Dynamics of solutions of the Einstein equations with twisted Gowdy symmetry

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Abstract

Some of the most interesting results on the global dynamics of solutions of the vacuum Einstein equations concern the Gowdy spacetimes whose spatial topology is that of a three-dimensional torus. In this paper certain of these ideas are extended to a wider class of vacuum spacetimes where the spatial topology is that of a non-trivial torus bundle over a circle. Compared to the case of the torus these are topologically twisted. They include inhomogeneous generalizations of the spatially homogeneous vacuum spacetimes of Bianchi types II and VI$_0$. Using similar procedures it is shown that the vacuum solutions of Bianchi type VII$_0$ are isometric to a class of Gowdy spacetimes, the circular loop spacetimes, thus establishing links between results in the literature which were not previously known to be related to each other.

1 Introduction

The Gowdy spacetimes are a class of solutions of the vacuum Einstein equations defined by certain symmetries. They are invariant under an action of the torus T^2 and, in addition, they possess a discrete symmetry. In what follows only those Gowdy spacetimes are considered which have a compact Cauchy surface and are such that the transformations belonging to the action of T^2 have no fixed points. It can be concluded that the Cauchy surface is diffeomorphic to the three-dimensional torus T^3 [2]. The discrete symmetry characterizes these spacetimes among the more general class of T^2-symmetric vacuum spacetimes whose dynamics is much less well understood. This characterization is explained in [9]. If it is assumed that in a solution of the Einstein equations coupled to matter the metric and the matter fields have the type of symmetry just described then they are said to have Gowdy symmetry. This paper is concerned with vacuum spacetimes but many of its results also apply to solutions with matter. The significance of the Gowdy solutions on T^3 for the more general task of

investigating the dynamics of solutions of the vacuum Einstein equations is that they represent the simplest class of inhomogeneous spatially compact solutions of these equations. As such they are an ideal laboratory for studying certain phenomena.

The subject of this paper is a generalization of Gowdy symmetry which may be called twisted Gowdy symmetry (as it is in this paper) or local Gowdy symmetry. The former name is related to the fact that in many cases the metric is naturally defined on a manifold which is topologically twisted. (This has nothing to do with the 'twist constants' whose vanishing is often used as a characterization of Gowdy spacetimes among T^2-symmetric spacetimes, cf. [11], section 4.4.) Spacetimes which generalize those with Gowdy symmetry in the way considered in what follows have previously been discussed in [10], [20], [21], [22] and [28]. Analogous symmetry assumptions have been studied for the Ricci flow in [4]. In many cases the group T^2 does not act on the solution itself but only on its pull-back to the universal covering manifold. The question of central interest in what follows is the dynamics of these solutions. There are two asymptotic regimes, the approach to the initial singularity and the late-time behaviour. The dynamics near the initial singularity in Gowdy spacetimes is very well understood [15], [18]. Many of the arguments used in that work are local in space and so can be applied to get analogous conclusions in the twisted Gowdy case. There are, nevertheless, important questions in this context which are still open. This is discussed at the end of the next section. In the case of the late-time behaviour there is no reason to expect spatial localization. This paper focusses on the late-time behaviour, examining to what extent it differs from the known facts about the Gowdy case obtained in [14].

The second section introduces the basic definitions and equations for twisted Gowdy spacetimes and explains how they can be regarded as generalizations of Bianchi models of types II and VI_0, just as the ordinary Gowdy spacetimes are generalizations of Bianchi type I. Section 3 discusses a topic related to the main theme of this paper, establishing a link between the circular loop Gowdy solutions and spacetimes of Bianchi type VII_0. Section 4 discusses the late-time asymptotic behaviour of the twisted models. In section 5 the nonlinear analysis done in this paper is compared to existing work on linear perturbations of Bianchi models. The last section presents some conclusions and an outlook on possible future developments.

2   Basic equations

In the Gowdy class the spacetime metric can be written in the form

$$t^{-\frac{1}{2}}e^{\frac{\lambda}{2}}(-dt^2 + d\theta^2) + t(e^P(dx + Qdy)^2 + e^{-P}dy^2). \qquad (1)$$

Here the functions P, Q and λ depend only on t and θ. The essential equations describing these spacetimes are a system of semilinear wave equations for two functions P and Q which are assumed to be periodic with period 2π in the

spatial coordinate θ. The equations are

$$P_{tt} + t^{-1}P_t = P_{\theta\theta} + e^{2P}(Q_t^2 - Q_\theta^2), \tag{2}$$

$$Q_{tt} + t^{-1}Q_t = Q_{\theta\theta} - 2(P_t Q_t - P_\theta Q_\theta) \tag{3}$$

where the subscripts denote partial derivatives. In what follows we call (2) and (3) the Gowdy equations.

It is useful to interpret these equations as a wave map, a concept which will now be recalled. Let (M, g) be a pseudo-Riemannian and (N, h) a Riemannian manifold and let Φ be a smooth mapping from M to N. Define a Lagrangian density by $L = \frac{\partial \Phi^I}{\partial x^\alpha} \frac{\partial \Phi^J}{\partial x^\beta} g^{\alpha\beta} h_{IJ}$. The corresponding Euler-Lagrange equations define what are called harmonic maps when g is Riemannian and wave maps when g is Lorentzian. The Riemannian manifold (N, h) is referred to as the target space.

