

Noise kernel for a quantum field in Schwarzschild spacetime under the Gaussian approximation

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A method is given to compute the noise kernel, defined as the symmetrized connected 2-point function of the stress tensor, for the conformally invariant scalar field in the Hartle-Hawking state in Schwarzschild spacetime. We consider the two points to be separated in a timelike or a spacelike direction and use a Gaussian approximation of the same type as that used by Page to compute an approximate form of the stress tensor for this field in Schwarzschild spacetime. Several components of the noise kernel have been explicitly computed and the expression for one of them is displayed.

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I. INTRODUCTION

Studies of the fluctuations in the stress tensors of quantum fields are playing an increasingly important role in investigations of quantum effects in curved spacetimes. In various forms they have provided criteria for and tests of the validity of semiclassical gravity [1–12]. They are relevant for the generation of cosmological perturbations during inflation [13–16] as well as the fluctuation and back-reaction problem in black hole dynamics [17, 18]. They also allow one to go beyond semiclassical gravity [19]. One important theory in which this is done systematically from first principles is stochastic semiclassical gravity [20–22], which takes into account fluctuations of the gravitational field that are induced by the quantum matter fields.

The centerpiece of stochastic semiclassical gravity is the noise kernel. The noise kernel is the symmetrized connected 2-point function of the stress tensor operator for a quantum field in curved spacetime. It plays a role in stochastic gravity [22, 23] similar to the expectation value of the stress tensor in semiclassical gravity [11, 24, 25]. The noise kernel characterizes the fluctuations of the stress tensor (4-point function for free fields) which are accounted for by the stochastic source of the Einstein-Langevin equation [26] and is thus an upgrade of semiclassical gravity, which includes only the expectation value of the stress tensor (2-point function for free fields). The construction of stochastic gravity was motivated by finding ways to include the fluctuations of the quantum fields [1, 2] and the dissipation of the gravitational field [27] as well as the induced metric fluctuations, properly [21, 26] in a covariant manner. For both black hole and early universe physics as well as in the viscosity-entropy bound via the fluid/gravity duality [28, 29] there is a clear need for a better understanding of the properties of the stress tensor correlators of quantum fields in curved spacetime.

Need for higher-order correlation functions of quantum fields

As pointed out by Hu and Roura [17] using the black-hole quantum back-reaction and fluctuation problems as examples, a consistent study of the horizon fluctuations requires a detailed knowledge of the stress tensor 2-point function. That is because, in contrast with the averaged energy flux, the existence of a direct correlation assumed in earlier studies between the fluctuations of the energy flux crossing the horizon and those far from it, is sim-

ply invalid. For cosmological structure formation problems, stochastic gravity gives at tree level a result equivalent [14] to the standard treatment of linear cosmological perturbations [30], but it is for one-loop corrections that the calculation via the noise kernel can provide particularly useful insight [15, 31]. In fact, the correlation functions in stochastic gravity, calculated as averages over stochastic distributions, have been shown [10, 32] to be equivalent, to leading order in the number of matter fields, to the quantum correlation functions for the metric perturbations around the semiclassical background that would follow from a purely quantum field theoretical treatment. Of current interest is the non-Gaussianity in the CMB anisotropies [33], where checking the trispectrum measurements requires high-precision information on the 4-point function of the inflaton field, which receives a dominant contribution already at tree level due to nonlinear interactions [34]. Compared to the last decade, progress of observational cosmology alone has advanced the need for the calculations of higher-order correlation functions such as the noise kernel beyond academic interest. (For a recent emphasis on the inadequacy of semiclassical gravity see, e.g., [35].) Beyond gravitational issues proper, one topic which has generated much recent interest is the proposal of a universal bound on the viscosity-to-entropy density ratio for strongly interacting gauge fluids [28]. The viscosity function is also directly connected with the 2-point function of the stress tensor of the quantum matter field (albeit in the fluid/gravity duality it is for strongly coupled fields). More precisely, it is linked to the dissipation kernel, which corresponds to the expectation value of the commutator of the stress tensor and in thermal equilibrium is tethered with the noise kernel through the fluctuation-dissipation relation (see, e.g., [21, 36–39]). For example, the calculation of the polarization tensor for a finite-temperature quantum field far away from a black hole (so the background metric is approximated by the Minkowski metric) and the derivation of the linear response theory from the Einstein-Langevin equation were carried out by Campos and Hu [40] (see also Ref. [41]).

