Decoherence Strength of Multiple Non-Markovian Environments

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It is known that one can characterize the decoherence strength of a Markovian environments by the product of its temperature and induced damping and order the decoherence strength of multiple environments by this quantity. For non-Markovian environments in the weak coupling regime we show that there exists a natural, albeit partial ordering of decoherence strengths via a perturbative treatment. This measure can be applied to both low-temperature and non-equilibrium environments.

I. INTRODUCTION

Environment-induced decoherence is an essential process for a quantum system to acquire classical attributes [1–3]. To characterize how strong an environment can induce decoherence in an open quantum system it is desirable to come up with a measure of the *"decoherence strength"* of each environment acting on the system. For a Markovian environment, which for quantum Brownian motion refers to high temperatures and Ohmic spectral density, such a measure can be constructed by the product of its temperature and damping rate. However, such a measure may not exist for a *general* environment (with non-Ohmic spectral density functions and under low temperature conditions, see [4–6]), or, even more challenging, for nonequilibrium environments where the notion of temperature loses meaning. We show in this letter that at least perturbatively under weak coupling between the system and these environments, low-temperature and non-equilibrium environments *can* be partially ordered. The object of comparison is, in fact, the correlation function of the collective coupling operators of the environment. As with matrices and kernels, quantum correlations can only be partially ordered. However, this does not rule out nontrivial comparisons; in our final section we explicitly detail how one can compare decoherence strengths for combinations of thermal reservoirs without resorting to the concocted notion of an effective temperature, as in general it does not exist. Our general relation include some recently reported results as special cases [7].

II. QUANTUM CORRELATIONS & DECOHERENCE STRENGTH

Consider a quantum system weakly interacting with an environment with interaction Hamiltonian:

$$\mathbf{H}_{\mathrm{I}} = \sum_n \mathbf{L}_n \otimes \mathbf{l}_n \,, \tag{1}$$

expanded as a sum of separable operators, where $\mathbf{L}_n$ and $\mathbf{l}_n$ are system and environment operators respectively. The environment coupling operators $\mathbf{l}_n$ will typically be collective observables of the environment, with dependence upon very many modes. The system-environment interaction will be treated perturbatively and so the central ingredient is the (multivariate) correlation function of the environment:

$$\alpha_{nm}(t,\tau) = \langle \underline{\mathbf{l}}_n(t)\, \underline{\mathbf{l}}_m(\tau) \rangle_{\mathrm{E}} \,, \tag{2}$$

where $\underline{\mathbf{l}}_n(t)$ represents the time-evolving $\mathbf{l}_n$ in the interaction (Dirac) picture. In the *influence functional* formalism [8] for the quantum Brownian model with bilinear couplings between the system and its environment [4, 5, 9] the correlation function appears as the kernel in the exponent of a Gaussian influence functional, called the influence kernel ζ in Refs. [10, 11]. Alternatively, in *quantum state diffusion* [12] this kernel takes the explicit role of a noise correlation for complex Gaussian noise. The influence kernel, or equivalently, the complex correlation function, can be written as a sum of two real parts corresponding to the noise and dissipation kernels [10, 11]:

$$\underbrace{\boldsymbol{\alpha}(t,\tau)}_{\text{complex noise}} = \underbrace{\boldsymbol{\nu}(t,\tau)}_{\text{noise}} + \imath \underbrace{\boldsymbol{\mu}(t,\tau)}_{\text{dissipation}} \,. \tag{3}$$

The noise kernel $\boldsymbol{\nu}$ appears in the influence kernel as the correlation of an ordinary real stochastic source, whereas the dissipation kernel $\boldsymbol{\mu}$ alone would produce a purely homogeneous (though not necessarily positivity preserving) evolution. These same roles can also be inferred from the Heisenberg equations of motion for the system operators after integrating the environment dynamics, producing the so-called *quantum Langevin equation* [13].