The equations (2) and (3) can be interpreted as the defining equations of wave maps from the auxiliary metric

$$- dt^2 + d\theta^2 + t^2 d\phi^2 \tag{4}$$

to the hyperbolic plane which are independent of the coordinate ϕ. In this context P and Q are interpreted as coordinates on the hyperbolic plane which put its metric into the form $dP^2 + e^{2P}dQ^2$. The remaining function λ in the spacetime metric is given by integrals where the integrands are determined by P and Q. This follows from the equations

$$\lambda_t = t[P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)], \tag{5}$$

$$\lambda_\theta = 2t(P_t P_\theta + e^{2P}Q_t Q_\theta). \tag{6}$$

To ensure that the metric coefficient λ is periodic it is necessary to require the condition

$$\int_0^{2\pi} P_t P_\theta + e^{2P}Q_t Q_\theta d\theta = 0. \tag{7}$$

If this condition is satisfied at one time it is satisfied at all times, since

$$\frac{d}{dt}\left(\int_0^{2\pi} P_t P_\theta + e^{2P}Q_t Q_\theta d\theta\right) = -\frac{1}{t}\left(\int_0^{2\pi} P_t P_\theta + e^{2P}Q_t Q_\theta d\theta\right). \tag{8}$$

In Gowdy spacetimes P and Q are periodic in θ with period 2π. The aim of this paper is to study the dynamics of solutions of equations (2) and (3) with other types of boundary conditions. Another way of describing the periodic boundary conditions is to say that the pull-back of the solution to the universal cover is invariant under a translation in θ by 2π. The more general boundary conditions correspond to replacing invariance under a translation by 2π in θ by equivariance under that translation. Let X be a Killing vector of the hyperbolic plane and Ψ_γ the one-parameter family of diffeomorphisms it generates. Equivariant wave maps are defined by the condition that

$$(P, Q)(t, \theta + \gamma) = \Psi_\gamma((P, Q)(t, \theta)). \tag{9}$$

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In what follows an equivariant wave map is defined to be one which satisfies this condition for all real numbers γ. The restriction of this to integer multiples of 2π defines the generalizations of periodic boundary conditions to be considered. The solutions of the Gowdy equations with these boundary conditions can be used to construct spacetimes defined on manifolds corresponding to Bianchi types II and VI$_0$ which are bundles over the circle S^1 whose fibre is the torus T^2. These solutions contain a topological twist and for this reason will be said to have twisted Gowdy symmetry. They are inhomogeneous generalizations of the corresponding Bianchi models.

Let the initial data for P, P_t, Q and Q_t on some hypersurface $t = t_0$ be denoted by P_0, P_1, Q_0 and Q_1 respectively. Consider data for which P_0, P_1 and Q_1 are periodic while $Q_0(\theta + 2\pi) = Q_0(\theta) + 2\pi\alpha$ for a constant $\alpha \neq 0$. This type of data will be referred to as twisted data of type II. It is related to equivariance with respect to the Killing vector $\alpha\frac{\partial}{\partial Q}$. It is known that any periodic data for the Gowdy equations prescribed at some time $t = t_0 > 0$ give rise to a unique corresponding global solution on the time interval $(0, \infty)$ [8]. Moreover, as a result of the domain of dependence, the solution at a point with coordinates (t, θ) is uniquely determined by the data on the interval $[\theta_1, \theta_2]$ of the initial hypersurface $t = t_0$, where $\theta_1 = \theta - |t - t_0|$ and $\theta_2 = \theta + |t - t_0|$. From this global existence and uniqueness can be concluded for solutions of the Gowdy equations corresponding to data on the real line without imposing any spatial boundary conditions. It suffices to piece together suitable solutions which are defined locally in space. Using uniqueness it can be shown that the solution corresponding to twisted data of type II is such that P is periodic in θ while $Q(t, \theta + 2\pi) = Q(t, \theta) + 2\pi\alpha$. For in this case $(P(t, \theta), Q(t, \theta))$ and $(P(t, \theta + 2\pi), Q(t, \theta + 2\pi) - 2\pi\alpha)$ are solutions of the Gowdy equations with the same initial data and hence must be equal. Note that the condition (7) generalizes in an obvious way to twisted data of type II since the integrand occurring there is periodic. When this condition is satisfied it is possible to find a solution λ of the equations (5) and (6) which is periodic in θ since the right hand side of (5) is periodic. The metric which has been given in local coordinates defines a spatially compact spacetime provided $\pi\alpha$ is an integer.

Next consider data satisfying $P_0(\theta + 2\pi) = P_0(\theta) + 2\pi\alpha$ for a constant $\alpha \neq 0$, $P_1(\theta + 2\pi) = P_1(\theta)$, $Q_0(\theta + 2\pi) = e^{-2\pi\alpha}Q_0(\theta)$, $Q_1(\theta + 2\pi) = e^{-2\pi\alpha}Q_1(\theta)$. This will be referred to as twisted data of type VI$_0$. It is related to equivariance with respect to the Killing vector $\alpha(\frac{\partial}{\partial P} - Q\frac{\partial}{\partial Q})$. Using the same methods as in the type II case it can be shown that there exist unique global solutions corresponding to data of this kind and that these solutions satisfy the conditions that $P(t, \theta + 2\pi) = P(t, \theta) + 2\pi\alpha$ and $Q(t, \theta + 2\pi) = e^{-2\pi\alpha}Q(t, \theta)$. As in type II the integrand in (7) is periodic for these solutions and so the restriction which ensures the periodicity of λ is well-defined. The metric which has been given in local coordinates defines a spatially compact spacetime provided $\pi\alpha$ is an eigenvalue of a matrix in $GL(2, Z)$. (For the significance of $GL(2, Z)$ in this context see section 2 of [10].) In the type VI$_0$ case there is a class of polarized solutions defined by the vanishing of Q. Note for comparison that the boundary conditions for twisted Gowdy solutions of type II are not consistent with setting

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$Q = 0$.

