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Inflationary observables in loop quantum cosmology

Martin Bojowald

Institute for Gravitation and the Cosmos, The Pennsylvania State University
104 Davey Lab, University Park, PA 16802, USA
E-mail: bojowald@gravity.psu.edu

Gianluca Calcagni

Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
Am Mühlenberg 1, D-14476 Golm, Germany
E-mail: calcagni@aei.mpg.de

Abstract: The full set of cosmological observables coming from linear scalar and tensor perturbations of loop quantum cosmology is computed in the presence of inverse-volume corrections. Background inflationary solutions are found at linear order in the quantum corrections; depending on the values of quantization parameters, they obey an exact or perturbed power-law expansion in conformal time. The comoving curvature perturbation is shown to be conserved at large scales, just as in the classical case. Its associated Mukhanov equation is obtained and solved. Combined with the results for tensor modes, this yields the scalar and tensor indices, their running, and the tensor-to-scalar ratio, which are all first order in the quantum correction. The latter could be sizable in phenomenological scenarios. Contrary to a pure minisuperspace parametrization, the lattice refinement parametrization is in agreement with both anomaly cancellation and our results on background solutions and linear perturbations. The issue of the choice of parametrization is also discussed in relation with a possible superluminal propagation of perturbative modes, and conclusions for quantum spacetime structure are drawn.

Keywords: Cosmology of Theories beyond the SM, Loop quantum gravity and cosmology.

Contents

1. Introduction

Loop quantum cosmology (LQC) [1] provides the framework for implementing several effects seen to arise for the quantum geometry of loop quantum gravity (LQG) [2, 4, 3] in a cosmological setting. Strict formulations of loop quantum cosmology are defined on minisuperspace, where one quantizes homogeneous spacetimes using the methods of loop quantum gravity. The characteristic effects of discrete spatial geometry then emerge also in the reduced context, changing the dynamics of expanding universe models. The dynamics changes in particular at high densities, giving rise to mechanisms avoiding classical singularities.

Quantum corrections to the classical spacetime structure not only exist at high densities but remain present, in weaker form, as the universe expands and dilutes. In such a regime, non-linearities as well as corrections from loop quantum gravity can be treated perturbatively in a gauge-invariant way. This has been done for linear perturbations around spatially flat FRW models with one particular class of corrections expected from loop quantum gravity. These corrections, related to spatial discreteness, arise whenever an inverse of a certain metric component (or rather, densitized-triad components) appears in a matter Hamiltonian or in the Hamiltonian constraint of gravity. Such inverse components are ubiquitous, for instance in kinetic matter terms, and thus the resulting corrections, called inverse-volume corrections, are an unavoidable consequence of loop quantum gravity. Specifically, the corrections are due to the fact that the quantized densitized triad has a discrete spectrum, with the value zero contained in the spectrum. Such an operator does not allow the existence of a densely-defined inverse, but an operator providing the inverse as the classical limit can nevertheless be defined [5]. When one departs from the classical regime, however, quantum corrections arise whose form can be computed in some models and which can be parametrized sufficiently generally for phenomenological investigations. In the presence of these corrections, provided they are small, consistent gauge-invariant cosmological perturbation equations have been determined [6].

Gauge invariance in general relativity and the candidates for its quantization is intimately related with spacetime structure because gauge transformations include changes of coordinates. Accordingly, quantum corrections to gauge-invariant perturbation equations show how fundamental quantum spacetime effects such as discrete geometry are reflected, in physical terms, by implications for cosmological observables. Cosmology then provides an intriguing link between the fundamental physics of spacetime and phenomenology, as it will be explored in this paper.

Looking at the issue from the phenomenological side, linearization of dynamical equations in the presence of inhomogeneous perturbations is one of the most extensively studied problems in cosmology. With recent progress in LQC, the context in which such questions can be addressed has been extended to a new class of quantum-gravity effects with the aim of making a prediction for early-universe spectra and, hopefully, constraining the theory. The perturbed equations contain quantum correction functions and are augmented by counterterms which guarantee cancellation of anomalies in the effective constraint algebra [7, 8, 9]. These contributions provide insights in the spacetime structure, modify the equations of motion and, eventually, imply characteristic signatures for physical observables.

The perturbed equations of motion for vector, tensor, and scalar modes have been computed, respectively, in [7, 8, 6], while the tensor spectrum and index have been found and explored in [10, 11]. To close the set of cosmological consistency relations, it remains to compute the scalar spectrum and its derived observables. Such is the *first goal* of this paper: We shall find the Mukhanov equation of scalar perturbations for the first time and solve it. With exactly the same procedure, we will rederive the solution of tensor modes with zero effort. The scalar and tensor spectra and spectral indices will be obtained together with a consistency relation between the tensor-to-scalar ratio and the tensor index. In doing so, we will derive a conservation law for the curvature perturbation, extending the well-known classical result to the quantum-corrected equations, and discuss its implications for quantum spacetime structure.

All cosmological observables, including index running and higher-order quantities, are linearly corrected by quantum terms $\delta_{\rm Pl}$ whose magnitude we cannot predict yet because we have no control over the details of the underlying full quantum theory. The presence of extra parameters would seem to at best make it possible to place upper bounds on the quantum corrections $\delta_{\rm Pl}$ for a given inflationary potential. For instance, one can arrange to have large-enough quantum corrections so that the scalar running be sizable. On the other hand, we can naturally envisage a situation where $\delta_{\rm Pl}$ is much smaller than the slow-roll parameters, and therefore completely negligible. Until better control over the theory is achieved, the phenomenological consequences of inverse-volume (or other) corrections will remain unclear but here, as our *second goal*, we highlight the following issue: Different parametrization schemes will lead to different background solutions and predictions for the size of $\delta_{\rm Pl}$. In particular, the usual minisuperspace parametrization of FRW loop quantum cosmology seems to be incompatible with anomaly cancellation in inhomogeneous LQC, as well as with the simplest power-law solutions. Conversely, the lattice refinement parametrization overcomes all these problems, can predict much larger quantum corrections, but it might indicate a problem related to superluminal propagation of the perturbations. We wish to stress all these features and the importance of further investigating the lattice refinement parametrization, which requires input from the full theory.

Since anomaly cancellation so far has been shown to occur only in the quasi-classical regime where inverse-volume corrections and counterterms are small, we shall concentrate on this case. Therefore it is not possible to draw comparisons with holonomy-correction results [10, 12, 13] or with previous works which partially fixed the gauge, considered test fields and neglected metric back-reaction, since they were devoted to the superinflationary regime of a near-Planckian epoch [14, 15, 16, 17, 18].

In section 3 we review the LQC background equations of motion for a flat FRW model and a scalar field, and the parametrizations arising in minisuperspace quantization and the lattice refinement phenomenological approach. Background solutions with de Sitter and exact or perturbed power-law expansion are found in section 4.

Section 5 is completely devoted to scalar perturbations. After a review and some important updates of the results of [6], we show that the comoving curvature perturbation is conserved at large scales, as in the classical case (section 5.3). The Mukhanov equation for a scalar perturbation variable is then found and solved in section 5.4, while the set of linear cosmological observables (power spectrum, index and index running) is derived in section 5.6. The set of observables is completed in section 6, where the same analysis of the previous section is applied *verbatim* to tensor perturbations. A discussion of the main achievements of this paper and future directions can be found in section 7.

2. Spacetime structure and phenomenology

Loop quantum gravity has provided results that show the discreteness of spatial quantum geometry: geometrical operators such as those for area and volume have discrete spectra [19, 20]. Taken by themselves, these features are not observable because the corresponding objects, the areas of spatial surfaces or the volumes of spatial regions, are not gauge invariant. However, these mathematical properties, derived from the underlying principle of background independence on which the theory is built, indicate new features of the quantum representation with important effects for the dynamics. The volume operator, for instance, enters matter Hamiltonians [5] and the gravitational Hamiltonian constraint operator [21], and thus influences their properties at the quantum level. In this way, once the dynamical equations are sufficiently well understood, physical observables are affected and a potential comparison with observations is made possible.

The dynamics of loop quantum gravity amounts to that of a coupled, interacting many-body problem in which the elementary constituents are the fundamental building blocks of space. Such equations governing the dynamics are difficult to solve exactly, but several crucial effects visible in them are generic and characteristic; they provide the basis for phenomenological evaluations. There are two main effects: (i) inverse-volume corrections and (ii) holonomy corrections due to the fact that the background-independent quantization used in LQG allows the representation only of exponentiated curvature components, gathered as the holonomies of connection variables. Inverse-volume corrections are currently under much better control in cosmological perturbation equations, and so they will be our main focus here.

To see implications of those effects in the dynamics, the many-body Hamiltonians must be analyzed. There is by now a systematic procedure to do so, based on effective canonical equations to describe semi-classical dynamics [22, 23]. Effective equations in this context, analogous to low-energy effective actions for expansions around the ground state of anharmonic systems, are obtained from expectation values of the Hamiltonian (constraint) operators in a generic class of semi-classical states. So far, these equations have been computed and analyzed only in isotropic cosmological models, in which a solvable system analogous to the harmonic oscillator is available [24]. The form of the equations, however, is more general and can be used also in the presence of inhomogeneous cosmological perturbations. To avoid bias, one only has to ensure that correction functions are parametrized sufficiently generally, since the control over the theory is not yet strong enough to provide unique predictions for them.

These equations, including parametrized quantum-gravity corrections, allow important conclusions at the fundamental and phenomenological levels by a clear line of arguments. First, the structure of spatial geometry changes according to LQG, affecting the form of the constraints generating gauge transformations. Secondly, in the presence of quantum corrections the algebra under Poisson brackets (or commutators when quantized) of the constraints as gauge generators is modified, amounting to a different realization of the classical transformations of spacetime and the underlying gauge behaviour. This abstract feature has several consequences. For instance, quantum-geometry corrections cannot be implemented by any higher-curvature effective action because those corrections would not change the classical constraint algebra and the underlying notion of gauge and covariance. An effective action that can include all effects of LQG must be of a more general form, for instance allowing for non-commutative geometry. Moreover, gauge transformations belonging to a deformed algebra no longer correspond to ordinary coordinate transformations on a manifold. Thus, effective line elements may be questionable because the transformations of metric components and coordinate differentials collected in $\mathrm{d}s^2$ no longer match to make the line element invariant. In such a context, physical information can be gained only from gauge-invariant variables that take into account the new gauge structure. With these additional effects, there is a chance that quantum gravity corrections may be stronger than usually expected, for instance, from naive arguments based on the size of higher-curvature corrections.

One possible new phenomenon is of particular interest. When the constraint algebra is deformed, the Bianchi identity or the conservation equation for stress-energy is modified. There are still analogous identities if the deformation is consistent and anomaly-free, but they may refer to different quantities than in the classical case. Then, the curvature perturbation is no longer guaranteed to be conserved, a possibility which has already been raised [6, 25]. If the curvature perturbation is no longer conserved, on the other hand, magnification effects for modes outside the Hubble radius can be expected. Even though deviations from conservation given by quantum gravity were small at any given (sufficiently late) time, expected to be determined by the tiny ratio of the Planck length by the Hubble radius, the lever arm of non-conservation during long times between horizon exit during inflation and re-entry might magnify those tiny effects. In this way, a tight link is obtained between fundamental spacetime structure and cosmological phenomenology.

Here we will provide results in both directions, fundamental physics as well as phenomenology. In particular, we will demonstrate that a subtle cancellation in the anomaly-free correction functions of LQC does make the curvature perturbation conserved in spite of the non-trivial deformation of the constraint algebra, a feature which has not been noticed before. As a consequence, effective linear perturbations of Friedmann–Robertson–Walker geometries can be meaningfully constructed even in the presence of a deformed gauge structure. Corrections to standard perturbation equations then follow naturally.

3. Background equations and parametrizations

To begin, we write down the LQC effective equations of motion for an FRW background $\mathrm{d}s^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}x^i \mathrm{d}x_i)$ in conformal time τ (see, e.g., the review in [11] for a detailed derivation of these results). We shall ignore holonomy corrections, which have not yet been considered in the perturbed dynamics. Notice, however, that these contributions in some parametrizations dominate over inverse-volume corrections as far as tensor modes are concerned [11, 13]; we will further discuss this point in the final section. For a scalar field φ with potential V, the effective Friedmann and Klein–Gordon equations read

$$\mathcal{H}^2 = \frac{8\pi G}{3}\,\alpha\left[\frac{\varphi'^2}{2\nu} + pV(\varphi)\right] \tag{3.1}$$

and

$$\varphi'' + 2\mathcal{H}\left(1 - \frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)\varphi' + \nu p V_{,\varphi} = 0\,, \tag{3.2}$$

where G is the gravitational constant, $\mathcal{H} \equiv a'/a$ is the Hubble parameter, primes denote derivatives with respect to conformal time, and $p = a^2$ (in comoving volume units) is the triad variable in minisuperspace. (Triad variables can take both signs depending on the orientation of space. Here we assume p to be positive without loss of generality for effective equations.) From equations (3.1) and (3.2) one obtains the Raychaudhuri equation

$$\mathcal{H}' = \left(1 + \frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}\right)\mathcal{H}^2 - 4\pi G\frac{\alpha}{\nu}\left(1 - \frac{1}{3}\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)\varphi'^2\,. \tag{3.3}$$

In these equations,

$$\alpha \approx 1 + \alpha_0\delta_{\mathrm{Pl}}, \tag{3.4}$$

$$\nu \approx 1 + \nu_0\delta_{\mathrm{Pl}}, \tag{3.5}$$

where

$$\delta_{\mathrm{Pl}} \equiv \left(\frac{p_{\mathrm{Pl}}}{p}\right)^{\frac{\sigma}{2}} = \left(\frac{a_{\mathrm{Pl}}}{a}\right)^{\sigma}\,, \tag{3.6}$$

and σ and $p_{\rm Pl} = a_{\rm Pl}^2$ are constant.[1] We will often need to switch from p to conformal derivatives via

$$' = 2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\ln p} \tag{3.7}$$

and the formulæ

$$\delta_{\rm Pl}' = -\sigma\mathcal{H}\delta_{\rm Pl}\,, \qquad \delta_{\rm Pl}'' = \sigma\mathcal{H}^2(\sigma - 1 + \epsilon)\delta_{\rm Pl}\,, \tag{3.8}$$

with $\epsilon = 1 - \mathcal{H}'/\mathcal{H}$ introduced as a slow-roll parameter below.