Using the notation of Ref. [14], the second-order master equation [15–17] of the reduced density matrix $\boldsymbol{\rho}$ can be represented in terms of the noise correlation as

$$\dot{\boldsymbol{\rho}} = [-\imath \mathbf{H}, \boldsymbol{\rho}] + \boldsymbol{\mathcal{L}}_2\{\boldsymbol{\rho}\} \,, \tag{4}$$

with the second-order contribution given by the operation

$$\boldsymbol{\mathcal{L}}_2\{\boldsymbol{\rho}\} \equiv \sum_{nm} \left[\mathbf{L}_n, \boldsymbol{\rho}\, (\mathbf{A}_{nm} \diamond \mathbf{L}_m)^\dagger - (\mathbf{A}_{nm} \diamond \mathbf{L}_m)\, \boldsymbol{\rho}\right] \,, \tag{5}$$

where the $\mathbf{A}$ operators and $\diamond$ product define the second-order operators

$$(\mathbf{A}_{nm}\diamond\mathbf{L}_m)(t) \equiv \int_0^t d\tau\, \alpha_{nm}(t,\tau)\, \{\mathbf{G}_0(t,\tau)\,\mathbf{L}_m(\tau)\}\,, \quad (6)$$

given the free system propagator $\mathbf{G}_0(t,\tau) : \boldsymbol{\rho}(\tau) \to \boldsymbol{\rho}(t)$. Because of the nonlocal character of the noise correlation, the *time-translation generator* in Eq. (5) is not generally of Lindblad form. Lindblad's theorem (also due to Gorini, Kossakowski and Sudarshan) specifically characterizes the *algebraic generators* $\boldsymbol{\Phi}$ for all completely-positive maps $e^{\eta\,\boldsymbol{\Phi}}$, where $\eta > 0$ parameterizes the semigroup [18, 19].

$$\boldsymbol{\Phi}\,\boldsymbol{\rho} = \underbrace{-\imath\,[\boldsymbol{\Theta},\boldsymbol{\rho}]}_{\text{unitary}} + \underbrace{\sum_{ij}\Xi_{ij}\left(\mathbf{e}_i\,\boldsymbol{\rho}\,\mathbf{e}_j^\dagger - \frac{1}{2}\left\{\mathbf{e}_j^\dagger\mathbf{e}_i,\boldsymbol{\rho}\right\}\right)}_{\text{decoherent}}\,. \quad (7)$$

Such generators and the dynamics they engender when the master equation has the Lindblad form have been extensively studied [20–29]. Here, Ξ_{ij}, the (algebraic) *dissipator*, is a positive-definite coefficient matrix and $\mathbf{e}_i$ denotes a particular basis of representation for the dissipator. The "dissipation" generated by the dissipator is that of states, including decoherence.

From Ref. [14], the second-order algebraic dissipator $\Xi_{ii';jj'}(t)$ evaluates to

$$\sum_{nm}\int_0^t d\tau\int_0^t d\tau'\,\langle i|\,\underline{\mathbf{L}}_m(\tau)\,|i'\rangle\,\alpha_{nm}(\tau',\tau)\overline{\langle j|\,\underline{\mathbf{L}}_n(\tau')\,|j'\rangle}\,, \quad (8)$$

in the interaction (Dirac) picture and evaluated in some basis $|i\rangle\langle j|$, though it should be noted that perturbation in this representation is somewhat secular in time. Expression (8) will be shown to be a positive-definite form for all microscopically derived noise correlations, thus agreeing with Lindblad's theorem in as much as is required.

The key to accomplishing our stated goal rests in the comparison of perturbative dissipators. We first note that, from its microscopic origins, Eq. (2), the environment correlation function is Hermitian in the sense of

$$\boldsymbol{\alpha}(t,\tau) = \boldsymbol{\alpha}^\dagger(\tau,t)\,, \quad (9)$$

and also positive definite in the sense of

$$\int_0^t d\tau_1\int_0^t d\tau_2\,\mathbf{f}^\dagger(\tau_1)\,\boldsymbol{\alpha}(\tau_1,\tau_2)\,\mathbf{f}(\tau_2) \geq 0\,, \quad (10)$$

for all vector functions $\mathbf{f}(t)$ indexed by the noise. All quantum correlations are at least *nonlocally decoherent* as their algebraic Lindblad dissipator, Eq. (8), is necessarily positive definite. Nonlocal decoherence only implies that there is more net decoherent evolution than recoherent evolution. The more strict property of instantaneously decoherent evolution, $\dot{\boldsymbol{\Xi}}(t) > 0$, can only be generally satisfied by *local decoherence* with delta correlations (Markovian processes) and would always produce a Lindblad master equation. However, some very restricted classes of system-environment interactions, such as the RWA-interaction Hamiltonian [30], can be constrained by their coupling to be locally decoherent. This characterizes the class of systems with non-Markovian dynamics whose master equation is, nevertheless, naturally of Lindblad form.