The terminology involving type II and type VI$_0$ is explained by the fact that there are special solutions satisfying these boundary conditions which correspond to spatially homogeneous solutions of the respective Bianchi types. The first solution is obtained by setting $P(t, \theta) = \bar{P}(t)$ and $Q(t, \theta) = \alpha\theta$. The metric takes the form

$$t^{-\frac{1}{2}}e^{\frac{\lambda}{2}}(-dt^2 + d\theta^2) + t\left[e^{\bar{P}(t)}(dx + \alpha\theta dy)^2 + e^{-\bar{P}(t)}dy^2\right]. \qquad (10)$$

With these assumptions equation (3) is satisfied automatically while (2) reduces to

$$\frac{d^2\bar{P}}{dt^2} + t^{-1}\frac{d\bar{P}}{dt} = -\alpha^2 e^{2\bar{P}}. \qquad (11)$$

Condition (7) is also satisfied because the integrand vanishes. The metric (10) is a spatially homogeneous spacetime which is expressed in terms of the one-forms $d\theta$, $dx + \alpha\theta dy$ and dy with dual basis $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y} - \alpha\theta\frac{\partial}{\partial x}$. Computing the commutators of these vector fields reveals that this is a metric of Bianchi type II. This metric corresponds directly to the expression for the Taub solutions on p. 196 of [25]. The latter is given by

$$- A^2 dt^2 + t^{2p_1}A^{-2}(dx + 4p_1 bz dy)^2 + t^{2p_2}A^2 dy^2 + t^{2p_3}A^2 dz^2 \qquad (12)$$

where p_1, p_2 and p_3 are constants satisfying the Kasner relations $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$ while $A^2 = 1 + b^2 t^{4p_1}$. In order that the solution be of Bianchi type II it is important that $p_1 \neq 0$ and $b \neq 0$. Otherwise a Bianchi I solution is obtained. When $p_1 = 0$ it is the flat Kasner solution. When $p_1 \neq 0$ it follows that $p_3 < 1$ and the metric (12) can be put into the Gowdy form (1) by defining $\tilde{t} = (1 - p_3)^{-1}t^{1-p_3}$, $\theta = z$, $\tilde{x} = (1 - p_3)^{\frac{1}{2}}x$, $\tilde{y} = (1 - p_3)^{\frac{1}{2}}y$, $\alpha = 4p_1 b$, $P = (p_1 - p_2)\log t - 2\log A$ and $\lambda = (3p_3 + 1)\log t + 4\log A$. Here A, P and λ should be thought of as functions of $\tilde{t}$ via the relation $t = [(1 - p_3)\tilde{t}]^{\frac{1}{1-p_3}}$. These are the most general vacuum solutions of Bianchi type II. More precisely, the pull-back of any vacuum solution of Bianchi type II to the universal covering manifold can be written globally in the form given in (12).

Another type of solution of equations (2) and (3) is obtained by setting $P(t, \theta) = \bar{P}(t) + \alpha\theta$, $Q(t, \theta) = e^{-\alpha\theta}\bar{Q}(t)$. The metric on the group orbits is then of the form

$$e^{\bar{P}(t)}(e^{\frac{1}{2}\alpha\theta}dx + \bar{Q}(t)e^{-\frac{1}{2}\alpha\theta}dy)^2 + e^{-\bar{P}(t)}(e^{-\frac{1}{2}\alpha\theta}dy)^2. \qquad (13)$$

Equations (2) and (3) become

$$\frac{d^2\bar{P}}{dt^2} + t^{-1}\frac{d\bar{P}}{dt} = e^{2\bar{P}}\left[\left(\frac{d\bar{Q}}{dt}\right)^2 - \alpha^2\bar{Q}^2\right], \qquad (14)$$

$$\frac{d^2\bar{Q}}{dt^2} + t^{-1}\frac{d\bar{Q}}{dt} = -2\frac{d\bar{P}}{dt}\frac{d\bar{Q}}{dt} - \alpha^2\bar{Q}. \qquad (15)$$

Provided the right hand side of (6) vanishes a spatially homogeneous spacetime is obtained expressed in terms of the one-forms $d\theta$, $e^{\frac{1}{2}\alpha\theta}dx$ and $e^{-\frac{1}{2}\alpha\theta}dy$ with dual basis $\frac{\partial}{\partial\theta}$, $e^{-\frac{1}{2}\alpha\theta}\frac{\partial}{\partial x}$, $e^{\frac{1}{2}\alpha\theta}\frac{\partial}{\partial y}$. Computing the commutators of these vector fields shows that this is a solution of Bianchi type VI_0.

The vanishing condition for the right hand side of (6) and the evolution equations for $\bar{P}$ and $\bar{Q}$ are rather complicated. A more transparent formulation can be obtained by introducing the variables

$$V = \log[e^P Q + \sqrt{1 + e^{2P}Q^2}], \tag{16}$$

$$W = -\frac{1}{2}\log[e^{-2P} + Q^2]. \tag{17}$$

The mapping $(P, Q) \mapsto (V, W)$ is smooth and has a smooth inverse which is given explicitly as follows:

$$P = W + \log\cosh V, \tag{18}$$

$$Q = e^{-W}\tanh V. \tag{19}$$

In the variables (V, W) the wave map equations take the form

$$W_{tt} + t^{-1}W_t = W_{\theta\theta} + \tanh V(-W_t V_t + W_\theta V_\theta), \tag{20}$$

$$V_{tt} + t^{-1}V_t = V_{\theta\theta} - \cosh V \sinh V(-W_t^2 + W_\theta^2). \tag{21}$$