Functional forms of $\alpha(p)$ and $\nu(p)$ are fully computable in general form from operators in exactly isotropic models [26] and for regular lattice states in the presence of inhomogeneities [27], with parametrizations of quantization ambiguities affecting the values of α_0, ν_0 and σ [28, 29]. However, only the expanded forms (3.4) and (3.5) are needed in the perturbative regime considered here. From these explicit calculations of inverse-volume operators and their spectra, one can derive further properties characteristic of loop quantum gravity. In particular, correction functions implementing inverse-volume corrections, when evaluated at large values of the densitized triad or the scale factor in a nearly isotropic geometry, approach the classical value always from above. This consequence, which is a robust feature under quantization ambiguities and will turn out to be important later, implies that the coefficients α_0 and ν_0 introduced in the parametrizations used here must be positive.

3.1 Parametrizations

The above equations are derived in a minisuperspace Hamiltonian formalism where the super-Hamiltonian (the only non-trivial constraint on homogeneous backgrounds) is first symmetry-reduced, and then quantized with LQG techniques. The resulting equations then constitute partial effective equations, which means that they capture the behaviour of expectation values of observables in semi-classical states, but without taking all quantum corrections into account. In particular, quantum back-reaction by fluctuations and the holonomy corrections of loop quantum gravity are not included at the present stage. The LQG techniques consist in a choice of canonical variables and operator ordering which make the final result quite different with respect to the traditional minisuperspace Wheeler–DeWitt quantum cosmology. At the semi-classical level, the main difference is the presence of correction functions (3.4) and (3.5). The constants α_0, ν_0 and σ will enter the cosmological observables and it is important to set their value range beforehand. This range strongly depends on the physical interpretation of the model. We can identify two views on the issue, one purely homogeneous and isotropic and the other associated with the natural presence of inhomogeneities.

[1] We put a subscript 'Pl' in the definition (3.6) in order to avoid confusion with perturbations such as $\delta\varphi$. However, the equations below do not rely on any particular characteristic scale $a_{\rm Pl}$, which may differ from the Planck length.

3.1.1 Minisuperspace parametrization

On an ideal FRW background, open and flat universes have infinite spatial volume and the super-Hamiltonian constraint is formally ill defined because it entails a divergent integration of a spatially constant quantity over a comoving spatial slice Σ,

$$\int_\Sigma \mathrm{d}^3x = +\infty\,.$$

To make the integral finite, it is customary to define the constraint on a freely chosen finite region of size $\mathcal{V} = a^3\mathcal{V}_0$, where $\mathcal{V}_0$ is the corresponding comoving volume:

$$\int_\Sigma \mathrm{d}^3x \to \int_{\Sigma(\mathcal{V}_0)} \mathrm{d}^3x = \mathcal{V}_0 < +\infty\,.$$

The volume appears in the correction function (3.6) as $\delta_{\rm Pl} \sim a^{-\sigma} \sim \mathcal{V}^{-\sigma/3}$. To make $\delta_{\rm Pl}$ adimensional, one can use the Planck length $\ell_{\rm Pl}$ to write

$$\delta_{\rm Pl} \sim \left(\frac{\ell_{\rm Pl}^3}{\mathcal{V}_0}\right)^{\frac{\sigma}{3}} a^{-\sigma}\,. \tag{3.9}$$

To specify the coefficient further, one sometimes introduces the area gap $\Delta_{\rm Pl} \equiv 2\sqrt{3}\pi\gamma\ell_{\rm Pl}^2$, where γ is the Barbero–Immirzi parameter.[2] A detailed calculation then shows that the constant coefficients α_0 and ν_0 are [11]

$$\alpha_0 = \frac{(3q-\sigma)(6q-\sigma)}{2^2 3^4}\left(\frac{\Delta_{\rm Pl}}{p_{\rm Pl}}\right)^2, \qquad \nu_0 = \frac{\sigma(2-l)}{54}\left(\frac{\Delta_{\rm Pl}}{p_{\rm Pl}}\right)^2. \tag{3.10}$$

They depend on two sets of ambiguities, one ($1/2 \leq l < 1$ and $1/3 \leq q < 2/3$) related to different ways of quantizing the classical Hamiltonian[3] and the other (σ) depending on which geometrical minisuperspace variable has an equidistant stepsize in the dynamics: in terms of the triad variable p, $p^{\sigma/2}$ is equidistant if inverse-volume corrections with exponent σ in $\Delta_{\rm Pl}$ appear. More physically, this parameter is related to how the number of plaquettes of an underlying discrete state changes with respect to the volume as the universe expands. The latter is a phenomenological prescription for the area of holonomy plaquettes, but ideally it should be an input from the full theory [34].

In the minisuperspace context, a natural choice of these parameters is

$$\sigma = 6\,, \qquad l = \tfrac{3}{4}\,, \qquad q = \tfrac{1}{2}\,, \tag{3.11}$$

[2]The use of twice this value according to recent findings may be better justified [30] but the resulting change in the values of α_0 and ν_0 is not relevant for what follows.

[3]The different interval for q with respect to the one given in [11] stems from the same argument in the full theory which constrains the range of l [31, 32]. We set the natural value of q to be $1/2$ rather than 1 [33] in equation (3.11).

so that, assuming $p_{\rm Pl} = \Delta_{\rm Pl}$ [33], one has

$$\alpha_0 = \tfrac{1}{24} \approx 0.04\,, \qquad \nu_0 = \tfrac{5}{36} \approx 0.14\,. \tag{3.12}$$

Notice that $\sigma = 6$, motivated by holonomy corrections not becoming large at small curvature, corresponds to the so-called 'improved quantization scheme' [33], a name which applies also to values of l and q different from equation (3.11). A specific example provided in [33], for instance, is $q = 1$ such that $\alpha_0 = 0$. Note, however, that the exact value $\alpha_0 = 0$ is obtained in this case only due to a spurious cancellation in isotropic settings; it seems to suggest negligible inverse-volume corrections but is unreliable compared with more general derivations. For phenomenology at the current level of precision, the most significant parameter is σ, which is not as much affected by different choices of the minisuperspace scheme.

Since $\delta_{\rm Pl}$ is $\mathcal{V}_0$-dependent, inverse-volume corrections cannot strictly be made sense of in a pure minisuperspace treatment. Inverse-volume corrections, as used here, have never been derived fully consistently in this context owing to the $\mathcal{V}$-dependence. In particular, as discussed in [35, 36], while the improved scheme does take into account refinement for holonomies in an ad-hoc manner, it ignores these effects for inverse-volume corrections. This failure to represent inverse-volume effects, which are crucial for well-defined Hamiltonians in loop quantum gravity, presents a serious limitation of pure minisuperspace models which can be overcome only by bringing in further ingredients to take into account the behaviour of inhomogeneities, as indicated in what follows. Precise derivations become more complicated in this situation, but by a combination with input from phenomenology one can obtain valuable restrictions on the possible realizations.

We wish to make a further comment on this issue. In the context of inflation, one has a set of observables given by the anisotropy spectra and their derivatives; all these quantities contain parameters of the inverse volume corrections. Since they are all evaluated at horizon crossing, the comoving fiducial volume therein (implicitly conceived as greater than the causal region $\sim \mathcal{H}^{-3}$) can be naturally set to be the Hubble volume. If one maintained the minisuperspace parametrization also in the presence of perturbations, the conclusion would be that the fiducial volume problem is less severe than expected. However, the minisuperspace setting pertains only to exactly isotropic models, and this solution of the problem is at best incomplete. To get a clearer picture, we should include inhomogeneities already at the fundamental level. The following argument shows how to do so qualitatively.

3.1.2 Lattice refinement parametrization

The chosen volume $\mathcal{V}$ is a purely mathematical object which should not appear in physical observables, but it does appear in equation (3.9). Since $\delta_{\rm Pl}$ will enter the observables, we might face a problem. To make the situation better behaved, we introduce generic effects of inhomogeneities. One example for doing so is the lattice parametrization discussed in [11] which, as one implication, extends the range of the parameters. Then, one has a large range of σ,

$$4 < \sigma \leq 6\,, \tag{3.13}$$

the value of the improved minisuperspace quantization scheme being included. When $p_{\rm Pl} = \Delta_{\rm Pl}$, for instance, varying over the range of σ (equation (3.13)), l and q (as specified below equation (3.1.1)),

$$0 < \alpha_0 \leq \tfrac{5}{81} \approx 0.06\,, \tag{3.14}$$

$$0.07 \approx \tfrac{2}{27} < \nu_0 < \tfrac{1}{6} \approx 0.17\,. \tag{3.15}$$

Instead of repeating the arguments leading to equation (3.13), we consider an alternative lattice parametrization where one uses the ‘patch’ volume of an underlying discrete state in correction functions, rather than the much larger volume $\mathcal{V}$ [36]. In this parametrization, motivated by key aspects of discrete spacetime dynamics, the ranges of parameters change more significantly than in (3.13), with important consequences for phenomenology. Now, corrections refer to the patch size

$$v \equiv \frac{\mathcal{V}}{\mathcal{N}}\,, \tag{3.16}$$

with $\mathcal{N}$, the main input from quantum gravity, the number of discrete patches in $\mathcal{V}$. By construction, v is independent of the size of the region, since both $\mathcal{V}_0$ and $\mathcal{N}$ scale in the same way when the size of the region is changed. Physical predictions should not feature the region one chooses unless one is specifically asking region-dependent questions (such as: What is the number of vertices in a given volume?).

Given the available parameters and their dimensions, the leading-order quantum correction in α and ν is then of the form

$$\delta_{\rm Pl} = \left(\frac{\ell_{\rm Pl}^3}{v}\right)^{m/3} = \left(\ell_{\rm Pl}^3\frac{\mathcal{N}}{\mathcal{V}}\right)^{m/3}\,, \tag{3.17}$$

where $m > 0$ is a positive integer parameter. It determines the power by which p appears in leading corrections of an expansion of inverse-volume correction functions. Primarily, the correction functions α and ν, and thus $\delta_{\rm Pl}$ as well, depend on flux values, corresponding to p for the isotropic background. Since p changes sign under orientation reversal but the operators are parity invariant, only even powers of p can appear, giving $m = 4$ as the smallest value. At this stage of development of the full theory, it is not entirely clear that general correction functions depend only on the fluxes (rather than, e.g., also on the eigenvalues of more complicated volume operators; for properties of their spectra see [37, 38]). Therefore we set $m \geq 4$.

A time-dependent $\mathcal{N}(t)$ corresponds to the dynamical 'lattice-refinement' behaviour [34]. For some stretches of time, one can choose to use the scale factor a as the time variable and represent $\mathcal{N}(a)$ as a power law

$$\mathcal{N} = \mathcal{N}_0 a^{-6x}\,, \tag{3.18}$$

where $\mathcal{N}_0$ is some (coordinate and $\mathcal{V}_0$-dependent) parameter and the power x describes different qualitative behaviours of changing lattices. Overall, we have

$$\delta_{\rm Pl} = \left(\ell_{\rm Pl}^3 \frac{\mathcal{N}_0}{\mathcal{V}_0}\right)^{\frac{m}{3}} a^{-(2x+1)m}\,. \tag{3.19}$$

This equation cannot be obtained in a pure minisuperspace setting (e.g., [11]) where only one parameter σ enters as in (3.9). The presence of an extra parameter, compared to the minisuperspace parametrization, may appear as a disadvantage, but we will see later that it is required for being able to match with phenomenology; otherwise the theory would be ruled out.

