* **543** 25–94 (arXiv:gr-qc/9905087v1)

[2] Barrett J and Naish-Guzman I 2009 The Ponzano–Regge model *Class. Quantum Grav.* **26** 155014 (arXiv:0803.3319 [gr-qc])

[3] Barrett J W and Crane L 1998 Relativistic spin-networks and quantum gravity *J. Math. Phys.* **39** 3296–302 (arXiv:gr-qc/9709028)

[4] Engle J, Livine E, Pereira R and Rovelli C 2008 LQG vertex with finite Immirzi parameter *Nucl. Phys.* B **799** 136–49 (arXiv:gr-qc/0711.0146v2)

[5] Engle J, Pereira R and Rovelli C 2008 Flipped spinfoam vertex and loop gravity *Nucl. Phys.* B **798** 251–90 (arXiv:gr-qc/0708.1236v1)

Livine E R and Speziale S 2007 A new spinfoam vertex for quantum gravity *Phys. Rev.* D **76** 084028 (arXiv:gr-qc/0705.0674v2)

[6] Freidel L and Krasnov K 2008 A new spin foam model for 4d gravity *Class. Quantum Grav.* **25** 125018 (arXiv:gr-qc/0708.1595v2)

Livine E R and Speziale S 2008 Consistently solving the simplicity constraints for spinfoam quantum gravity *Europhys. Lett.* **81** 50004 (arXiv:0708.1915 [gr-qc])

Bojowald M and Perez A 2010 Spin foam quantization and anomalies *Gen. Rel. Grav.* **42** 877–907 (arXiv:gr-qc/0303026)

[7] Kamiński W, Kisielowski M and Lewandowski J 2010 Spin-foams for all loop quantum gravity *Class. Quantum Grav.* **27** 095006 (arXiv:0909.0939v2)

[8] Bahr B 2001 On knottings in the physical Hilbert space of LQG as given by the EPRL model arXiv:1006.0700

[9] Kamiński W, Kisielowski M and Lewandowski J 2010 The EPRL intertwiners and corrected partition function *Class. Quantum Grav.* **27** 165020 (arXiv:0912.0540v1)

[10] Baez J C and Barrett J W 2001 Integrability for relativistic spin networks *Class. Quantum Grav.* **18** 4683 (arXiv:gr-qc/0101107)

Engle J and Pereira R 2009 Regularization and finiteness of the Lorentzian LQG vertices *Phys. Rev.* D **79** 084034 (arXiv:0805.4696 [gr-qc])

Kamiński W 2010 All 3-edge-connected relativistic BC and EPRL spin-networks are integrable arXiv:1010.5384v1

[11] Bianchi E, Modesto L, Rovelli C and Speziale S 2006 Graviton propagator in loop quantum gravity *Class. Quantum Grav.* **23** 6989–7028 (arXiv:gr-qc/0604044)

Alesci E and Rovelli C 2007 The complete LQG propagator: I. Difficulties with the Barrett–Crane vertex *Phys. Rev.* D **76** 104012 (arXiv:gr-qc/0708.0883)

Alesci E and Rovelli C 2007 The complete LQG propagator: II. Asymptotic behavior of the vertex *Phys. Rev.* D **77** 044024 (arXiv:0711.1284)

Alesci E, Bianchi E and Rovelli C 2008 LQG propagator: III. The new vertex arXiv:0812.5018

Mamone D and Rovelli C 2009 Second-order amplitudes in loop quantum gravity arXiv:0904.3730

Bianchi E, Magliaro E and Perini C 2009 LQG propagator from the new spin foams *Nucl. Phys.* B **822** 245–69 (arXiv:0905.4082)

[12] Reisenberger M P 1994 World sheet formulations of gauge theories and gravity arXiv:gr-qc/9412035

Reisenberger M P and Rovelli C 1997 'Sum over surfaces' form of loop quantum gravity *Phys. Rev.* D **56** 3490–508 (arXiv:gr-qc/9612035v)

[13] Perez A 2003 Spinfoam models for quantum gravity *Class. Quantum Grav.* **20** R43 (arXiv:gr-qc/0301113v2)