When P and Q satisfy the type VI_0 boundary conditions V is periodic while $W(t, \theta+2\pi) = W(t, \theta)+2\pi\alpha$. The polarized class is characterized by $V = 0$. The restricted class of solutions transforms to solutions of the form $V(t, \theta) = \bar{V}(t)$ and $W(t, \theta) = \bar{W}(t) + \alpha\theta$. They are equivariant with respect to the vector field $\alpha\frac{\partial}{\partial W}$. For this class the evolution equations for V and W reduce to

$$\bar{W}_{tt} + t^{-1}\bar{W}_t = -\tanh\bar{V}\bar{W}_t\bar{V}_t, \tag{22}$$

$$\bar{V}_{tt} + t^{-1}\bar{V}_t = -\cosh\bar{V}\sinh\bar{V}(-\bar{W}_t^2 + \alpha^2). \tag{23}$$

The condition for the periodicity of λ simplifies to $\bar{W}_t = 0$. From now on it will be assumed that $\bar{W} = 0$. Then (23) simplifies to

$$\bar{V}_{tt} + t^{-1}\bar{V}_t = -\alpha^2\cosh\bar{V}\sinh\bar{V}. \tag{24}$$

The evolution equation for λ is

$$\lambda_t = t[\cosh^2 V(W_t^2 + W_\theta^2) + (V_t^2 + V_\theta^2)]. \tag{25}$$

The metric on the group orbits takes the form

$$\frac{1}{2}e^{\bar{V}}(e^{\frac{1}{2}\alpha\theta}dx + e^{-\frac{1}{2}\alpha\theta}dy)^2 + \frac{1}{2}e^{-\bar{V}}(e^{\frac{1}{2}\alpha\theta}dx - e^{-\frac{1}{2}\alpha\theta}dy)^2. \tag{26}$$

Here the metric is diagonal in a left-invariant basis, in contrast to the metric (13). In the terminology of [17] the basis used in (26) is canonical and the fact

that the metric is diagonal may be put into context by comparing with Corollary 19.14 of that reference which says that any solution of Bianchi class A can be diagonalized in a canonical frame. The special case obtained by setting $\bar{Q} = 0$ in the metric (13) (polarized case) gives rise to a class of spacetimes which can be identified with the Ellis-MacCallum solutions given on p. 197 of [25]. Note that in this case the condition (7) forces $\bar{P} = 0$.

A large class of twisted Gowdy solutions of Bianchi type II can be obtained starting from ordinary Gowdy solutions using the Gowdy-to-Ernst transformation. This transformation was introduced in the study of spikes in Gowdy spacetimes [12] and was later used in the study of the initial singularity [15]. The definition of the transformation is as follows. Given a solution (P, Q) of the Gowdy equations define a new solution $(\tilde{P}, \tilde{Q})$ by the relations

$$\tilde{P} = -\log t - P, \tag{27}$$
$$\tilde{Q}_t = te^{2P}Q_\theta, \quad \tilde{Q}_\theta = te^{2P}Q_t. \tag{28}$$

Determining $\tilde{Q}$ requires some integration and while this is always possible locally it is only possible globally on the torus if

$$\int_0^{2\pi} te^{2P}Q_t d\theta = 0. \tag{29}$$

If the integral in this equation has a suitable non-zero value then $\tilde{P}$ and $\tilde{Q}$ define a twisted Gowdy solution of type II. Note that

$$\frac{d}{dt}\left(\int_0^{2\pi} te^{2P}Q_t d\theta\right) = 0, \tag{30}$$

so that it is enough to require the condition at one time in order to ensure that it is satisfied at all times. The condition (7) is preserved by this transformation and so if a periodic λ exists before transformation the same is true after transformation. Not all solutions of type II are obtained in this way. A necessary and sufficient condition for a solution to be contained in the image of this transformation is that it satisfies (29).

It will now be shown how certain statements about the behaviour of solutions near the initial singularity can be transferred from the standard Gowdy case to twisted Gowdy solutions. Consider a solution of the Gowdy equations defined on a region of the form $S = (0, t_1) \times I$ for a open interval I. Consider a point (t_0, θ_0) for which t is so small that the interval $(\theta_0 - 2t_0, \theta_0 + 2t_0)$ is contained in I. Then the part of the solution in the past of (t_0, θ_0) is determined by data on the part of $t = t_0$ which is contained in I. Suppose in addition that the length of I is less than 2π. Then it is elementary to see that the solution on the past of (t_0, θ_0) can be embedded into a solution with periodic boundary conditions of period 2π. Define the asymptotic velocity at θ_0 to be

$$\lim_{t\to 0}[t(P_t^2 + e^{2P}Q_t^2)^{1/2}](t, \theta_0). \tag{31}$$

The asymptotic velocity defines a function of θ which is periodic. It is proved in [15] that in a Gowdy solution this limit exists at any point θ_0. From the remarks just made about embeddings it follows that the same conclusion holds for twisted Gowdy solutions. The notions of true and false spikes as defined in [18] make sense for twisted Gowdy solutions. Thus as in that paper it is possible to define the set $\mathcal{G}_c$ of twisted Gowdy solutions which satisfy (7), have non-degenerate true and false spikes at a finite number of values of θ and are such that the asymptotic velocity is strictly between zero and one everywhere else. To define a notion of genericity it is necessary to define a suitable topology on the set of twisted solutions of a given type. This can be done by using the standard C^∞ topology on an interval of length 2π and noting that the result does not depend on which interval is chosen. The relevant seminorms are equivalent. The arguments of [15] show that the set $\mathcal{G}_c$ is open in the C^∞ topology. For the usual Gowdy case it is shown in [18] that $\mathcal{G}_c$ is also dense. It has not been verified whether the analogous statement is true for twisted Gowdy solutions.