In the lattice-refinement derivation, the parameter a plays two roles, one as a dynamical geometric quantity and the other as internal time. While writing down the semi-classical Hamiltonian with inverse-volume (and holonomy) corrections, one is at a non-dynamical quantum-geometric level. Then, internal time is taken at a fixed value but the geometry still varies on the whole phase space. In this setting, we must keep $\mathcal{N}$ fixed while formulating the constraint as a composite operator. The net result is the Hamiltonian constraint operator of the basic formulation of loop quantum cosmology [39, 40] not taking into account any refinement, corresponding to $x = 0$ and $m = \sigma$. However, when one solves the constraint or uses it for effective equations, one has to bring in the dynamical nature of $\mathcal{N}$ from an underlying full state. This is the motivation for promoting $\mathcal{N}$ to a time-dependent quantity, a step which captures operator as well as state properties of the effective dynamics. Its parametrization as a power law of the scale factor is simply a way to encode the qualitative (yet robust, see below) phenomenology of the theory. The general viewpoint is similar to mean-field approximations which model effects of underlying degrees of freedom by a single, physically motivated function.

Comparing with the earlier minisuperspace parametrization, equation (3.19) gives $\sigma = (2x+1)m$ as far as the a-dependence is concerned. The number of vertices $\mathcal{N}$ must not decrease with the volume, so $x \leq 0$; it is constant for $x = 0$. Also, $v = \mathcal{V}/\mathcal{N} \sim a^{3(1+2x)}$ is the elementary geometry as determined by the state; in a discrete geometrical setting, this quantity has a lower non-zero bound which requires $-1/2 \leq x \leq 0$. In particular, for $x = -1/2$ we have a constant patch volume, corresponding to what is assumed in the improved minisuperspace quantization scheme [33]. In contrast with the minisuperspace parametrization (3.11), in the effective parametrization of equation (3.19) we have $\sigma = 0$ in this case. Thus, even for $x = -1/2$ is the parametrization new and different from the minisuperspace representation, overcoming the problem of representing inverse-volume effects in a pure minisuperspace treatment.

To summarize the general lattice-refinement scheme, $\sigma = (2x+1)m$ is a time-independent or slowly changing parameter[4] given by the reduction from the full theory and with range

$$\sigma \geq 0\,. \tag{3.20}$$

Assuming this range will be of utmost importance for justifying the validity of the cosmological perturbation spectra. In fact, we shall find that σ must be small in order for the spectra to be almost scale invariant, a range of values that cannot occur for either the minisuperspace parametrization or the first lattice parametrization (3.13). Responsible for the better matching is the new parameter x, while m alone (or $x = 0$) would give a range similar to the minisuperspace parametrization.

Note that, in principle, σ may be different in α and ν for an inhomogeneous model. However, here we assume that the background equations (3.4) and (3.5) with the same $\delta_{\rm Pl}$ are valid also in the perturbed case. Not only is this choice natural whenever background quantities are considered, but it is also crucial for several simplifications to follow.

Before moving on, a remark is in order. The patches of volume v find a most natural classical analog in inhomogeneous cosmologies, in particular within the separate universe picture [41]. For quantum corrections, the regions of size v are provided by an underlying discrete state and thus correspond to quantum degrees of freedom absent classically. However, the discrete nature of the state implies that inhomogeneities are unavoidable and no perfectly homogeneous geometry can exist. Given these inhomogeneities and their scale provided by the state, one can reinterpret them in a classical context, making use of the separate universe picture. There, the volume $\mathcal{V}$ can be regarded as a region of the universe where inhomogeneities are non-zero but small. This region is coarse grained into smaller regions of volume v, each centered at some point $\mathbf{x}$, wherein the universe is FRW and described by a 'local' scale factor $a(t,\mathbf{x}) = a_{\mathbf{x}}(t)$. The difference between scale factors separated by the typical perturbation wavelength $|\mathbf{x}' - \mathbf{x}| \sim \lambda \ll \mathcal{V}^{1/3}$ defines a spatial gradient interpreted as a metric perturbation. In a perfectly homogeneous context, $v \sim \mathcal{V}$ and there is no sensible notion of cell subdivision of $\mathcal{V}$; this is tantamount to stating that only the fiducial volume will enter the quantum corrections and the observables, $\mathcal{N} = \mathcal{N}_0$. On the other hand, in an inhomogeneous universe the quantity v carries a time dependence which, in turn, translates into a momentum dependence. The details of the cell subdivision (number of cells per unit volume) are intimately related with the

[4]The creation or subdivision of new cells in a discrete state depends on the spatial geometry and can thus be considered as changing more slowly than other processes in an expanding universe. On large time scales, the parameter σ may change, distinguishing different microscopic epochs in the history of the universe.

structure of the small perturbations and their spectrum. Thus, lattice refinement is better suitable in the cosmological perturbation analysis. As long as perturbations are linear and almost scale invariant, the size of volume within which the study is conducted is totally irrelevant.

3.1.3 Parameter estimates

Since the lattice refinement picture is phenomenological, presently we are unable to determine the quantum correction $\delta_{\rm Pl}$ from first principles and, at this stage, the latter is regarded as a free parameter which can be constrained by experiments. In the minisuperspace and lattice parametrization (3.13), on the other hand, there may be an argument which estimates the magnitude of $\delta_{\rm Pl}$ heuristically. In fact, in those parametrizations $\delta_{\rm Pl} = (\delta_0/\mathcal{V})^{\sigma/3}$, where δ_0 is some constant volume and $\mathcal{V}$ is the fiducial volume; for lack of better knowledge, one often assumes $\delta_0 \sim \ell_{\rm Pl}^3$. All inflationary observables are evaluated at horizon crossing, so the volume $\mathcal{V}$ is very naturally fixed by the size of the Hubble horizon at that moment (denoted with a $*$) [11]:

$$\mathcal{V} \sim H_*^{-3}\,, \tag{3.21}$$

where $H = \dot{a}/a = \mathcal{H}/a$. (This equation is invariant under isotropic rescalings of the coordinates. In terms of $a_{\rm Pl}$ introduced via $\delta_{\rm Pl} = (a_{\rm Pl}/a)^\sigma$ in the minisuperspace parametrization, we may write $a_{\rm Pl}/a = \ell_{\rm Pl} H_*$.) The point here is that so far $\mathcal{V}$ has been arbitrary, the only requirement being that it contains the Hubble region at any given moment. Provided δ_0 is fixed *a priori*, this equation fixes $\mathcal{V}$ once and for all because one is not at the liberty of changing the numerical factor in (3.21), which is O(1). Slightly different definitions of the Hubble horizon differ only for O(1) coefficients, which do not affect the discussion qualitatively (on the other hand, O(10) or O(0.1) coefficients are unacceptable because the observables here are defined at, not before or after, horizon crossing).

To estimate $\delta_{\rm Pl}$ during inflation, we could take the grand-unification scale $H_* \sim 10^{14} \div 10^{17}$ GeV. As $\delta_{\rm Pl} = (\ell_{\rm Pl} H_*)^\sigma$ and $4 < \sigma \leq 6$, one has the upper bound $\delta_{\rm Pl} \lesssim {\rm O}(10^{-8})$. Typical prefactors in observable quantities may even carry an extra ${\rm O}(10^{-1})$ suppression, as we will find, so none of the inverse volume corrections with these choices are observable, even in the scalar running, as the slow-roll parameters are ${\rm O}(10^{-2})$. In comparison, the inflationary tensor index gets an extra contribution $\delta_{\rm hol} \propto (\ell_{\rm Pl} H_*)^2 \lesssim {\rm O}(10^{-4})$ from holonomy corrections [13], which dominate over $\delta_{\rm Pl}$. Holonomy corrections in the perturbed scalar sector has never been computed, but we expect a similar hierarchy of scales.

Unfortunately, these estimates rely on a particular choice for δ_0 and the argument cannot be regarded as robust. The size of this dimensionful constant strongly depends on the underlying theory, and a change in magnitude of δ_0 would modify the above results. Moreover, we will see that the minisuperspace parametrizations are not compatible with requirements on background solutions combined with anomaly freedom. The estimate of $\delta_{\rm Pl}$ could thus at best be used as an external input for the lattice-refinement parametrization. Then, once the scale of inverse-volume corrections is fixed, a further consistency condition must be satisfied because inverse-triad corrections and holonomy corrections are determined by the same parameter that specifies the underlying discreteness scale [36]. To evaluate this condition, we use the alternative form of $\delta_{\rm Pl} = (\ell_{\rm Pl}^3/v)^{m/3} = \ell_{\rm Pl}^4/v^{4/3}$ for $m = 4$. The underlying discreteness scale, as a distance parameter, is then the linear dimension of patches, $L = v^{1/3} = \ell_{\rm Pl}/\delta_{\rm Pl}^{1/4} = \ell_{\rm Pl}/(\ell_{\rm Pl}H_*)^{\sigma/4}$. On the other hand, the strength of holonomy corrections can be expressed in terms of the critical density $\rho_{\rm crit} = 3/(8\pi G\gamma^2 L^2)$;[5] holonomy corrections are weak when the matter density satisfies $\rho \ll \rho_{\rm crit}$. With L as assumed here,

$$\rho_{\rm crit} = \frac{3H_*^2(\ell_{\rm Pl}^2 H_*^2)^{\sigma/4-1}}{8\pi G\gamma^2} \sim \rho_* \left(\frac{\rho_*}{\rho_{\rm Pl}}\right)^{\sigma/4-1}, \tag{3.22}$$

with the matter density ρ_* at the time of horizon crossing. Thus, for $\rho_{\rm crit} \gg \rho_*$ to ensure that holonomy corrections do not significantly alter the classical behaviour at horizon crossing, we must require $\sigma < 4$. Then, $\delta_{\rm Pl}$ becomes larger than estimated above. For a critical density of Planckian size, which is often desired so as to have strong quantum-gravity corrections only in the Planckian regime, σ must be close to zero. For such values, $\delta_{\rm Pl} \sim {\rm O}(1)$ with the above estimate of δ_0, clearly dominating holonomy corrections at horizon crossing. (For $\sigma = 2$, the critical density is the geometric mean $\rho_{\rm crit} \sim \sqrt{\rho_{\rm Pl}\rho_*}$ and we have $\delta_{\rm Pl} \sim \delta_{\rm hol}$.)

3.2 Slow-roll parameters

For later convenience, we define the first three slow-roll parameters as

$$\epsilon \equiv 1 - \frac{\mathcal{H}'}{\mathcal{H}^2}, \tag{3.23}$$

$$\eta \equiv 1 - \frac{\varphi''}{\mathcal{H}\varphi'}, \tag{3.24}$$

$$\xi^2 \equiv \frac{1}{\mathcal{H}^2}\left(\frac{\varphi''}{\varphi'}\right)' + \epsilon + \eta - 1, \tag{3.25}$$

which coincide with the standard definitions in synchronous time

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \qquad \eta \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}}, \qquad \xi^2 \equiv \frac{1}{H^2}\left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right)^{\cdot}.$$

[5] As an explicit calculation shows, the true critical density is actually $\alpha\rho_{\rm crit}$ [11]. Setting $q = 1$ and $\sigma = 6$ in the lattice parametrization one gets $\alpha = 1$, as in [33]. In general α does appear, but the arguments in this discussion are qualitative and we can ignore this issue.

The parameter ϵ will be especially important later on and we can rewrite it as

$$\begin{aligned}
\epsilon &= 4\pi G\frac{\alpha}{\nu}\frac{\varphi'^2}{\mathcal{H}^2}\left(1-\frac{1}{3}\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)-\frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}\\
&= 4\pi G\frac{\varphi'^2}{\mathcal{H}^2}\left\{1+\left[\alpha_0+\nu_0\left(\frac{\sigma}{6}-1\right)\right]\delta_{\mathrm{Pl}}\right\}+\frac{\sigma\alpha_0}{2}\delta_{\mathrm{Pl}}\,,
\end{aligned} \tag{3.26}$$

using the Raychaudhuri equation (3.3) in the first step. Notice that the symbol $=$ in the last line of equation (3.26) implicitly hides the $\mathrm{O}(\delta_{\mathrm{Pl}})$ truncation. This note of caution applies to any of the equations below, where $\mathrm{O}(\delta_{\mathrm{Pl}}^2)$ terms are dropped as required for self-consistency of perturbed equations. In contrast, the slow-roll approximation will always be invoked explicitly and indicated with the symbol $\approx$.

The derivatives of ϵ and η are

$$\begin{aligned}
\epsilon' &= 2\mathcal{H}\left(\epsilon+\frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}\right)\left[\epsilon-\eta+\frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}-\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}-\frac{1}{3}\frac{\mathrm{d}^2\ln\nu}{\mathrm{d}\ln p^2}\left(1-\frac{1}{3}\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)^{-1}\right]\\
&\quad -2\mathcal{H}\frac{\mathrm{d}^2\ln\alpha}{\mathrm{d}\ln p^2}\\
&= 2\mathcal{H}\epsilon(\epsilon-\eta)-\sigma\mathcal{H}\tilde{\epsilon}\delta_{\mathrm{Pl}}\,, \qquad (3.27)\\
\eta' &= \mathcal{H}(\epsilon\eta-\xi^2)\,, \qquad (3.28)
\end{aligned}$$

where

$$\tilde{\epsilon}\equiv\alpha_0\left(\frac{\sigma}{2}+2\epsilon-\eta\right)+\nu_0\left(\frac{\sigma}{6}-1\right)\epsilon\,. \tag{3.29}$$

While in standard inflation ϵ is almost constant whenever it is small (since the classical part of ϵ' is quadratic in the parameters), depending on the size of the quantum correction $\sigma\tilde{\epsilon}\delta_{\mathrm{Pl}}$ the quantity ϵ' could be of the same order as ϵ. However, we expect δ_{Pl} to be small in the typical setting.