[14] Alesci E, Bianchi E, Magliaro E and Perini C 2008 Asymptotics of LQG fusion coefficients arXiv:0809.3718v2

[15] Baratin A, Flori C and Thiemann T 2008 The Holst spin foam model via cubulations arXiv:gr-qc/0812.4055v2

Han M and Thiemann T 2009 On the relation between operator constrain, master constrain, reduced phase space and path integral quantisation arXiv:0911.3428v1

Engle J, Han M and Thiemann T 2009 Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation arXiv:0911.3433v1

[16] Ashtekar A 1991 *Lectures on Non-perturbative Canonical Gravity* (Singapore: World Scientific) (Notes prepared in collaboration with R S Tate)

[17] Thiemann T 2007 *Introduction to Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)

[18] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)

[19] Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report *Class. Quantum Grav.* **21** R53 (arXiv:gr-qc/0404018)

[20] Han M, Huang W and Ma Y 2007 Fundamental structure of loop quantum gravity *Int. J. Mod. Phys.* D **16** 1397–474 (arXiv:gr-qc/0509064)

[21] Oeckl R and Pfeiffer H 2001 The dual of pure non-Abelian lattic gauge theory as a spin foam model *Nucl. Phys.* B **598** 400–26 (arXiv:hep-th/0008095)

[22] Barrett J and Crane L 2000 A Lorentzian signature model for quantum general relativity *Class. Quantum Grav.* **17** 3101–18 (arXiv:gr-qc/9904025)

[23] Oeckl R 2003 Generalized lattice gauge theory, spin foams and state sum invariants *J. Geom. Phys.* **46** 308 (arXiv:hep-th/0110259)

[24] Conrady F 2005 Geometric spin foams, Yang–Mills theory and background-independent models arXiv:gr-qc/0504059

[25] Freidel L and Krasnov K 1999 Spin foam models and the classical action principle *Adv. Theor. Math. Phys.* **2** 1183 (arXiv:hep-th/9807092)

[26] Pfeiffer H 2002 Dual variables and a connection picture for the Euclidean Barrett–Crane model *Class. Quantum Grav.* **19** 1109 (arXiv:gr-qc/0112002)

[27] Han M and Thiemann T 2010 Commuting simplicity and closure constraints for 4D spin foam models arXiv:1010.5444 [gr-qc]

[28] Bonzom V and Livine E R 2009 A Lagrangian approach to the Barrett–Crane spin foam model *Phys. Rev.* D **79** 064034 (arXiv:0812.3456 [gr-qc])

Bonzom V 2009 From lattice BF gauge theory to area-angle Regge calculus *Class. Quantum Grav.* **26** 155020 (arXiv:0903.0267 [gr-qc])

Bonzom V 2009 Spin foam models for quantum gravity from lattice path integrals *Phys. Rev.* D **80** 064028 (arXiv:0905.1501 [gr-qc])

[29] Bianchi E, Magliaro E and Perini C 2010 Spinfoams in the holomorphic representation *Phys. Rev.* D **82** 124031 (arXiv:1004.4550 [gr-qc])

[30] Dittrich B and Ryan J P 2008 Phase space descriptions for simplicial 4d geometries arXiv:0807.2806 [gr-qc]
Dittrich B and Ryan J P 2010 Simplicity in simplicial phase space *Phys. Rev.* D **82** 064026 (arXiv:1006.4295 [gr-qc])

[31] Bahr B and Dittrich B 2009 Breaking and restoring of diffeomorphism symmetry in discrete gravity arXiv:0909.5688 [gr-qc]

[32] Krajewski T, Magnen J, Rivasseau V, Tanasa A and Vitale P 2010 Quantum corrections in the group field theory formulation of the EPRL/FK models arXiv:1007.3150 [gr-qc]

[33] Geloun J B, Gurau R and Rivasseau V 2010 arXiv:1008.0354 [hep-th]

[34] Barett J, Dowdall R, Fairbairn W, Gomez H and Hellmann F 2009 Asymptotic analysis of the EPRL four-simplex amplitude *J. Math. Phys.* **50** 112504 (arXiv:0902.1170 [gr-qc])