It is the key point of this work that the environment correlation function itself provides a natural comparison of state dissipation or *decoherence strength*. If two correlation functions are ordered $\boldsymbol{\alpha}_+(t,\tau) > \boldsymbol{\alpha}_-(t,\tau)$ in the sense of positivity (10), then their corresponding second-order Lindblad dissipators are also ordered $\boldsymbol{\Xi}_+(t) > \boldsymbol{\Xi}_-(t)$, and we can, therefore, say that one environment generates more state dissipation than the other, regardless of the system. For instance, the set of univariate Markov processes is totally ordered by the scalar magnitude of the respective delta correlations, e.g. $2\,\delta(t-\tau) > 1\,\delta(t-\tau)$. In general, the set of all quantum correlations is only partially ordered, but nontrivial orderings do exist. We illustrate this principle with a specific problem below.

III. THERMAL CORRELATIONS

A. Individual Reservoirs

Constant coupling to a thermal reservoir will always produce environment correlations which can be expressed in the Fourier domain as

$$\tilde{\boldsymbol{\alpha}}(\omega) = \tilde{\boldsymbol{\gamma}}(\omega)\,[\tilde{\kappa}_T(\omega) - \omega]\,, \quad (11)$$

$$\tilde{\kappa}_T(\omega) \equiv \omega\,\coth\!\left(\frac{\omega}{2T}\right)\,, \quad (12)$$

in terms of the damping kernel $\tilde{\boldsymbol{\gamma}}(\omega)$ (anti-derivative of the dissipation kernel) and fluctuation-dissipation kernel $\tilde{\kappa}_T(\omega)$. This can be derived directly from first principles in Eq. (2), by demanding a coupling-invariant fluctuation-dissipation relation (FDR), or by demanding coupling-invariant detailed balance in the master equation [14].

Note that a Markovian quantum regime (complex white noise) necessarily implies a local damping kernel and high temperature (local FDR kernel). The Markovian regime is reached when the system time scales are much slower than those of the environment, so that we can take the zero-frequency approximation

$$\lim_{\omega\to 0}\tilde{\boldsymbol{\alpha}}(\omega) = \tilde{\boldsymbol{\gamma}}(0)\,2T\,. \quad (13)$$

Markovian processes can, therefore, be ordered in their decoherence strength by the product of their damping and temperature, a result which is well known. If one inquires as to the temperature of an unknown Markovian

process, the FDR kernel (or noise-to-damping ratio) will always reveal this.

In general, nonlocal correlations (e.g. finite temperature) and thus decoherence strengths are not totally ordered. For a fixed temperature thermal correlations can be ordered by damping. On the other hand we have the inequality

$$\tilde{\kappa}_{\rm hot}(\omega) > \tilde{\kappa}_{\rm cold}(\omega) \geq |\omega| \,, \tag{14}$$

and so for fixed damping, correlations can also be ordered by their temperature. Therefore, finite-temperature thermal correlations are partially ordered by damping and temperature.

$$\tilde{\gamma}_{\rm strong}(\omega)\,[\tilde{\kappa}_{\rm hot}(\omega) - \omega] > \tilde{\gamma}_{\rm weak}(\omega)\,[\tilde{\kappa}_{\rm cold}(\omega) - \omega] \,, \tag{15}$$

and so it immediately follows that

$$\tilde{\boldsymbol{\alpha}}^{\rm hot}_{\rm strong}(\omega) > \tilde{\boldsymbol{\alpha}}^{\rm cold}_{\rm weak}(\omega) \,. \tag{16}$$

If one environment has a higher temperature but weaker damping, then the two correlations cannot be ordered – the implication is that different systems would decohere faster or slower for each environment, but not in a manner which can be strictly ordered.