3   Bianchi type VII_0 and the circular loop space-times

This section is concerned with the circular loop spacetimes (see [3], Appendix B) and their relation to spacetimes of Bianchi type VII_0. In contrast to the spacetimes of Bianchi types II and VI_0 considered elsewhere in this paper there is no topological twist in this case. The spatial topology is T^3 but there is a geometrical twist. In terms of the wave map formulation the solutions to be considered here are again equivariant with respect to a Killing vector of the hyperbolic plane. This Killing vector looks complicated when expressed in terms of the coordinates P and Q and so it is convenient at this point to convert to coordinates adapted to the disc model of the hyperbolic plane. They are defined by the relations that $\Phi \cos \Theta$ and $\Phi \sin \Theta$ are the real and imaginary parts of the complex quantity

$$\frac{Q + i(e^{-P} - 1)}{Q + i(e^{-P} + 1)} \tag{32}$$

respectively. The image of the (P, Q) plane under this mapping is the region given by $0 \leq \Phi < 1$. This is a polar coordinate system with origin at $\Phi = 0$.

The wave map equations take the form

$$\Phi_{tt} + t^{-1}\Phi_t - \Phi_{\theta\theta} = \frac{1}{2}\sinh 2\Phi(\Theta_t^2 - \Theta_\theta^2), \tag{33}$$

$$\sinh^2 \Phi(\Theta_{tt} + t^{-1}\Theta_t - \Theta_{\theta\theta}) = \sinh 2\Phi(-\Phi_t\Theta_t + \Phi_\theta\Theta_\theta) \tag{34}$$

and the evolution equation for λ is

$$\lambda_t = t[(\Phi_t^2 + \Phi_\theta^2) + \sinh^2 \Phi(\Theta_t^2 + \Theta_\theta^2)]. \tag{35}$$

In these variables the metric on the group orbits is

$$e^{\bar\Phi}((\cos \frac{1}{2}\alpha\theta)dx + (\sin \frac{1}{2}\alpha\theta)dy)^2 + e^{-\bar\Phi}((-\sin \frac{1}{2}\alpha\theta)dx + (\cos \frac{1}{2}\alpha\theta)dy)^2. \tag{36}$$

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Let the initial data for Φ, Φ_t, Θ and Θ_t be denoted by Φ_0, Φ_1, Θ_0 and Θ_1 respectively. The circular loop spacetimes are defined by the conditions $\Phi(t,\theta) = \bar{\Phi}(t)$ and $\Theta(t,\theta) = \alpha\theta$ for a constant $\alpha \neq 0$ or by corresponding conditions on the initial data. The equation for Θ is satisfied identically by this ansatz while that for Φ reduces to

$$\frac{d^2\bar{\Phi}}{dt^2} + t^{-1}\frac{d\bar{\Phi}}{dt} = -\frac{\alpha^2}{2}\sinh 2\bar{\Phi}. \tag{37}$$

The metric on the group orbits takes the form

$$e^{\bar{\Phi}}((\cos\frac{1}{2}\alpha\theta)dx + (\sin\frac{1}{2}\alpha\theta)dy)^2 + e^{-\bar{\Phi}}((-\sin\frac{1}{2}\alpha\theta)dx + (\cos\frac{1}{2}\alpha\theta)dy)^2 \tag{38}$$

and the spacetime metric is of Bianchi type VII_0. The wave map is equivariant with respect to the Killing vector $\alpha\frac{\partial}{\partial\Theta}$. In this way it can seen that the circular loop form defines the same class of solutions of the vacuum Einstein equations as Bianchi type VII_0. Notice also the remarkable fact that with a change of notation (37) is identical to (24). The author has found no explanation for this coincidence which means that the dynamics of solutions of Bianchi types VI_0 and VII_0 are controlled by the same ODE.

In [3] the late-time behaviour of the circular loop spacetimes was determined. Among other things it was shown that $\bar{\Phi}$ and $\bar{\Phi}_t$ are $O(t^{-\frac{1}{2}})$ as $t \to \infty$, that $tH(t)$ converges to a constant H_∞ as $t \to \infty$ and that the spacetimes are future geodesically complete. The constant H_∞ is strictly positive for any circular loop spacetime. (Bianchi type I solutions are not considered to belong to this class.) All these statements were later extended to general Gowdy spacetimes in [14]. Independently of this the late-time behaviour of vacuum spacetimes of Bianchi type VII_0 was analysed in [13]. Knowing the relation between the Bianchi type VII_0 and the circular loop spacetimes, those results in [13] which concern solutions of Bianchi type VII_0 can easily be deduced from the results of [3]. In [13] it was shown that two of the Wainwright-Hsu variables N_1 and N_2 tend to the same constant value. It turns out that this constant is equal to $12H_\infty^{-1}$.

4   Late time dynamics

An important property of Gowdy models is the existence of a functional, often called energy, whose dependence on time is monotone. It is given by

$$H = \frac{1}{2}\int_0^{2\pi} P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)d\theta. \tag{39}$$

In the Bianchi type II case the same quantity is monotone non-increasing. The proof is essentially the same. It is just necessary to check that no additional boundary terms arise during partial integration. The relevant identity is

$$\frac{dH}{dt} = -t^{-1}\int_0^{2\pi} P_t^2 + e^{2P}Q_t^2 d\theta + [P_t P_\theta + e^{2P}Q_t Q_\theta]_0^{2\pi}. \tag{40}$$

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Since P, P_t, Q_t and Q_θ are periodic in the twisted type II case the boundary term vanishes. The boundary term also vanishes in the twisted type VI_0 case. It should be noted that in both cases the energy density is periodic so that the apparently arbitrary choice of $\theta = 0$ as the starting point of integration has no effect on the value of the integral. An analogous definition using a different starting point gives the same answer.