Physical meaning of metric fluctuations from integrating the noise kernel with smearing fields

Fluctuations, which play a central role in the above discussion, like noise a decade or two ago, are often used but little understood. The situation with metric fluctuations, even disregarding their linkage to spacetime foam at the Planck scale, is even worse. This is

partly due to the highly nontrivial task of constructing appropriate fully gauge-invariant (diffeomorphism-invariant) observables in quantum gravity even when focusing on its infrared regime (by treating it as a low-energy effective field theory [42]). One needs to envisage suitable ways of probing metric fluctuations and extracting physically meaningful information from formal objects like the noise kernel and related mathematical constructions. This entails developing satisfactory operational definitions [43] of phenomena like quantum light-cone [44] and horizon fluctuations [4, 45]. An inquiry of what metric fluctuations mean beyond the classical notion where spacetime is a sharply defined entity was made in Ref. [17] in the context of black-hole horizon fluctuations. Their analysis shows that when trying to localize the horizon as a three-dimensional hypersurface, as in the classical case, the amplitude for the fluctuations can become infinite. The quantum event horizon should possess a finite effective width due to its quantum fluctuations (*intrinsic* or *induced* by the fluctuations of the matter fields). Such a picture is borne out by integrating the noise kernel with smearing functions over different kinds of spacetime regions around the horizon. In order to characterize its width one must find a sensible way of probing the metric fluctuations near the horizon and extracting physically well-defined and unambiguous information, such as their effect on the Hawking radiation emitted by the black hole. One way to probe the metric fluctuations is to analyze their effects on the 2-point quantum correlation functions of a test field. The 2-point functions characterize the response of a particle detector for that field and can be used to obtain the expectation value and the fluctuations of the stress tensor of the test field.

Noise kernel with two separate points is needed for analyzing fluctuation phenomena

The noise kernel is finite for spacelike and timelike separations of the points, but it diverges for null separations and in all cases in the coincidence limit. Despite this fact, a careful analysis reveals that the noise kernel can be naturally regarded as a well-defined distribution (generalized function) which leads to finite results when integrated with appropriate test functions [17, 41]. In particular this means that one should obtain a finite result when integrating the noise kernel with smooth smearing functions without the need for any subtraction that removes the divergences in the coincidence limit. Calculations involving such an integration of the noise kernel with suitable smearing functions have been performed,

for instance, to study the energy density fluctuations in Minkowski and Casimir spacetimes [7, 46]. Similarly, the computation of correlation functions for the metric perturbations in stochastic gravity involves integrating the noise kernel with the retarded propagator associated with the Einstein-Langevin equation. (Because the retarded propagator is not a smooth function at the initial time, one gets divergent boundary terms when specifying initial conditions at a finite initial time; these can be interpreted as a consequence of having considered an unphysical factorized initial state [10, 47] and could be avoided by dealing with a suitably correlated initial state for the metric perturbations and the matter fields.) In [48, 49] attention was paid to the regularization of the noise kernel at the coincident limit by subtracting state-dependent terms which rendered such a limit finite. Motivated by the discussion above, we carry out here a different line of investigation and focus our attention on the noise kernel for separate points and without performing any subtraction. (In order to get the right prescription at the coincidence limit when integrating with test functions, one can follow the procedure laid out in Appendices B and C of Ref. [17].)