For a given background $a(\tau)$ and $\varphi(\tau)$, the slow-roll parameters are functionally identical to the classical case. Clearly, the potential required to give rise to such an evolution is different, as one can see also from the (later useful) relations

$$\begin{aligned}
V_{,\varphi} &= \frac{\mathcal{H}\varphi'}{\nu p}\left(\eta-3+2\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)\\
&= \frac{\mathcal{H}\varphi'}{\nu p}\left(\eta-3-\sigma\nu_0\delta_{\mathrm{Pl}}\right)\,, \qquad (3.30)\\
V_{,\varphi\varphi} &= \frac{\mathcal{H}^2}{\nu p}\left[3(\epsilon+\eta)-\eta^2-\xi^2+2(3-\epsilon-2\eta)\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}+4\frac{\mathrm{d}^2\ln\nu}{\mathrm{d}\ln p^2}-4\left(\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)^2\right]\\
&= \frac{1}{\nu p}\left(-m_\varphi^2+\mathcal{H}^2\sigma\mu_\varphi\delta_{\mathrm{Pl}}\right)\,, \qquad (3.31)
\end{aligned}$$

where

$$m_\varphi^2\equiv\mathcal{H}^2[\eta^2+\xi^2-3(\epsilon+\eta)]\,, \tag{3.32}$$

$$\mu_\varphi\equiv\nu_0\left(\sigma-3+\epsilon+2\eta\right)\,. \tag{3.33}$$

4. Background solutions

Let ϕ be a set of generic scalar variables and let us write the background equations of motion, as well as the soon-to-be-found Mukhanov equation for the scalar perturbation, as $\mathcal{O}[\phi] = 0$, where $\mathcal{O}$ is a (possibly non-linear) differential operator. One can drop quantum terms of order higher than $\delta_{\rm Pl}$ and split each variable into a classical part $\phi_{\rm c}$ and a quantum correction $\phi_{\rm q}\delta_{\rm Pl}$ [11],

$$\phi = \phi_{\rm c} + \phi_{\rm q}\delta_{\rm Pl}\,, \tag{4.1}$$

so that each equation becomes

$$\mathcal{O}_{\rm c}[\phi_{\rm c}] + \{\mathcal{O}_{\rm c}[\phi_{\rm q}] + \mathcal{O}_{\rm q}[\phi_{\rm c}]\}\,\delta_{\rm Pl} = 0\,. \tag{4.2}$$

Requiring that the classical and quantum part vanish separately (a condition which defines what is meant by $\phi_{\rm c}$) yields two equations:

$$\mathcal{O}_{\rm c}[\phi_{\rm c}] = 0\,, \qquad \mathcal{O}_{\rm c}[\phi_{\rm q}] + \mathcal{O}_{\rm q}[\phi_{\rm c}] = 0\,. \tag{4.3}$$

This splitting strongly resembles the one into coarse- and fine-grained perturbations in stochastic inflation [42, 43, 44, 45]; in fact, for a Klein–Gordon scalar the second equation (4.3) is nothing but a 'complementary' Langevin-type equation for a quantum variable with a noise term sourced by the classical part.

For example, consider the scalar field and scale factor profiles

$$\varphi = \varphi_{\rm c} + \varphi_{\rm q}\delta_{\rm Pl}\,, \qquad a = a_{\rm c} + a_{\rm q}\delta_{\rm Pl}\,. \tag{4.4}$$

The Hubble parameter can be written as

$$\mathcal{H} = \mathcal{H}_{\rm c} + \mathcal{H}_{\rm q}\delta_{\rm Pl}\,, \qquad \mathcal{H}_{\rm q} = \frac{a_{\rm q}}{a_{\rm c}}\left[\frac{a_{\rm q}'}{a_{\rm q}} - (1+\sigma)\mathcal{H}_{\rm c}\right]. \tag{4.5}$$

Also, the scalar potential V is expanded in a Taylor series around $\varphi_{\rm c}$,

$$V(\varphi) = V(\varphi_{\rm c}) + V_{,\varphi}(\varphi_{\rm c})\varphi_{\rm q}\delta_{\rm Pl} \equiv V_{\rm c} + V_{\rm q}\delta_{\rm Pl}\,, \tag{4.6}$$

$$V_{,\varphi}(\varphi) = V_{,\varphi}(\varphi_{\rm c}) + V_{,\varphi\varphi}(\varphi_{\rm c})\varphi_{\rm q}\delta_{\rm Pl} \equiv V_{,\varphi\,{\rm c}} + V_{,\varphi\,{\rm q}}\delta_{\rm Pl}\,. \tag{4.7}$$

Plugging these expressions into the Friedmann and Klein–Gordon equations (3.1) and (3.2), we obtain a pair of classical equations,

$$\mathcal{H}_{\rm c}^2 = \frac{8\pi G}{3}\left(\frac{\varphi_{\rm c}'^{\,2}}{2} + a_{\rm c}^2 V_{\rm c}\right), \tag{4.8}$$

$$\varphi_{\rm c}'' + 2\mathcal{H}_{\rm c}\varphi_{\rm c}' + a_{\rm c}^2 V_{,\varphi\,{\rm c}} = 0\,, \tag{4.9}$$

plus another pair of relations involving the correction functions φ_q and a_q:

$$\mathcal{H}_\mathrm{c}\mathcal{H}_\mathrm{q} = \frac{4\pi G}{3}\left[\varphi_\mathrm{c}'(\varphi_\mathrm{q}'-\sigma\mathcal{H}_\mathrm{c}\varphi_\mathrm{q})+\frac{\alpha_0-\nu_0}{2}\,\varphi_\mathrm{c}'^{\,2}+a_\mathrm{c}^2(\alpha_0 V_\mathrm{c}+V_\mathrm{q})+2a_\mathrm{q}V_\mathrm{c}\right], \tag{4.10}$$

$$0 = \varphi_\mathrm{q}''+2\mathcal{H}_\mathrm{c}(1-\sigma)\varphi_\mathrm{q}'+\sigma\mathcal{H}_\mathrm{c}^2(\sigma+\epsilon_\mathrm{c}-3)\varphi_\mathrm{q}+a_\mathrm{c}^2(\nu_0 V_{,\varphi\,\mathrm{c}}+V_{,\varphi\,\mathrm{q}}) +2a_\mathrm{q}V_{,\varphi\,\mathrm{c}}+\varphi_\mathrm{c}'(2\mathcal{H}_\mathrm{q}+\sigma\nu_0\mathcal{H}_\mathrm{c})\,. \tag{4.11}$$

At this point we look for special background solutions with exactly constant slow-roll parameter ϵ, i.e., with a scale factor expanding as a power-law:

$$a = a_\mathrm{c} = |\tau|^n\,,\qquad n\leq -1\,, \tag{4.12}$$

where $\tau<0$ and the limit $n\sim -1$ corresponds to de Sitter spacetime. By this definition, the quantum corrections a_q and $\mathcal{H}_\mathrm{q}$ vanish identically. At the classical level, one gets the power-law solution [46]

$$\varphi_\mathrm{c} = \varphi_0\ln|\tau| = \pm\sqrt{\frac{n(n+1)}{4\pi G}}\ln|\tau|\,, \tag{4.13}$$

$$V = V_0\,\mathrm{e}^{-2(n+1)\varphi/\varphi_0} = \frac{n(2n-1)}{8\pi G}\,\mathrm{e}^{-2(n+1)\varphi/\varphi_0}\,, \tag{4.14}$$

for which $V_\mathrm{c}=V(\varphi_\mathrm{c})=V_0|\tau|^{-2n-2}$ and

$$\epsilon = \eta_\mathrm{c} = \xi_\mathrm{c} = 1+\frac{1}{n}. \tag{4.15}$$

Let us see if there exist solutions of the form $\varphi_\mathrm{q}=\varphi_{\mathrm{q}0}|\tau|^b$, distinguishing two cases:

- If $n\neq -1$,

$$V_\mathrm{q} = -2(n+1)V_0\frac{\varphi_{\mathrm{q}0}}{\varphi_0}|\tau|^{b-2n-2}\,, \tag{4.16}$$

  for V_q as defined in (4.6). Equations (4.10) and (4.11) become

$$0 = \varphi_0\varphi_{\mathrm{q}0}\left[(b-\sigma n)-2(n+1)\frac{V_0}{\varphi_0^2}\right]|\tau|^{b-2}+\varphi_0^2\left[\frac{\alpha_0-\nu_0}{2}+\alpha_0\frac{V_0}{\varphi_0^2}\right]|\tau|^{-2}\,, \tag{4.17}$$

$$0 = \varphi_{\mathrm{q}0}\left[b(b-1)+2bn(1-\sigma)+\sigma n(\sigma n+1-2n)+4(n+1)^2\frac{V_0}{\varphi_0^2}\right]|\tau|^{b-2} +\nu_0\varphi_0\left[\sigma n-2(n+1)\frac{V_0}{\varphi_0^2}\right]|\tau|^{-2}\,. \tag{4.18}$$

  It turns out that, if $\varphi_{\mathrm{q}0}\neq 0$, the solution requires $b=0$ and the equalities

$$\frac{\varphi_{\mathrm{q}0}}{\varphi_0} = \left(\frac{3n}{1+n}\alpha_0-\nu_0\right)\frac{1}{2(\sigma n+2n-1)} = \left(\nu_0-\frac{3}{\epsilon}\alpha_0\right)\frac{1-\epsilon}{2(3+\sigma-\epsilon)}\,, \tag{4.19}$$

$$0 = \nu_0\left[6(3-\epsilon)-\sigma(3+\sigma-\epsilon)\right]-3\alpha_0\left[2(3-\epsilon)-\frac{\sigma}{\epsilon}(3-\sigma-\epsilon)\right]. \tag{4.20}$$

The potential then reads

$$V = \left[1 - 2(n+1)\frac{\varphi_{q0}}{\varphi_0}\delta_{\rm Pl}\right] V_{\rm c}\,, \tag{4.21}$$

while the second and first slow-roll parameters are

$$\eta = \eta_{\rm c} - \sigma^2 n\frac{\varphi_{q0}}{\varphi_0}\delta_{\rm Pl}\,, \tag{4.22}$$

$$\xi^2 = \xi_{\rm c}^2 - \sigma^2(\sigma n + 1 + n)\frac{\varphi_{q0}}{\varphi_0}\delta_{\rm Pl}\,. \tag{4.23}$$

An exact solution is

$$\sigma = 0\,, \qquad \alpha_0 = \nu_0\,, \qquad V = (1-\alpha_0\delta_{\rm Pl})V_{\rm c}\,, \tag{4.24}$$

while for $\sigma \gtrsim {\rm O}(1)$ and small ϵ the expression (4.20) is satisfied if α_0 and ν_0 obey

$$\nu_0\left[18 - \sigma(3+\sigma)\right] - 3\alpha_0\left[6 - \frac{\sigma}{\epsilon}(3-\sigma)\right] \approx 0\,. \tag{4.25}$$

For $0 < \sigma < 3$, the solution has $\alpha_0, \nu_0 > 0$ if $\epsilon > \sigma(3-\sigma)/6$ (e.g., $\epsilon > 1/3$ if $\sigma = 1$ or $\sigma = 2$). Because of the lower bound on ϵ, this solution prefers the limiting values $\sigma \sim 0$, $\sigma \sim 3$ if extreme slow roll is to be realized.

When $\sigma \geq 3$, α_0 and ν_0 have opposite sign and $|\nu_0| \gg |\alpha_0|$. This case is excluded by the above considerations on inverse-volume operators, which require α_0 and ν_0 to be both positive.

If $\varphi_{q0} = 0$, equations (4.17) and (4.18) are solved for

$$\alpha_0 = \frac{\epsilon}{3}\,\nu_0\,, \qquad \sigma = 3-\epsilon\,, \qquad V = V_{\rm c}\,. \tag{4.26}$$

This solution has $\nu_0 \gg \alpha_0$ and $2 < \sigma < 3$. For a given background, ϵ is constant per (4.15); a relation to the constant α_0, ν_0 and σ may thus be acceptable. However, the required tuning of general quantization parameters to background parameters makes this solution very special.

By construction and consistently, all these power-law solutions obey equation (3.26). Their qualitative features are summarized in table 1.