In Gowdy models it has been proved that $H(t) = O(t^{-1})$ as $t \to \infty$ [14]. This is done in two steps. First, it is shown that if the energy is ever smaller than a certain threshold H_1 then it is $O(t^{-1})$ as $t \to \infty$. Second, it is shown that the energy tends to zero for all solutions. Twisted Gowdy solutions of type VI_0 satisfy the inequality $H(t) \geq \pi\alpha^2$ and so in that case this strategy must be modified if it is to have a chance of success. It may be conjectured that the energy tends to zero as $t \to \infty$ for twisted Gowdy solutions of type II and to $\pi\alpha^2$ for solutions of type VI_0. In the homogeneous case this follows from known results. It is also not difficult to treat the homogeneous case directly, as will now be shown.

In the type II case, suppose first that $\bar{P}_t(t_1) > 0$ for some t_1. Then from (11) for $t \geq t_1$

$$(t\bar{P}_t)_t(t) \leq -\alpha^2 t_1 e^{2\bar{P}(t_1)} \tag{41}$$

as long as $\bar{P}_t$ stays positive. It follows that $\bar{P}_t$ must become zero after a finite time. When $\bar{P}_t$ is zero its derivative is negative. This means that once it reaches zero $\bar{P}_t$ can never become positive again. Moreover it must become negative immediately after the time when it is zero. Thus to study the late time behaviour it may be assumed without loss of generality that $\bar{P}_t$ is always negative. In particular $\bar{P}$ tends to a limit $\bar{P}^\infty$, finite or infinite, as $t \to \infty$. Suppose now that $\bar{P}^\infty > -\infty$. Then

$$(t\bar{P}_t)_t(t) \leq -\alpha^2 e^{2\bar{P}^\infty} t. \tag{42}$$

Integrating this twice shows that $\bar{P} \to -\infty$ as $t \to \infty$, contradicting the assumption on $\bar{P}^\infty$. Thus in fact $\bar{P}(t) \to -\infty$ as $t \to \infty$. Now H tends to a limit $H_0 \geq 0$ as $t \to \infty$ and it follows that $\lim_{t\to\infty} \bar{P}_t = -\sqrt{2H_0}$. If H_0 were positive then $-\bar{P}$ would grow at least linearly and $e^{2\bar{P}}$ would decay at least exponentially. This would imply that $t\bar{P}_t$ tends to a finite limit, a contradiction. Hence $H_0 = 0$.

In the type VI_0 case the boundedness of the energy shows that $\bar{V}$ and $\bar{V}_t$ are bounded. Using the evolution equation (24) this implies the boundedness of $\bar{V}_{tt}$. This allows the use of a compactness argument. Let $\{t_n\}$ be a sequence of times tending to infinity. Define translated quantities by $\bar{V}^n(t) = \bar{V}(t - t_n)$. Then $\bar{V}^n$ and $\bar{V}_t^n$ satisfy uniform C^1 bounds on any compact time interval. By the Arzela-Ascoli theorem [19] it follows that, possibly after passing to a subsequence, $\bar{V}^n$ converges uniformly on finite time intervals to a limit $\bar{V}^*$ which satisfies the equation $\bar{V}_{tt}^* = -\frac{1}{2}\alpha^2 \sinh 2\bar{V}^*$. If the limit H_0 of H as $t \to \infty$ is greater than $\pi\alpha^2$ then the limiting solution is not identically zero. Moreover it is straightforward to show that it is periodic. Thus there exists $T > 0$ such that

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$\bar{V}^*(t + T) = \bar{V}^*(t)$ for all t. Let $\eta = \int_{t_0}^{t_0+T}(\bar{V}_t^*(t))^2 dt$. Then

$$\int_{t_0}^{t_0+mT} t^{-1}(\bar{V}_t^*(t))^2 dt \geq \eta \sum_{k=1}^{m}(t_0 + kT)^{-1} \tag{43}$$

Since the sum on the right hand side diverges as $m \to \infty$ there exists an integer M such that

$$\int_{t_0}^{t_0+MT} t^{-1}(\bar{V}_t^*(t))^2 dt \geq 2(H(t_0) - H_0). \tag{44}$$

Using the uniform convergence of the sequence $\bar{V}^n$ to its limit $\bar{V}^*$ on the interval $[t_0, t_0 + MT]$ shows that

$$\int_{t_0}^{\infty} t^{-1}(\bar{V}_t^n(t))^2 dt > (H(t_0) - H_0) \tag{45}$$

for n sufficiently large and this contradicts the fact that H is always positive. Thus in fact $\bar{V}^*$ is identically zero. From this it can be concluded that $H \to \pi\alpha^2$ as $t \to \infty$. Recall that the asymptotics of vacuum solutions of types VI_0 and VII_0 are governed by the same basic equation. It would be interesting to do a detailed comparison between the results in the literature of relevance to the detailed asymptotics of solutions of this equation. Apart from the papers [3] and [13] mentioned in the last section there is the work of Heinzle and Ringström [6] on the late-time behaviour of solutions of Bianchi type VI_0 based on the Wainwright-Hsu system [26]. Due to the variety of formulations of the equations and notations used in the different papers a comparison of this type would involve heavy computations.

A relatively simple case to start with in studying the behaviour of the energy as $t \to \infty$ in inhomogeneous spacetimes is that of polarized twisted Gowdy solutions of type VI_0 since there the main field equation is linear. Let $P(t, \theta)$ be a solution of this type. Let $\tilde{P}$ be the explicit solution defined by $\alpha\theta$. Due to the linearity of the equation for P the difference $P - \tilde{P}$ is also a solution. Moreover $P - \tilde{P}$ is periodic in θ and so is an ordinary polarized solution. Hence its asymptotics can be deduced from results proved in [7]. It follows in particular that the energy of any polarized Gowdy solution of type VI_0 tends to $\pi\alpha^2$ as $t \to \infty$. The energy $\tilde{H}$ of a twisted solution of type II obtained from a Gowdy solution by the Gowdy-to-Ernst transformation need not be equal to the energy H of the original solution but it does satisfy $\tilde{H}(t) = H(t) + O(t^{-2})$. It follows from the known results on Gowdy solutions that $\tilde{H}(t) = O(t^{-1})$ for twisted type II solutions obtained in this way. Note that there are ordinary Gowdy solutions which have $H = 0$. They are isometric to the Kasner solution with Kasner exponents $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. A twisted Gowdy solution of type II never has $H = 0$ and this applies in particular to the homogeneous solutions obtained by transforming the solutions of type I which satisfy $H = 0$.