A sobering conclusion from the work of Hu and Roura in Ref. [17] is that a detailed calculation of the noise kernel is unavoidable (it should be clear from the beginning that a mean field cannot possibly provide information about fluctuations, except for very special theories where this information is subsumed.). For studying the effect of Hawking radiation emitted by a black hole on its evolution and the metric fluctuations driven by the quantum field (the “back-reaction and fluctuation” problem [50]) the need for the noise kernel of a quantum field near a black hole horizon has been pronounced earlier. For example, Sinha, Raval and Hu [51] have outlined a program for such a study, which is the stochastic gravity upgrade (Einstein-Langevin equation) of those carried out for the mean field in semiclassical gravity (semiclassical Einstein equation) by York [52, 53] and by York and his collaborators [54]. The results of our paper will serve that purpose and more. The challenge that we face here is to come up with an economic way of computing the noise kernel, which we describe in the following subsection along with our findings.

Our approach and findings

An expression for the noise kernel for free fields in a general curved spacetime in terms of the corresponding Wightman function was obtained a decade ago [21, 48]. Since then

this general result has been employed to obtain the noise kernel in Minkowski [41], de Sitter [15, 55, 56] and anti-de Sitter [56, 57] spacetimes, as well as in Schwarzschild spacetime in the coincidence limit [49]. This paper reports on work which continues this effort. Specifically, we calculate the noise kernel for the conformally invariant scalar field in Schwarzschild spacetime when the points are separated by a small geodesic distance and the field is in the Hartle-Hawking state. To do so we use the same method that Page [58] used to compute the stress tensor for this field in Schwarzschild spacetime. What he did was to work in an optical Schwarzschild spacetime, which is ultra-static and conformal to Schwarzschild. There he used the Gaussian approximation to compute the Euclidean Green function when the field is in a thermal state at the black hole temperature. As he points out, this approximation corresponds to taking the first term in the DeWitt-Schwinger expansion for the Green function. In most spacetimes that would not be sufficient to generate an approximation to the stress tensor which could be renormalized correctly. However, in the optical Schwarzschild spacetime (and for any other ultra-static metric conformal to an Einstein metric in general) the second and third terms in the DeWitt-Schwinger expansion vanish identically, so that the approximation is much better than it would usually be. Page then conformally transformed his result to Schwarzschild spacetime by finding the nontrivial conformal transformation which works for the renormalized stress tensor, $\langle \hat{T}_{ab}(x) \rangle_{\text{ren}}$, of the conformally invariant scalar field.

Here we use the approximation that Page found for the Euclidean Green function to compute the noise kernel in the optical Schwarzschild spacetime when the points are separated (and non-null related). Because we do not attempt to take the limit in which the points come together (or are null related) the result is finite without the need for any subtraction. As we show, this makes the transformation of the noise kernel for this field to Schwarzschild spacetime trivial.

Specifically, we find an explicit expression for the Euclidean Green function in the ultra-static metric and then analytically continue it to the Lorentzian sector. We show that in the ultra-static spacetime with which we are working, the Wightman function when the points are non-null related is given to $O[(x - x')^2]$ by the real part of the Gaussian approximation to the Euclidean Green function when it is analytically continued to the Lorentzian sector.

In our approximation the Wightman function is made up of one part which is necessarily coordinate dependent and one part which can be expressed in a covariant form. It is the non-

vanishing temperature of the state restricted to a static region of Schwarzschild spacetime which prevents us from writing it in a completely covariant form. We describe a way in which the partially covariant form can be used to formally compute an approximation for the noise kernel. Then we turn to a more direct calculation of the Wightman function in the Gaussian approximation which is completely coordinate dependent. This has the advantage of being an easier method to use. By substituting it into the equation satisfied by the Wightman function we show that it is valid to order $O[(x - x')^2]$ as expected. This implies that expression for the noise kernel obtained with it is valid through $O[(x - x')^{-4}]$, while the leading terms in the noise kernel are $O[(x - x')^{-8}]$.

We have used our direct calculation of the Wightman function to compute explicitly some of the components of the noise kernel in the optical Schwarzschild spacetime. Due to the length of the expressions for most of the nonzero components of the noise kernel, we explicitly display only one component in this paper.