- The last exact power-law case we consider is de Sitter, $n = -1$, whose classical solution is $\varphi_{\rm c} = $ const. There we cannot use equation (4.16) because the potential (4.16) was derived using (4.13) which is ill-defined for $n = -1$. From equation (4.10) we simply obtain $V_{\rm q} = -\alpha_0 V_{\rm c}$. Since $V_{\rm q} = V_{,\varphi}(\varphi_c)\varphi_{\rm q}$, this implies that φ_q is constant. But $\delta_{\rm Pl}(a)$ is not constant, so that φ is not constant and $V(\varphi)$ can be reconstructed from the evolution. Combining (4.1) and (4.6), we find

$$V(\varphi) = V_{\rm c}(1-\alpha_0\delta_{\rm Pl}) = V_{\rm c} + V_{,\varphi\,{\rm c}}(\varphi - \varphi_{\rm c})\,. \tag{4.27}$$

φ	σ	α_0, ν_0
	$\sigma = 0$	$\alpha_0 = \nu_0$
$\varphi_{\rm c}(t) + \varphi_{\rm q0}\delta_{\rm Pl}$	$0 < \sigma < 3$	$\alpha_0, \nu_0 > 0$ if $\epsilon > \frac{\sigma(3-\sigma)}{6}$
	$\sigma \geq 3$	$\mathrm{sgn}(\alpha_0) = -\mathrm{sgn}(\nu_0)$, $\lvert\nu_0\rvert \gg \lvert\alpha_0\rvert$
$\varphi_{\rm c}(t)$	$2 < \sigma \lesssim 3$	$\alpha_0 = \frac{\epsilon}{3}\nu_0 \ll \nu_0$

Table 1: Inflationary power-law solutions $a = |\tau|^n$ with $0 < \epsilon = 1 + 1/n < 1$ and exponential potential. $\varphi_{\rm c}(t)$ and $\varphi_{\rm q0}$ are given by equations (4.13) and (4.19), respectively.

Equation (4.11) becomes $\sigma(\sigma-3)\tau^{-2}\varphi_{\rm q0} + a^2m^2\varphi_{\rm q0} = 0$, where $m^2 \equiv V_{,\varphi\varphi}(\varphi_{\rm c})$. The reconstructed potential above is linear in φ, so $m^2 = 0$. If $\varphi_{\rm q0} \neq 0$, this implies that either $\sigma = 0$ or $\sigma = 3$.

In the next section we will find that not all of these solutions will be compatible with a certain consistency relation on quantum counterterms. We have not shown that the above solutions are attractors in configuration space, a necessary condition for adopting them as valid backgrounds. In the next section we will assume this is the case, since in the quasi-classical regime the dynamics is very close to general relativity. Anyway, the structure of perturbation equations and observables does not change qualitatively if one expands about a more general quasi-de Sitter solution.

Also, setting $a = a_{\rm c}$ as in equation (4.12) will not result in any loss of generality in solving the Mukhanov equation. Since the coefficients in the second equation (4.3) will depend only on $\mathcal{H}_{\rm c}$, the structure of the quantum corrections in the solution will be always the same, regardless of $\mathcal{H}_{\rm q}$. However, here we see a possible drawback of assuming an exact power-law expansion, equation (4.12): these background solutions constrain the range of σ, α_0, and ν_0, and from this analysis it is not obvious whether more general, quasi-power-law backgrounds will admit a different parameter space. We will leave also this question to future investigations. For the time being, we show that there exist quasi-power-law expansions as exact solutions and we briefly sketch a profile corresponding to a perturbed de Sitter background, $a_{\rm c} = -1/\tau = \mathcal{H}_{\rm c}$ and $a_{\rm q} \neq 0$. Assuming that $\varphi_{\rm q} = \varphi_{\rm q0}$, $V_{\rm q} = \text{const}$, equation (4.11) yields, as before, either $\varphi_{\rm q0} = 0$ or $m^2 = \sigma(3-\sigma)$, while equation (4.10) becomes

$$a_{\rm q}' + \left(\frac{1+\sigma}{\tau} - \frac{8\pi G}{3}V_{\rm c}\right)a_{\rm q} - \frac{4\pi G}{3}\frac{\alpha_0 V_{\rm c} + V_{\rm q}}{\tau^2} \equiv a_{\rm q}' + \left(\frac{1+\sigma}{\tau} - b_1\right)a_{\rm q} - \frac{b_2}{\tau^2} = 0\,. \tag{4.28}$$

The solution is

$$a_{\rm q}(t) = \frac{e^{b_1\tau}}{\tau}\left[\frac{a_0}{\tau^\sigma} - b_2 E_{1-\sigma}(b_1\tau)\right], \tag{4.29}$$

where E is the exponential integral function. For $b_1 > 0$ and integer $\sigma > 0$,

$$a_{\rm q} = \frac{a_0 e^{b_1\tau} + \mathrm{Pol}[\mathrm{O}(\tau^{\sigma-1})]}{\tau^{\sigma+1}}\,.$$

The last term, a polynomial of degree $\sigma - 1$, dominates at early times ($\tau \to -\infty$) and $a_{\rm q} \sim |\tau|^{-2}$, while at late times ($\tau \to 0$) one has $a_{\rm q} \sim |\tau|^{-1-\sigma}$.

When $\sigma = 0$, at early times $E_1(b_2\tau) \sim {\rm e}^{-b_2\tau}/(b_2\tau)$ and $a_{\rm q} \sim |\tau|^{-2}$, while at late times $E_1(b_2\tau) \sim -\ln|\tau|$ and $a_{\rm q} \sim \ln|\tau|/|\tau|$.

5. Scalar perturbations

5.1 Counterterms

As mentioned in the introduction, the corrected perturbed equations feature counterterms proportional to $\delta_{\rm Pl}$ in addition to the primary correction functions α and ν, which guarantee consistency of the constraint algebra at $O(\delta_{\rm Pl})$ order. Consistency in a given scheme then uniquely relates the counterterms, which all vanish classically, to the primary correction functions, but also restricts the range of parameters in α and ν. Before using the counterterms in perturbation equations, we evaluate these consistency conditions in relation with table 1. In the following, we shall rewrite the counterterms and equations of motion of [6] according to the $\delta_{\rm Pl}$-expansion. To keep notation light, background quantities will not be denoted with bars as in [6]. Also, contrary to this reference we shall expand all intermediate expressions to linear order in counterterms (for instance, $(1+f)(1+h) = 1+f+h+O(\delta_{\rm Pl}^2)$, and so on). Explicitly, the counterterms are

$$\begin{aligned} f &= \frac{1}{\sigma}\frac{{\rm d}\ln\alpha}{{\rm d}\ln p} \\ &= -\frac{\alpha_0}{2}\delta_{\rm Pl}\,, \end{aligned} \tag{5.1}$$

$$\begin{aligned} f_1 &= f - \frac{1}{3}\frac{{\rm d}\ln\nu}{{\rm d}\ln p} \\ &= \frac{1}{2}\left(\frac{\sigma\nu_0}{3} - \alpha_0\right)\delta_{\rm Pl}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} h &= 2\frac{{\rm d}\ln\alpha}{{\rm d}\ln p} - f \\ &= \alpha_0\left(\frac{1}{2} - \sigma\right)\delta_{\rm Pl}\,, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} g_1 &= \frac{1}{3}\frac{{\rm d}\ln\alpha}{{\rm d}\ln p} - \frac{{\rm d}\ln\nu}{{\rm d}\ln p} + \frac{2}{9}\frac{{\rm d}^2\ln\nu}{{\rm d}\ln p^2} \\ &= \frac{\sigma}{2}\left(\frac{\sigma\nu_0}{9} + \nu_0 - \frac{\alpha_0}{3}\right)\delta_{\rm Pl}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} f_3 &= f_1 - g_1 \\ &= \frac{1}{2}\left[\alpha_0\left(\frac{\sigma}{3} - 1\right) - \frac{2\sigma\nu_0}{3}\left(\frac{\sigma}{6} + 1\right)\right]\delta_{\rm Pl}\,. \end{aligned} \tag{5.5}$$

There is also the extra consistency condition

$$2\frac{\mathrm{d}f_3}{\mathrm{d}\ln p}+3(f_3-f)=0\,,$$

which makes some of the parameters dependent:

$$\alpha_0\left(\frac{\sigma}{6}-1\right)-\nu_0\left(\frac{\sigma}{6}+1\right)\left(\frac{\sigma}{3}-1\right)=0\,, \tag{5.6}$$

so that

$$g_1 = \left[\nu_0\left(\frac{\sigma}{3}+1\right)-\alpha_0\right]\delta_{\rm Pl}\,, \tag{5.7}$$

$$f_3 = \left[\frac{\alpha_0}{2}-\nu_0\left(\frac{\sigma}{6}+1\right)\right]\delta_{\rm Pl} \tag{5.8}$$

$$= \frac{1}{2}\frac{3\alpha_0}{\sigma-3}\delta_{\rm Pl}\,, \tag{5.9}$$

the last expression being valid only if $\sigma\neq 3$.

It is interesting to notice that, for the second but not last time, the minisuperspace and first lattice parametrization (3.13)–(3.15) show an incompatibility with independent results: in these parametrizations, equation (5.6) is never respected. However, with the new range (3.20) for σ, the equation can easily be satisfied by the solutions of table 1. Let us compare case by case with equation (5.6). The solution in the first line of the table (equation (4.24)) is also an exact solution of (5.6). This is the limiting case of the solution in the second line (equation (4.25)) for $\sigma\ll 1$, giving $\alpha_0\approx\nu_0$; for general values $\sigma<3$ (e.g., $x=0$ with $m<3$ or $x=-1/4$ with $m<6$) solutions exist in this class for positive α_0 and ν_0, while $\alpha_0=0$ when $\sigma=3$. The solutions in the third line of table 1 are already excluded by the general constraint $\alpha_0,\nu_0>0$.[6] Finally, the last solution in the table, equation (4.26) combined with (5.6), is non-trivial and inflationary only if $\sigma=3$, but this collapses to de Sitter, $\epsilon=0$. To summarize, our solutions will span the range

$$0\leq\sigma\leq 3\,, \tag{5.10}$$

with preference to the extremum values if α_0 and ν_0 are positive. In fact, $\epsilon\gtrsim 1/3$ is not very small for $1\lesssim\sigma\lesssim 2$, which should lead to unviable deviations from scale invariance.

[6] Even ignoring the constraint on the sign, these solutions would be inconsistent or trivial. For $3<\sigma<6$, both (4.25) and (5.6) require α_0 and ν_0 to have opposite sign, but while $|\nu_0|\gg|\alpha_0|$ for (4.25), equation (5.6) asks them to be of about the same magnitude. When $\sigma=6$, $a_0=0=\nu_0$. If $\sigma>6$, equation (5.6) requires α_0 and ν_0 to have the same sign, in contrast with (4.25); so again $a_0=0=\nu_0$.

5.2 Scalar perturbation equations

An inhomogeneous perturbation $\delta\varphi$ in the scalar field induces two gauge-invariant scalar modes Φ and Ψ in the metric, which are proportional to each other [6]:

$$\Phi = (1+h)\Psi\,. \tag{5.11}$$

After solving the equations for Ψ, the expression for Φ will be readily obtained via equation (5.11). The scalar-field perturbation and Ψ are related by the diffeomorphism constraint (equation (90) of [6])

$$4\pi G\frac{\alpha}{\nu}\varphi'\delta\varphi = \Psi' + (1+f+h)\mathcal{H}\Psi\,. \tag{5.12}$$

Using this equation, one can show that the perturbed equation for Ψ is[7]

$$\Psi'' + \mathcal{H}F\Psi' - \left(s^2\Delta^2 + m_\Psi^2\right)\Psi = 0. \tag{5.14}$$

The friction term is

$$\begin{aligned} F &= 2\left(1-\frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}\right) + (1+f+h) + 3(1+f-f_3) - 2\left(3-2\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right) + 2\eta \\ &= 2\eta + \sigma F_0\delta_{\rm Pl}, \end{aligned} \tag{5.15}$$

where

$$F_0 \equiv \nu_0\left(\frac{\sigma}{6}-1\right) - \frac{\alpha_0}{2}\,. \tag{5.16}$$

The (squared) propagation speed of the perturbation is

$$s^2 = \alpha^2(1-f_3) = 1+\chi\delta_{\rm Pl}\,, \tag{5.17}$$

where

$$\chi \equiv \frac{\sigma\nu_0}{3}\left(\frac{\sigma}{6}+1\right) + \frac{\alpha_0}{2}\left(5-\frac{\sigma}{3}\right)\,. \tag{5.18}$$

Finally, the effective mass term is

$$\begin{aligned} m_\Psi^2 &= \mathcal{H}^2\left[2(\epsilon-\eta) - 2\frac{df}{\mathrm{d}\ln p} + 3\frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p} - 3(f-f_3) - 4\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right. \\ &\qquad \left. +\epsilon\left(\frac{1}{3}\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p} + f_1 + f + 2h\right) - 2(f+h)\eta - \frac{h'}{\mathcal{H}}\right] \\ &= \mathcal{H}^2\left[2(\epsilon-\eta) - \sigma\mu_\Psi\delta_{\rm Pl}\right]\,, \end{aligned} \tag{5.19}$$