It is worth noting that the results on the dynamics of Gowdy solutions proved by Ringström are results on solutions of the Gowdy equations and are not dependent on the restriction (7) arising from the equation for λ. In this

context it is interesting to remark that there is an explicit class of solutions of the Gowdy equations whose dynamics is easy to analyse but which almost always violate (7). In these solutions $P(t,\theta) = -\frac{1}{2}\log t$ and $Q(t,\theta) = q(t-\theta)$ for an arbitrary function q of one variable. They are characterized by the fact that they are fixed points of the Gowdy-to-Ernst transformation. If q is periodic these solutions have Gowdy symmetry while if $q(x+2\pi) = q(x)+2\pi\alpha$ they have twisted Gowdy symmetry of type II. They never satisfy (7) unless q is constant. The known results on the decay of solutions of the Gowdy equations apply to these solutions and the fact that $H(t) = O(t^{-1})$ can be read off directly in this case. In addition it is seen that solutions of this kind with type II symmetry have the same decay of H. The asymptotics of these spacetimes near the singularity can also be read off directly. In the notation of equations (12) and (13) of [15] the function q of that paper coincides with that used here, the function ϕ vanishes identically, $v_a(\theta) = \frac{1}{2}$ and $\psi(\theta) = q'(\theta)$. It does not seem that solutions of the Gowdy equations of this kind can be interpreted as coming from spatially compact spacetimes. These solutions belong to a class discussed by Wainwright and Marshman [27] [1]. To see the relation set $m = -\frac{3}{16}$ (which is the vacuum condition) and $q = 1$ in Case I of [27].

Some evidence has now been collected that in twisted models of type II and VI$_0$ the energy tends to the limits zero and $\pi\alpha^2$ respectively as $t \to \infty$. Unfortunately this has not yet been proved in general, even in the case that the energy is initially small. The proofs of these statements for the usual Gowdy spacetimes make extensive use of the averages of the unknowns in space. In the twisted models global averages do not always make sense but it is possible to define analogous quantities by the formula

$$\langle f(t,\theta)\rangle = \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} f(t,\sigma)d\sigma. \tag{46}$$

If f is periodic then $\langle f\rangle$ is equal to the average value of f and, in particular, independent of θ. For the functions P and Q in twisted models this is in general no longer the case. The averaged quantities satisfy the same boundary conditions as the original ones. This means in particular that in type II the difference $Q - \langle Q\rangle$ is periodic with integral zero. We also have the identity $\langle Q\rangle_\theta = \alpha$. The starting point for the proof of the asymptotics in the case of initial data with small energy is a differential inequality for a suitable corrected energy of the form $H + \Gamma^P + \Gamma^Q$ where

$$\Gamma^P = \frac{1}{2t}\int_0^{2\pi}(P-\langle P\rangle)P_t d\theta, \tag{47}$$

$$\Gamma^Q = \frac{1}{2t}\int_0^{2\pi}e^{2\langle P\rangle}(Q-\langle Q\rangle)Q_t d\theta. \tag{48}$$

The integrands in these formulas are periodic in both types II and VI$_0$. Many of the calculations which lead to the important differential inequality work just

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[1]I thank Woei-Chet Lim for drawing my attention to this

as well in the twisted type II case but there is one problematic term which is left over. This is of the form $\int_0^{2\pi} e^{2\langle P \rangle} \langle Q(t,\theta) \rangle_\theta Q_\theta(t,\theta) d\theta$. For an ordinary Gowdy solution it is zero but it the type II twisted case it is equal to $\alpha^2 e^{2\langle P \rangle}$. It is difficult to see how the latter expression could be estimated in way which would lead to a useful differential inequality similar to that obtained in the ordinary Gowdy case. Thus this technique of proof seems to fail in the type II case.

Since P is periodic in type II it is possible to show that $|P - \langle P \rangle| \leq CH^{\frac{1}{2}}$ for a constant C which only depends on the initial vakue of H and that

$$C^{-1}e^P \leq e^{\langle P \rangle} \leq Ce^P. \tag{49}$$

Using this it can be shown that

$$e^{2\langle P \rangle} \leq CH\alpha^{-2} \tag{50}$$

It follows that there is one important way in which the type II twisted case differs from the untwisted case: if H tends to zero in a type II solution as $t \to \infty$ then $\langle P \rangle$ tends to $-\infty$ in that limit. For ordinary Gowdy solutions, on the other hand, there are solutions where $\langle P \rangle$ tends to $-\infty$ but also solutions where it tends to $+\infty$ and solutions where it remains bounded for all time. If $\langle P \rangle$ tends to $-\infty$ then P tends uniformly to $-\infty$.