From Eq. (15) we can now compare environments of low temperature and nonlocal damping. For fixed damping, the monotonic ordering of temperature is no surprise. While for fixed temperature, the ordering of damping is more subtle though also not surprising. The damping can be increased by an overall rescaling, say $\tilde{\gamma}(\omega) \to 2\,\tilde{\gamma}(\omega)$, but also by increasing the high-frequency response of the environment, say $\Lambda \to 2\Lambda$ for any monotonic regulator (not necessarily Ohmic) with cutoff Λ.

B. Multiple environments

Here we wish to duplicate the work of Beer & Lutz [7] wherein they compared the decoherence rates of collective environments with different temperatures and Ohmic cutoff frequencies, specifically for linear coupling to an oscillator and with both reservoirs at a relatively high temperature. For multiple environments we can make the same comparison by using the natural measure of decoherence strength from the Lindblad dissipator and environment correlation function. First we note that for coupling to one reservoir, the individual thermal correlations can be expressed via Eq. (11). Next we note that for any monotonic cutoff regulator, with fixed local limit $\tilde{\gamma}(0)$ and variable cutoff Λ, then

$$\tilde{\gamma}_{\rm high}(\omega) \geq \tilde{\gamma}_{\rm low}(\omega) \,, \tag{17}$$

where $\Lambda_{\rm high} > \Lambda_{\rm low}$ (referred to as "fast" and "slow" in Ref. [7]). We can also compare the individual FDR kernels as per Eq. (14). Finally we can use the above relations construct the mathematical inequality

$$[\tilde{\gamma}_{\rm high}(\omega) - \tilde{\gamma}_{\rm low}(\omega)]\{[\tilde{\kappa}_{\rm hot}(\omega) - \omega] - [\tilde{\kappa}_{\rm cold}(\omega) - \omega]\} > 0 \,, \tag{18}$$

which can then be rearranged to show that

$$\tilde{\boldsymbol{\alpha}}^{\rm hot}_{\rm high}(\omega) + \tilde{\boldsymbol{\alpha}}^{\rm cold}_{\rm low}(\omega) > \tilde{\boldsymbol{\alpha}}^{\rm hot}_{\rm low}(\omega) + \tilde{\boldsymbol{\alpha}}^{\rm cold}_{\rm high}(\omega) \,, \tag{19}$$

which is consistent with the results of Ref. [7], when we interpret the left and right-hand sides of Eq. (19) as comparing the decoherence strengths of two different collective (non-equilibrium) environments. Note that as the individual reservoirs are Gaussian and independent, one may simply add their correlations in determining the collective correlation. Our result applies more generally than that of Ref. [7], in terms of coupling and temperature, though we do not calculate a specific decoherence time. Their work relied upon what is essentially the exact FDR kernel, but which has been referred to as an *effective temperature* [31] in the classical regime. We would rather avoid this nomenclature given that such environments will lead to an asymptotic stationary state which is not thermal in general (sufficiently simple systems can reach a thermal state, but the corresponding temperature will be different for different systems).

IV. DISCUSSION

In this work we have motivated a general notion of decoherence strength as that generated by the quantum correlations of the environment, which in turn determine the magnitude of the algebraic Lindblad dissipator and thus state dissipation itself. The ordering of decoherence strengths in this formalism is only partial, though when it occurs it is system and state independent. However, this is not to say that comparison of environment correlations is not as useful when there is no strict ordering. State-dependent decoherence is of particular interest in the search for decoherence-free states and how they emerge in certain classes of environments. If for two environments $\tilde{\boldsymbol{\alpha}}_1(\omega) - \tilde{\boldsymbol{\alpha}}_2(\omega)$ is indefinite, then there *could* be states corresponding to the vectors $\mathbf{f}$ in Eq.(10) which exploit this. As an explicit example, if two atoms are brought close together in a quantum field, then rapidly and slowly decaying states emerge which mirror the behavior of the eigenvalues of the correlation function [32].

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