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[7] This is obtained by combining our equations (5.11) and (5.12) with equation (82) of [6]. In equation (82) one should correct the typographical error $\bar\alpha^2\Delta\Phi \to \bar\alpha^2\Delta\Psi$ [47]:

$$\alpha^2\Delta\Psi - 3\mathcal{H}(1+f)[\Psi' + (1+f+h)\mathcal{H}\Psi] = 4\pi G\frac{\alpha}{\nu}(1+f_3)\left[\varphi'\delta\varphi' - \varphi'^2(1+f_1+h)\Psi + \nu p V_{,\varphi}\delta\varphi\right]\,, \tag{5.13}$$

where Δ is the Laplacian in comoving spatial coordinates. Also, in equation (91) of [6] one must replace $\bar\alpha^2(1+h)\Delta\Psi$ with $\bar\alpha^2\Delta\Psi$. Unfortunately, this typo propagated in some of the other equations, so our results supersede those of [6] when in disagreement.

where

$$\mu_\Psi \equiv [2(\epsilon-\eta)+(1+\sigma)]\alpha_0+\nu_0\left(\frac{\sigma}{6}-1\right)\,. \tag{5.20}$$

Taking equation (84) of [6], expanding it to leading order in quantum corrections, and making use of equations (3.30), (3.31), and (5.11), one obtains the perturbed Klein–Gordon equation for the gauge-invariant perturbation $\delta\varphi$:

$$\delta\varphi''+2\mathcal{H}B_1\delta\varphi'-(s^2\Delta^2-\nu pV_{,\varphi\varphi})\delta\varphi-B_2\varphi'\Psi'+2B_3\mathcal{H}\varphi'\Psi=0\,, \tag{5.21}$$

where

$$\begin{aligned} B_1 &= 1-\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}-\frac{\mathrm{d}g_1}{\mathrm{d}\ln p}\\ &= 1+B_{10}\delta_{\mathrm{Pl}}, && (5.22\mathrm{a})\\ B_2 &= 4+f_1+h+3g_1\\ &= 4+B_{20}\delta_{\mathrm{Pl}}, && (5.22\mathrm{b})\\ B_3 &= (1+f_1+h)\frac{\nu pV_{,\varphi}}{\mathcal{H}\varphi'}-\frac{\mathrm{d}h}{\mathrm{d}\ln p}-\frac{\mathrm{d}f_3}{\mathrm{d}\ln p}\\ &= \eta-3+B_{30}\delta_{\mathrm{Pl}}, && (5.22\mathrm{c}) \end{aligned}$$

and

$$\begin{aligned} B_{10} &\equiv \sigma\left[\nu_0\left(\frac{\sigma}{6}+1\right)-\frac{\alpha_0}{2}\right]\,, && (5.22\mathrm{d})\\ B_{20} &\equiv \frac{\sigma}{2}\left(\frac{\sigma\nu_0}{3}+\frac{10\nu_0}{3}-3\alpha_0\right)\,, && (5.22\mathrm{e})\\ B_{30} &\equiv \sigma\left[\left(\frac{\nu_0}{6}-\alpha_0\right)\eta-\nu_0\left(\frac{\sigma}{12}+2\right)+\frac{\alpha_0}{2}(7-\sigma)\right]\,. && (5.22\mathrm{f}) \end{aligned}$$

Before proceeding, we notice a potentially serious problem. In order to avoid superluminal propagation of signals, one should impose

$$s^2<\alpha^2\,, \tag{5.23}$$

where we used the fact that photons propagate with speed α greater than the classical one [8]. Then, it should be $f_3>0$. For this to happen, we can have:

- $0\leq\sigma<3$: equation (5.9) imposes $\alpha_0<0$.
- $\sigma=3$: equations (5.5) and (5.6) impose, respectively, $\nu_0<0$ and $\alpha_0=0$.
- $3<\sigma<6$: equation (5.6) imposes α_0 and ν_0 to have opposite sign.
- $\sigma=6$: equations (5.9) and (5.6) impose, respectively, $\alpha_0>0$ and $\nu_0=0$; this case is allowed.
- $\sigma>6$: this case, too, is allowed, with both α_0 and ν_0 strictly positive.

Unfortunately, for non-negative and α_0 and ν_0, f_3 is negative unless σ be large enough, and this condition is hardly compatible with inflation; see table 1. (In [47], the values of parameters given for subluminal evolution correspond to the case $\sigma > 6$ here.)

To check whether superluminal propagation is an artifact of linear perturbation theory or of the expansion in $\delta_{\rm Pl}$, one should go beyond linear order in both expansions. The covariant formalism of non-linear perturbation theory could be a useful tool for analyzing the consistency of the effective constraint algebra. A possibility is that holonomy corrections, which we have ignored, would play an important role in this issue, which we shall put aside in this paper. However, even if this were the case in some regimes, one can always find initial conditions so as to have dominant inverse-volume corrections; thus, superluminal velocities might constitute a conceptual problem with implications for the stability of the theory as a whole. On the other hand, we note that inflationary models based on superluminally propagating fields have been consistently formulated [48]. The case of superluminal motion found here therefore does not necessarily mean a severe problem.

We reemphasize the importance of equation (5.6) and of the counterterms it comes from. It rules out the minisuperspace-related parametrizations and severely restricts the lattice one. In this way, consistency alone already subjects the theory to strict tests even before evaluating the phenomenology, to which we turn now.

5.3 Conservation of curvature perturbation

The gauge-invariant linear comoving curvature perturbation is [6]

$$\begin{aligned}\mathcal{R} &= \Psi + \frac{\mathcal{H}}{\varphi'}(1+f-f_1)\,\delta\varphi &(5.24)\\ &= \Psi + \frac{\mathcal{H}}{\varphi'}\left(1-\frac{\sigma\nu_0}{6}\delta_{\rm Pl}\right)\delta\varphi\,. &(5.25)\end{aligned}$$

In the absence of counterterms, conservation of the energy-momentum tensor implies that $\mathcal{R}$ is constant at large scales [41]. One may ask if this result, which is not obvious in Hamiltonian formalism and for equation (5.24), holds also in semi-classical LQC. To check it, we invert equation (5.13) with respect to $\delta\varphi'$ and employ (5.12). Differentiating $\mathcal{R}$ with respect to conformal time, we obtain

$$\mathcal{R}' = (\alpha\nu+f-f_1-f_3)\frac{\mathcal{H}}{4\pi G\varphi'^2}\Delta\Psi + C\delta\varphi\,,$$

where

$$\begin{aligned}C &= 4\pi G\frac{\alpha}{\nu}\varphi' + \frac{\mathcal{H}^2}{\varphi'}\left[\frac{f'-f_1'}{\mathcal{H}} - (1+f-f_1)\left(\epsilon+2\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}+3f-3f_3\right)\right]\\ &= \frac{\mathcal{H}^2}{\varphi'}\left[\frac{f'-f_1'}{\mathcal{H}} + \frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p} + \left(\frac{1}{3}\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p} - f + f_1\right)\epsilon - 2\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p} - 3(f-f_3)\right]\\ &= 0\,. &(5.26)\end{aligned}$$

Here, after using (3.26), the prefactor of the parameter ϵ (which we are not assuming to be small) vanishes by virtue of equation (5.2), which also implies $(f'-f_1')/\mathcal{H} = (2/3)\mathrm{d}^2\ln\nu/\mathrm{d}\ln p^2$. Together with $f-f_3 = -(2/3)\mathrm{d}\ln\nu/\mathrm{d}\ln p + (1/3)\mathrm{d}\ln\alpha/\mathrm{d}\ln p + (2/9)\mathrm{d}^2\ln\nu/\mathrm{d}\ln p^2$ from (5.5) and (5.4), all terms are zero. The resulting conservation of power is consistent with the picture of an effective perturbed FRW geometry that models the dynamics of a nearly isotropic universe in the presence of corrections from loop quantum gravity. Quantum-geometry corrections from this theory, with the perturbation equations used here, have been shown to deform not just the dynamics but also the underlying spacetime structure, inferred by an analysis of the algebra of constraints. The gauge transformations they generate no longer correspond to pure coordinate transformations because they do not obey strictly the classical algebra of spacetime deformations. As a consequence, classical results about the conservation of power may no longer apply. As seen here, the linear curvature perturbation is nevertheless conserved on large scales. This observation demonstrates that perturbations in the presence of quantum corrections can still be seen as those of an effective line element: The large-scale curvature perturbation of an FRW line element in conformal time amounts simply to a spatially constant rescaling of the scale factor, which should not be subject to non-trivial dynamics. By being conserved also in the presence of quantum corrections, the interpretation of the effective geometry as a line element is still meaningful.

The result (5.26) is due to the delicate cancellations between counterterms. Therefore,

$$\boxed{\mathcal{R}' = \left[1+\left(\frac{\alpha_0}{2}+2\nu_0\right)\delta_{\rm Pl}\right]\frac{\mathcal{H}}{4\pi G\varphi'^2}\Delta\Psi}\,, \tag{5.27}$$

and *the curvature perturbation is conserved at large scales.*

5.4 Mukhanov equation

Conservation of $\mathcal{R}$ strongly suggests that one can write a simple Mukhanov equation in the variable

$$u = z\mathcal{R}\,, \tag{5.28}$$

where z is some background function. We can anticipate the main result with a very efficient trick, and then confirm it via a standard but tedious calculation. The trick is to notice that, at super-horizon scales, the comoving curvature perturbation is approximately constant, so that $u'' \approx z''\mathcal{R}$ and

$$u'' - \frac{z''}{z}u \approx 0\,.$$

The objective now is to find this friction-free Mukhanov equation from the perturbed equations of motion. Start from equation (5.21) and choose for simplicity a spatially flat slice where $\Psi \approx 0$. In order to remove the friction term, we need to define a field $u = a(1 - \beta\delta_{\rm Pl})\delta\varphi$, where $\beta = B_{10}/\sigma$. Then,

$$\frac{u''}{1-\beta\delta_{\rm Pl}} = \delta\varphi'' + 2\mathcal{H}(1 + B_{10}\delta_{\rm Pl})\delta\varphi' + \dots .$$

Comparing with the Mukhanov variable (5.28) and equation (5.25), one finds

$$z \equiv \frac{a\varphi'}{\mathcal{H}}\left[1 + \left(\frac{\sigma\nu_0}{6} - \beta\right)\delta_{\rm Pl}\right] = \frac{a\varphi'}{\mathcal{H}}\left[1 + \left(\frac{\alpha_0}{2} - \nu_0\right)\delta_{\rm Pl}\right] . \tag{5.29}$$

The only missing term in the Mukhanov equation is the Laplacian, with coefficient s^2 as an inspection of equation (5.21) immediately shows. Thus we obtain

$$u'' - \left(s^2\Delta + \frac{z''}{z}\right) u = 0 \,, \tag{5.30}$$

a result valid exactly at all scales and at the linear perturbative level. The rigorous calculation begins with the Mukhanov variable (5.28) and equation (5.29) with unknown β. Differentiating u twice, using equations (5.14) and (5.21), and using equation (5.12) to develop the Ψ' term, we obtain equation (5.30) plus just one extra term:

$$u'' = \left(s^2\Delta + \frac{z''}{z}\right) u + 2\sigma\mathcal{H}\delta_{\rm Pl}\left[\nu_0\left(\frac{\sigma}{6}+1\right) - \frac{\alpha_0}{2} - \beta\right](\mathcal{H}z\Psi - \eta\mathcal{H}a\delta\varphi - a\delta\varphi') + {\rm O}(\delta_{\rm Pl}^2) \,.$$

The extra term vanishes if β is chosen as above.

It may seem that equation (5.30) is not covariant since only the spatial-derivative term is corrected. However, despite appearance this is not the case: The quantum-corrected equations of motion correspond to a deformed algebra of constraints as found in [9], and the constraints determine what form gauge transformations take. In general relativity, the gauge transformations are spacetime diffeomorphisms or changes of coordinates whose classical form implements the usual notion of covariance. With corrected constraints obeying a deformed algebra, the gauge transformations are not of the classical form, and they do not correspond to the usual notion of coordinate changes. Even though the underlying structure of a 'quantum manifold' (perhaps non-commutative) on which the modified transformations could be interpreted as simple coordinate changes is unknown, the (generalized) covariance of (5.30) under these deformed transformations is guaranteed by the derivation of the equations of motion used here from an anomaly-free set of constraints.

It is quite remarkable that scalar perturbations are ultimately governed by such a simple equation as (5.30). However, the existence of one Mukhanov variable obeying one equation in closed form is not unexpected, as it could have been inferred by using the Hamilton–Jacobi method for constrained Hamiltonian systems developed in [49, 50]. In particular, the reduced phase space obtained after solving the constraints and factoring out their gauge flows has one local degree of freedom, parametrized by the curvature perturbation and its conjugate momentum. There must be a closed form for the dynamics on this reduced phase space, such that Hamiltonian first-order equations of motion exist involving only $\mathcal{R}$ and its momentum, and they are linear thanks to the linear perturbation scheme used. As always, first-order Hamiltonian equations of motion can be expressed as one second-order equation for the configuration variable, here $\mathcal{R}$. The second-order equation in general may have terms involving $\mathcal{R}''$, $\mathcal{R}'$, as well as $\mathcal{R}$, which on large scales is an ordinary differential equation with gradient-free coefficients (in momentum space, they are independent of the wave number k defined below). Thus, one may eliminate the last term involving $\mathcal{R}$ by substituting $y\mathcal{R}$ for $\mathcal{R}$ for a suitable background function y, and a constant mode for $y\mathcal{R}$ results. As a consequence, there must be a conserved quantity $y\mathcal{R}$ whose existence can be seen without any detailed calculations. Details are required to derive the form of y, and the non-trivial result found here is that $y = 1$.