5   Comparison with linearized perturbations

Linearizing the full vacuum Einstein equations about the background given by a solution of Bianchi type II gives rise to a perturbation problem which has been studied by Tanimoto [23], [24]. In fact these papers deal mainly with the model problems where the linearized Einstein equations are replaced by a scalar wave equation or the Maxwell equations on the given background. If Tanimoto's work is specialized to the case with a symmetry corresponding to perturbations belonging to the class of twisted Gowdy solutions then it should be possible to compare the result with the full nonlinear theory developed in Section 4. The analysis of [23] and [24] is based on thinking of the spatial manifold of the Bianchi type II solution as a circle bundle over T^2 rather than a T^2 bundle over the circle. In the latter interpretation θ is a coordinate on the base manifold and x and y are coordinates on the fibres. In the former θ and y are coordinates on the base manifold while x is a coordinate on the fibre. The case analysed in Section 4 relates to perturbations which depend only on θ and, in particular, not on x. The coordinate x in this paper corresponds to z in [23]. Hence in Tanimoto's notation they satisfy the condition $m = 0$. This case is not included in the theorems of [23] as a result of a genericity assumption. It is discussed in section 7 of [24].

With this motivation in mind, consider the dynamics of a solution of the wave equation $\nabla^\alpha \nabla_\alpha \psi = 0$ on a background solution of Bianchi type II and, due to the subject of interest in this paper, restrict consideration to solutions which only depend on the coordinates t and θ. The wave equation on a spatially

homogeneous spacetime takes the form $\alpha^{-2}\psi_{TT} + (-\alpha^{-3}\alpha_T + \mathrm{tr}k)\psi_T = \Delta\psi$ with respect to a time coordinate T which is constant on the hypersurfaces of homogeneity and has lapse function α. Consider for a moment the case of a Kasner solution expressed in terms of an areal time coordinate. The wave equation takes the form

$$\psi_{tt} + t^{-1}\psi_t = \psi_{\theta\theta} \tag{51}$$

for any Kasner solution. This is just the polarized Gowdy equation. Since the wave equation is the simplest example of a wave map it is not surprising that this coincidence is related to the representation of the Gowdy equations in terms of a wave map with the domain metric (4). In fact more is true. In any Gowdy or twisted Gowdy spacetime the wave equation for a function ψ depending only on t and θ takes the form (51). In particular this statement holds for any spatially homogeneous solution of Bianchi type II or VI_0. The linearization of the Gowdy equations about a homogeneous solution of type II (or even type I) are more complicated. They read

$$\tilde{P}_{tt} + t^{-1}\tilde{P}_t = \tilde{P}_{\theta\theta} - 2\alpha^2 e^{2\bar{P}}\tilde{P}, \tag{52}$$

$$\tilde{Q}_{tt} + t^{-1}\tilde{Q}_t = \tilde{Q}_{\theta\theta} - 2\bar{P}_t\tilde{Q}_t + 2\alpha\tilde{P}_\theta \tag{53}$$

where $\tilde{P}$ and $\tilde{Q}$ are the linearized variables corresponding to P and Q.

6 Conclusions and outlook

In this paper the dynamics of solutions of the Gowdy equations with unconventional boundary conditions corresponding to topologically twisted manifolds was studied. A central question concerns the late-time behaviour of the energy functional $H(t)$. In the untwisted case $H(t) = O(t^{-1})$ as $t \to \infty$ and this estimate is in general sharp. By analogy we conjecture that the quantity $H(t)$, which is known to be non-increasing, tends to zero as $t \to \infty$ in the type II case and to $\pi\alpha^2$ in the type VI_0 case. These statements were proved in some special cases including infinite dimensional families of solutions. Unfortunately a general proof of these statements was not found. Trying to apply the techniques which were successful in the usual Gowdy case runs up against obstacles and it seems that some essentially new ideas are needed to make more progress on this question.

The results of this paper concern solutions of the vacuum Einstein equations. If instead the Einstein-Maxwell equations are considered then interesting new issues arise. Even if the metric quantities satisfy the standard periodic boundary conditions there is a topological feature which can have an important effect on the dynamics. If we define a Maxwell field to be a field tensor which satisfies the Maxwell equations then this issue is not visible. It becomes so if we think of the Maxwell tensor as the curvature of a connection of a circle bundle over the spacetime manifold. An alternative approach is to ask whether the field can be derived from a global smooth vector potential. When it can it is possible to extend the techniques from the vacuum case to the Einstein-Maxwell case to

prove that the natural energy functional tends to zero as $t \to \infty$ [16]. In the work of [9] on strong cosmic censorship in solutions of the Einstein-Maxwell equations with polarized Gowdy symmetry the existence of a global potential satisfying periodic boundary conditions was assumed. Nothing was proved about the case where no potential exists. Using the concepts of the present paper a result on this question can be obtained. It was shown in [9] that in solutions of the Einstein-Maxwell equations with polarized Gowdy symmetry the metric function P and a potential χ satisfy the polarized Gowdy equations. If instead of assuming that χ is periodic in θ it is assumed that it satisfies the boundary condition $\chi(t, \theta + 2\pi) = \chi(t, \theta)$ a problem is obtained which is equivalent to the vacuum Gowdy case with twisted type II symmetry. These Einstein-Maxwell solutions can be interpreted as corresponding to a situation on the torus with a Maxwell field which does not come from a potential. We intend to develop these ideas concerning the Einstein-Maxwell equations further in a separate publication.

Another possible direction in which the results of this paper can be extended is to go to solutions of the vacuum Einstein equations in higher dimensions. Assume that a solution in $n+1$ dimensions has $n-1$ commuting Killing vectors and satisfies a suitable condition of reflection symmetry generalizing that defining the Gowdy class in four dimensions. The field equations can be written in a way closely analogous to that in four dimensions with the central equations defining a wave map with values in a suitable target space. There is a natural energy functional. It has been shown in [1], generalizing the proofs of [14], that this energy tends to zero as $t \to \infty$ in any dimension. There are many connections between homogeneous models and models of Gowdy type in higher dimensions which remain to be explored. Some more remarks on this subject can be found in section 5 of [5].

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