From now on we expand linear perturbations in momentum space, a subscript k indicating modes with comoving wavelength $2\pi/k$. The Laplacian becomes $\Delta \to -k^2$, and the Mukhanov equation

$$\boxed{u_k'' + \left(s^2k^2 - \frac{z''}{z}\right) u_k = 0}\,. \tag{5.31}$$

The effective mass term is a combination of slow-roll parameters and quantum corrections. In fact,

$$\frac{z'}{z} = \mathcal{H}(1+\epsilon-\eta) + \sigma\left(\nu_0 - \frac{\alpha_0}{2}\right)\mathcal{H}\delta_{\rm Pl}\,, \tag{5.32}$$

$$\frac{z''}{z} = \mathcal{H}^2\left(2+2\epsilon-3\eta-4\epsilon\eta+2\epsilon^2+\eta^2+\xi^2-\sigma\mu_u\delta_{\rm Pl}\right)\,, \tag{5.33}$$

$$\mu_u \equiv \frac{3\alpha_0}{2} + \nu_0(\sigma-3) + \left(\frac{5\alpha_0}{2} + \frac{\sigma\nu_0}{6} - 2\nu_0\right)\epsilon + 2(\nu_0-\alpha_0)\eta\,. \tag{5.34}$$

When the slow-roll parameters are constant classically, as in any of the solutions of section 4, one has

$$\frac{z''}{z} = \frac{4\mu_1^2 - 1 + 4\mu_2\delta_{\rm Pl}}{4\tau^2}\,, \tag{5.35}$$

where

$$\mu_1 = \frac{1}{2} - n\,, \qquad \mu_2 = \sigma n^2\left[\sigma(4n-\sigma n+1)\frac{\varphi_{\rm q0}}{\varphi_0} - \mu_u\right]\,. \tag{5.36}$$

An exact solution of the Mukhanov equation does exist but it is too complicated and not very instructive. We proceed to solve this equation asymptotically.

5.5 Asymptotic solutions

The moment of horizon crossing is, as usual, defined when the effective mass term equals the Laplacian term. Up to numerical factors, this happens when

$$k|\tau| = 1\,, \tag{5.37}$$

as in standard inflation. Super-horizon modes are characterized by $k|\tau| \ll 1$, while modes well inside the horizon have $k|\tau| \gg 1$. At large scales, we can ignore the k^2 term in equation (5.31), so that

$$u_k \overset{k|\tau|\ll 1}{\sim} C(k) z\,, \tag{5.38}$$

where $C(k)$ is a normalization constant. To determine it, we must find the asymptotic behaviour of u at small scales. There, one can ignore the mass term and consider the equation

$$u_k'' + (1+\chi\delta_{\rm Pl})k^2 u_k \approx 0\,. \tag{5.39}$$

Since all the analysis is valid only at first order in the quantum corrections, it is consistent to look for short-wavelength solutions of the form

$$u_{k\gg\mathcal{H}}(\tau) = u_{\rm c}(k,\tau)[1 + y(k,\tau)\delta_{\rm Pl}]\,, \tag{5.40}$$

where $u_{\rm c}$ is the solution of the classical Mukhanov equation and y is some function. In particular, the only choice compatible with the Bunch–Davies vacuum in the infinite past is an incoming plane wave,

$$u_{\rm c} = \frac{{\rm e}^{-{\rm i}k\tau}}{\sqrt{2k}}\,. \tag{5.41}$$

The normalization here is the classical one, which one might have to change for a vacuum matter state in a quantum geometry. In particular, the correction function ν multiplies the kinetic term of the scalar Hamiltonian, and thus affects the value of vacuum fluctuations. By the ansatz (5.40), all these effects will be included once the equation of motion for y is solved.

Plugging the ansatz (5.40) into (5.39) we obtain an inhomogeneous equation for the function y:

$$y'' - 2(\sigma\mathcal{H} + {\rm i}k)y' + 2{\rm i}k\sigma\mathcal{H}y + \chi k^2 = 0\,, \tag{5.42}$$

where, for consistency, we have dropped the mass term $\sigma\mathcal{H}^2(\sigma+\epsilon-1)y$. At this point we expand y in a power series,

$$y = \sum_{m=0}^{+\infty} y_m \tau^m\,, \tag{5.43}$$

and we pick a power-law background, $\mathcal{H}=n/\tau$. Then, equation (5.42) is

$$\begin{aligned}0=&\,2\sigma n(\mathrm{i}ky_0-y_1)\frac{1}{\tau}+[2\mathrm{i}k(\sigma n-1)y_1+2(1-2\sigma n)y_2+\chi k^2]\\&+\sum_{m=2}^{+\infty}[2\mathrm{i}k(\sigma n-m)y_m-(m+1)(2\sigma n-m)y_{m+1}]\tau^{m-1}\,.\end{aligned}\tag{5.44}$$

These terms must vanish order by order separately. If $\sigma=0$, then y_0 is unconstrained, while $y_1=-\mathrm{i}(y_2/k+k\chi/2)$ and $y_m=2(2\mathrm{i}k)^{m-2}y_2/m!$ for all $m\geq 2$. Summing the series, one obtains

$$\begin{aligned}y&=y_0-\mathrm{i}\left(\frac{y_2}{k}+\frac{k\chi}{2}\right)\tau+\frac{y_2}{2k^2}\left(1+2\mathrm{i}k\tau-\mathrm{e}^{2\mathrm{i}k\tau}\right)\\&=\left(y_0+\frac{y_2}{2k^2}\right)-\frac{\mathrm{i}k\chi}{2}\tau-\frac{y_2}{2k^2}\mathrm{e}^{2\mathrm{i}k\tau}\,.\end{aligned}$$

We can argue that $y_2=0$ because otherwise u in equation (5.40) would contain also an outgoing mode $\mathrm{e}^{+\mathrm{i}k\tau}$. If $\sigma\neq 0$, one obtains the following conditions:

$$y_1=\mathrm{i}ky_0\,,\tag{5.45}$$

$$y_2=\frac{k^2}{2(2\sigma n-1)}[\chi-2(\sigma n-1)y_0]\,,\tag{5.46}$$

$$y_{m+1}=\frac{2\mathrm{i}k(\sigma n-m)}{(m+1)(2\sigma n-m)}y_m\,.\tag{5.47}$$

The recursive relation would determine the sum of the series (5.43), but analytic continuation to the case $\sigma=0$ requires $y_m=0$ for $m\geq 2$. This fixes both y_0 and y_1 and the result is

$$y=\frac{\chi}{2(\sigma n-1)}(1+\mathrm{i}k\tau)\,.\tag{5.48}$$

The normalization of equation (5.38) is thus obtained by imposing the junction condition $|u_{k\gg\mathcal{H}}|=|u_{k\ll\mathcal{H}}|$ at horizon crossing. Then,

$$|u_{k\ll\mathcal{H}}|^2=\frac{1}{2k}\left[1+\frac{\chi}{\sigma n-1}\delta_{\rm Pl}(k)\right]\left[\frac{z}{z(k)}\right]^2\,,\tag{5.49}$$

where $z(k)=z(\tau=-1/k)$ and $\delta_{\rm Pl}(k)=\delta_{\rm Pl}(\tau=-1/k)\propto k^{n\sigma}$.

5.6 Scalar spectrum, spectral index and running

The scalar spectrum is defined as the two-point correlation function of the curvature perturbation $\mathcal{R}$ over a momentum ensemble at large scales, evaluated at horizon crossing:

$$\mathcal{P}_{\rm s}\equiv\frac{k^3}{2\pi^2z^2}\left\langle|u_{k\ll\mathcal{H}}|^2\right\rangle\Big|_{k|\tau|=1}\,.\tag{5.50}$$

The scalar spectral index is defined as

$$n_{\rm s}-1\equiv\frac{{\rm d}\ln\mathcal{P}_{\rm s}}{{\rm d}\ln k}\,. \tag{5.51}$$

For a power-law background, we have

$$\mathcal{P}_{\rm s}(k) = \frac{G}{\pi}k^{2(1+n)}\left[1+\left(\frac{\chi}{\sigma n-1}-\alpha_0+2\nu_0+2\sigma n\frac{\varphi_{\rm q0}}{\varphi_0}\right)\delta_{\rm Pl}\right]\,, \tag{5.52}$$

$$n_{\rm s}-1 = 2(1+n)+\sigma n\left(\frac{\chi}{\sigma n-1}-\alpha_0+2\nu_0+2\sigma n\frac{\varphi_{\rm q0}}{\varphi_0}\right)\delta_{\rm Pl}\,. \tag{5.53}$$

We can obtain more portable expressions by writing the spectrum on a general quasi-de Sitter background. Since, using (3.26),

$$z^2=\frac{a^2}{4\pi G}\left\{\epsilon-\left[\nu_0\left(\frac{\sigma}{6}+1\right)\epsilon+\frac{\sigma\alpha_0}{2}\right]\delta_{\rm Pl}\right\}\,,$$

we get

$$\boxed{\mathcal{P}_{\rm s}=\frac{G}{\pi}\frac{\mathcal{H}^2}{a^2\epsilon}\left(1+\gamma_{\rm s}\delta_{\rm Pl}\right)}\,, \tag{5.54}$$

where we used $k=\mathcal{H}$ and

$$\gamma_{\rm s}\equiv\nu_0\left(\frac{\sigma}{6}+1\right)+\frac{\sigma\alpha_0}{2\epsilon}-\frac{\chi}{\sigma+1}\,. \tag{5.55}$$

Notice that if $\sigma=0$, the quantum correction is constant and the only change with respect to the classical case is the normalization of the spectrum. In that case, $\gamma_{\rm s}=\nu_0-5\alpha_0/2$ could be of either sign. If $\gamma_{\rm s}\neq 0$, there is a large-scale enhancement of power because $\delta_{\rm Pl}\sim a^{-\sigma}\sim(1/|\tau|)^{-\sigma}\sim k^{-\sigma}$ at horizon crossing. The magnitude of the effect depends on the value of $\gamma_{\rm s}$ but we notice that, if $\sigma\neq 0$, $\gamma_{\rm s}\sim 3\alpha_0/(2\epsilon)$ unless $\nu_0\sim\alpha_0/\epsilon\gg\alpha_0$. Therefore, one could obtain a sizable enhancement unless $\alpha_0\lesssim {\rm O}(\epsilon)$ (which is the case, typically). Since this enhancement is of potential interest for comparisons with observations, we trace back where the inverse of the slow-roll parameter in the expression for $\gamma_{\rm s}$ came from. It arises due to the ϵ-independent term in z^2 above, which in turn is a direct consequence of the presence of gravity corrections in the Raychaudhuri equation (3.3), as opposed to just stress-energy modifications. As with several other key phenomena pointed out here, this feature is a consequence of corrections to the structure of spacetime geometry: corrections in the terms $\mathcal{H}'$, $\mathcal{H}^2$ of the Raychaudhuri equation (or the isotropic Einstein tensor) can be obtained only by changing the geometrical form of gravity.

Momentum derivatives are converted into conformal time derivatives via

$$\frac{\rm d}{{\rm d}\ln k}\approx\frac{1}{\mathcal{H}}\frac{\rm d}{{\rm d}\tau}\,,$$

so that the scalar index is

$$\boxed{n_{\mathrm{s}} - 1 = 2\eta - 4\epsilon + \sigma\gamma_{n_{\mathrm{s}}}\delta_{\mathrm{Pl}}}\,, \tag{5.56}$$

where

$$\gamma_{n_{\mathrm{s}}} \equiv \frac{\tilde{\epsilon}}{\epsilon} - \alpha_0\left(1 - \frac{\eta}{\epsilon}\right) - \gamma_{\mathrm{s}} = \alpha_0 - 2\nu_0 + \frac{\chi}{\sigma+1}\,. \tag{5.57}$$

Since the quantum correction is small, the scalar index does not deviate too much from scale invariance. If $\sigma = 0$, there are no corrections at all. If $\sigma \neq 0$, the sign of the correction depends on the choice of the parameters in the parameter space. We have seen that power-law/quasi de Sitter solutions have $\sigma \lesssim 3$, so it is immediate to associate scale invariance with small values of σ. The naturalness of this range is further stressed in the concluding section by an independent argument.

Interestingly, the running of the spectral index is dominated by the quantum correction (unless $\sigma = 0$):

$$\alpha_{\mathrm{s}} \equiv \frac{\mathrm{d}n_{\mathrm{s}}}{\mathrm{d}\ln k} \tag{5.58}$$

$$= 2(5\epsilon\eta - 4\epsilon^2 - \xi^2) + \sigma(4\tilde{\epsilon} - \sigma\gamma_{n_{\mathrm{s}}})\delta_{\mathrm{Pl}} \sim \delta_{\mathrm{Pl}}\,. \tag{5.59}$$

This result signals a qualitative departure from classical inflation, since the quantum correction may be larger than $\mathrm{O}(\epsilon^2)$. The details will depend on the chosen background, as the slow-roll parameter themselves can contain quantum corrections.

6. Tensor perturbations

The linearized equation of motion for tensor modes has been computed in [8] and solved in [11] for quasi-classical inverse volume corrections. In this section we review and improve these results, eventually obtaining the cosmological consistency relations.

6.1 Mukhanov equation

When only inverse-volume corrections are taken into account and in the absence of anisotropic stress, the equation of motion for the individual tensor mode h_k is [8]

$$h_k'' + 2\mathcal{H}\left(1 - \frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}\right)h_k' + \alpha^2 k^2 h_k = 0\,. \tag{6.1}$$

Defining

$$w_k \equiv \tilde{a}h_k\,, \qquad \tilde{a} \equiv a\left(1 - \frac{\alpha_0}{2}\delta_{\mathrm{Pl}}\right)\,, \tag{6.2}$$

we get the Mukhanov equation

$$\boxed{w_k'' + \left(\alpha^2 k^2 - \frac{\tilde{a}''}{\tilde{a}}\right) w_k = 0}, \tag{6.3}$$

where

$$\frac{\tilde{a}'}{\tilde{a}} = \mathcal{H}\left(1 + \frac{\sigma\alpha_0}{2}\delta_{\rm Pl}\right), \tag{6.4}$$

$$\frac{\tilde{a}''}{\tilde{a}} = \mathcal{H}^2\left[2 - \epsilon + (3 - \sigma - \epsilon)\frac{\sigma\alpha_0}{2}\delta_{\rm Pl}\right]. \tag{6.5}$$

Equation (6.3) is formally identical to the scalar Mukhanov equation and the analysis is exactly the same up to the substitutions

$$z \to \tilde{a}\,, \qquad \chi \to 2\alpha_0\,.$$

The final result is the analogue of equation (5.49),

$$|w_{k\ll\mathcal{H}}|^2 = \frac{1}{2k}\left[1 + \frac{2\alpha_0}{\sigma n - 1}\delta_{\rm Pl}(k)\right]\left[\frac{\tilde{a}}{\tilde{a}(k)}\right]^2. \tag{6.6}$$

6.2 Tensor spectrum, spectral index and running

The tensor spectrum is

$$\mathcal{P}_{\rm t} \equiv \frac{32G}{\pi}\frac{k^3}{\tilde{a}^2}\left\langle|w_{k\ll\mathcal{H}}|^2\right\rangle\Big|_{k|\tau|=1}, \tag{6.7}$$

so that in de Sitter ($n = -1$)

$$\boxed{\mathcal{P}_{\rm t} \equiv \frac{16G}{\pi}\frac{\mathcal{H}^2}{a^2}(1 + \gamma_{\rm t}\delta_{\rm Pl})}, \tag{6.8}$$

where

$$\gamma_{\rm t} \equiv \frac{\sigma - 1}{\sigma + 1}\alpha_0\,. \tag{6.9}$$

As for the scalar spectrum, barring special values of the parameters ($\gamma_{\rm t} = 0$ when $\sigma = 1$ or $\alpha_0 = 0$) there is a power enhancement at large scales because of $\delta_{\rm Pl} \sim k^{-\sigma}$, albeit the prefactor might not be as large as in equation (5.55). This type of enhancement has been seen in earlier numerical studies of the LQC tensor power spectrum, but it is difficult to exploit it observationally due to limitations by cosmic variance.

The tensor index and its running are

$$n_{\rm t} \equiv \frac{{\rm d}\ln\mathcal{P}_{\rm t}}{{\rm d}\ln k}\,, \qquad \alpha_{\rm t} \equiv \frac{{\rm d}n_{\rm t}}{{\rm d}\ln k}\,, \tag{6.10}$$

so that

$$\boxed{n_{\rm t} = -2\epsilon - \sigma\gamma_{\rm t}\delta_{\rm Pl}}\,, \tag{6.11}$$

and

$$\alpha_{\rm t} = -4\epsilon(\epsilon - \eta) + \sigma(2\tilde{\epsilon} + \sigma\gamma_{\rm t})\delta_{\rm Pl}\,. \tag{6.12}$$

6.3 Tensor-to-scalar ratio

The last piece of information we want to extract is the tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_{\rm t}}{\mathcal{P}_{\rm s}}\,. \tag{6.13}$$

From equations (5.54) and (6.8) one obtains

$$r = 16\epsilon[1 + (\gamma_{\rm t} - \gamma_{\rm s})\delta_{\rm Pl}]\,, \tag{6.14}$$

which yields the consistency relation

$$\boxed{r = -8\{n_{\rm t} + [n_{\rm t}(\gamma_{\rm t} - \gamma_{\rm s}) + \sigma\gamma_{\rm t}]\delta_{\rm Pl}\}}\,. \tag{6.15}$$

Here we implicitly assumed that $\gamma_{\rm s}$ is not too large, so that the expansion in $\delta_{\rm Pl}$ is still meaningful. In quasi de Sitter regime $\epsilon \ll 1$, so that $\gamma_{\rm s} = {\rm O}(1)$ if $\sigma\alpha_0 \sim {\rm O}(\epsilon)$. This means that either σ or α_0 or both should be small.

Unless $\sigma = 0$ or $\sigma = 1$ (for which $\gamma_{\rm t} = 0$), the tensor-to-scalar ratio is no longer proportional to the tensor index. Detection of a non-zero r would require either a consistent deviation from de Sitter in standard cosmology or a sufficiently large quantum correction in de Sitter LQC.

As already explained for (5.30), equations (5.31) and (6.3) are covariant under the deformed transformations generated by the anomaly-free set of corrected constraints. The deformation gives rise to a new type of quantum effects which could not be present for higher-curvature effective actions usually expected of quantum gravity; it is possible only thanks to quantum corrections to the geometry of space or even the manifold structure. The Mukhanov equations for scalar and tensor modes are not only corrected, they also acquire corrections of different forms. The scalar equations has a correction given by $s^2 = \alpha^2(1 - f_3)$, while the tensor equation is corrected by α^2. The counterterm f_3 cannot typically be set to zero, and so the corrections for scalar and tensor modes differ. This difference, in turn, makes possible changes to the tensor-to-scalar ratio which may provide a key signature of loop quantum gravity.

7. Discussion

As long as the slow-roll approximation is valid, the structure of the cosmological observables found above is valid for any background, although the coefficients of the quantum corrections themselves do depend on the background. In this final section we discuss how they can be used to restrict models of loop quantum cosmology, making the framework falsifiable. Details will be provided in a separate publication [51]. For such an endeavor, it is crucial to obtain independent information on the main correction parameter $\delta_{\rm Pl}$ and on different versions of the parametrization. For instance, as seen in section 3.1.3, a combination with holonomy corrections is interesting and shows a powerful interplay between these main two types of quantum-geometry corrections.

If inflation is assumed, the minisuperspace parametrization is under tight pressure: neither consistent power-law background solutions nor a nearly scale-free spectrum can be found in that case (unless $\delta_{\rm Pl}$ be very small; see below). On the other hand, the minisuperspace parametrization may still be viable if an alternative scenario of structure formation can be found. In this context, holonomy corrections are of particular interest not only by providing an additional consistency condition in combination with inverse-volume corrections, but also because they can easily trigger bounces at least in isotropic models whose matter energy is dominated by the kinetic term. (In general, it has not been shown that isotropic bounces occur as a natural consequence of holonomy corrections.) It would thus be of interest to develop linear perturbation equations around those models and analyze the structure evolution through the bounce, or perhaps new scenarios providing the generation of structure during a phase before (not after) the big bang. However, compared to inverse-triad corrections such ideas are currently hampered by several major difficulties: (i) Holonomy corrections have so far not been implemented in consistent deformations of linear perturbation equations. (ii) Strong quantum-geometry corrections are required to evolve through the bounce; no expansion in parameters such as $\delta_{\rm Pl}$ used here could be done. (iii) There are several indications as to the strong sensitivity of evolution through the bounce to initial conditions of perturbations [52] or even the quantum state [53], discussed in the context of cosmic forgetfulness.

The lattice parametrization is consistent with inflation and holonomy corrections at the homogeneous level and could yield strong effects according to section 3.1.3 because σ can be small, but that again depends on the details of $\delta_{\rm Pl}$. We have seen that $\delta_{\rm Pl}$ is an eigenfunction of the operator $\mathrm{d}/\mathrm{d}\ln k$ with eigenvalue $-\sigma$, so observables of higher order in the slow-roll parameters (e.g., the index running) are corrected by a term which is always of the form $\mathrm{O}(\sigma^n)\delta_{\rm Pl}$: it is first-order in $\delta_{\rm Pl}$ and n-th order in σ. If $\sigma = \mathrm{O}(1)$, this quantum correction is equally important at any slow-roll order, if not increasing with the order. This situation does not seem natural inasmuch as it would imply that higher-order k derivatives of the inflationary spectra are all on the same footing. Then, the notion that the spectra can be approximated by a power law would have to be abandoned. On the other hand, if $\sigma \ll 1$ the quantum correction is suppressed by higher and higher powers of σ, so there is a sort of balancing effect which keeps $O(\sigma^n)\delta_{\rm Pl}$ small at all orders in the slow-roll parameters. This leads to the speculation that small values of σ are more sensible, because for large σ quantum corrections would dominate in higher-order observables.

With $\sigma \ll 1$ preferred, inverse-volume corrections are of the form $1 + cp^{-\sigma/2}$ with a small exponent σ. They affect not only the equations for an expanding universe but also the dispersion relations of waves propagating in a quantum spacetime. Corrections for these equations are more difficult to derive because the situation is not as symmetric as the one of perturbations of an isotropic spacetime. But if they turned out to be of a similar form ($\delta_{\rm Pl}$ with small exponent σ) for a variable related to the particle's energy, severe observational pressure could be put on loop quantum gravity by a combination of cosmological and astroparticle observations.

The first-order cosmological observables already give a wealth of information about the early universe but it is natural to ask oneself what happens at second order, e.g., when looking at the trispectrum and possible non-Gaussian signatures. Posed in LQC, this question is less harmless than in classical general relativity. In fact, when going to higher orders in perturbation theory one would get more parameters initially, because there are more options for counterterms. The counterterms would then be fixed by consistency conditions. This would not necessarily add extra conditions for the parameters arising at lower orders, but something like this might happen. We do not know yet whether loop quantum gravity as a whole is consistent, so at some point parameters might be overconstrained. If that were the case, at least in LQC, one would have to understand if a non-perturbative cancellation of anomalies (which one did not see at the perturbative level) takes place. Therefore, an extension of our results to a second-order analysis would be most welcome not only for the purpose of finding the trispectrum, but also in order to further check the self-consistency of the theory.

Before we conclude, we emphasize that we have considered in detail only one type of corrections (inverse-volume) and no complete set of effective equations implementing all the effects expected from loop quantum gravity. Even so, the conclusions we draw are reliable because they point out characteristic phenomena from the corrections considered. An elimination of these effects by including other phenomena (most importantly, those due to holonomies) can be expected only under very fine-tuned conditions. The equations provided here can thus be used to place bounds on the free parameters of loop quantum gravity, and to rule out some parametrizations as extensively done in this paper. In all these cases, we see the importance of sufficiently general parametrizations for models to be able to stand up to phenomenological pressure; conceptual preferences or 'natural' choices of parameters may not always be the ones that survive stringent analysis. As our examples demonstrate, it is important to combine constraints from different sources. For instance, the minisuperspace parametrization is ruled out by (i) the consistency condition provided by anomaly-freedom, (ii) the interplay of inverse-volume with holonomy corrections, and (iii) the phenomenological requirement of a nearly scale-invariant spectrum. A single inconsistency could always be evaded by questioning the condition violated, especially in a situation in which no tight derivations from an underlying full theory exist. But as inconsistencies pile up, models should eventually be dropped. In this way, models of loop quantum gravity and loop quantum cosmology are already falsifiable not just by internal consistency considerations but also by comparison with observations.

Our results are interesting also because they highlighted a number of issues which are definitely worthy of further attention:

- The consistency of the slow-roll background solutions, of the anomaly cancellations, and of the physical observables only with respect to the second lattice parametrization urges us to study the latter in greater detail. Such an endeavor requires a better understanding of the full theory and its reduction to perturbations around isotropic models.

- Perturbations can propagate with superluminal speed. Either this is an artifact of linear perturbation theory, is curable with a particular choice of the parameters, or may give rise to non-standard inflationary scenarios as in [48]. The minisuperspace parametrization is safe as $f_3 > 0$ in that case (large $\sigma > 3$), while the lattice refinement parametrization with $\sigma \ll 1$ requires $\alpha_0 \leq 0$. This observation may be the one putting the most severe constraints on the lattice refinement parametrization, while the minisuperspace parametrization is under much stronger pressure from other consistency conditions.

We believe that addressing these points and the mutual tension between different parametrizations and the physical viability of the perturbations will stimulate the advance in the field.

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