In Sec. II we review the noise kernel for a conformally invariant scalar field in a general spacetime and discuss its properties including the transformation of the noise kernel from the optical spacetime to Schwarzschild spacetime. In Sec. III we review the relationship between the Wightman and Euclidean Green functions, the relevant parts of the formalism for the DeWitt-Schwinger expansion [59], and its use in the Gaussian approximation for the Euclidean Green function in the optical Schwarzschild spacetime which Page derived [58]. We show that the resulting expression for the Wightman function is $O[(x - x')^2]$. This expression is in a partially covariant form. In Sec. IV a method is given which allows for the computation of the noise kernel using the expression of the Wightman function derived in the previous section. Then a more direct method of obtaining the Wightman function in the Gaussian approximation which is not covariant at all is given. The computation of the noise kernel using this Wightman function is described. One component of the noise kernel in Schwarzschild spacetime is explicitly displayed. Sec. V contains a summary and discussion of our main results. In the Appendix two proofs are given for the way in which the noise kernel for the conformally invariant scalar field changes under conformal transformations. Throughout we use units such that $\hbar = c = G = k_B = 1$ and the conventions of Misner, Thorne, and Wheeler [60].

II. NOISE KERNEL FOR THE CONFORMALLY INVARIANT SCALAR FIELDS

In this section we review the general properties of the noise kernel for the conformally invariant scalar field in an arbitrary spacetime. The definition of the noise kernel for any quantized matter field is

$$N_{abc'd'}(x, x') = \frac{1}{2} \langle \{ \hat{t}_{ab}(x), \hat{t}_{c'd'}(x') \} \rangle, \quad (2.1)$$

with

$$\hat{t}_{ab}(x) \equiv \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle. \quad (2.2)$$

Here $\langle \dots \rangle$ denotes the quantum expectation value with respect to a normalized state of the matter field [more generally, $\langle \dots \rangle = \text{Tr}(\hat{\rho} \dots)$ for a mixed state] and \hat{T}_{ab} is the stress tensor operator of the field.

The classical stress tensor for the conformally invariant scalar field is

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi + \xi (g_{ab} \square - \nabla_a \nabla_b + G_{ab}) \phi^2, \quad (2.3)$$

with $\xi = (D - 2)/4(D - 1)$, which becomes $\xi = 1/6$ in $D = 4$ spacetime dimensions. Note that since the stress tensor is symmetric, the noise kernel is also symmetric under exchange of the indices a and b , or c' and d' . To compute the noise kernel one promotes the field $\phi(x)$ in Eq. (2.3) to an operator in the Heisenberg picture while still treating g_{ab} as a classical background metric. The result is then substituted into Eq. (2.1).

Given a Gaussian state of the quantum matter field, one can express the noise kernel in terms of products of two Wightman functions by applying Wick's theorem. (Gaussian states include for instance the usual vacua, thermal and coherent states. In general most states will not be Gaussian and Wick's theorem will not apply: it will not be possible to write the 4-point function of the field in terms of 2-point functions and expectation values of the field. That is for example the case for all eigenstates of the particle number operators other than the corresponding vacuum.) The Wightman function is defined as

$$G^+(x, x') = \langle \phi(x) \phi(x') \rangle. \quad (2.4)$$

The result for a scalar field with arbitrary mass and curvature coupling in a general spacetime has been obtained in Refs. [21, 48]. For the conformally invariant scalar field in a general

spacetime the noise kernel is [48]

$$N_{abc'd'} = \bar{N}_{abc'd'} + g_{ab}\bar{N}_{c'd'} + g_{c'd'}\bar{N}'_{ab} + g_{ab}g_{c'd'}\bar{N} \quad (2.5)$$

with¹

$$\begin{aligned} 72\bar{N}_{abc'd'} = & 4 (G_{;c'b} G_{;d'a} + G_{;c'a} G_{;d'b}) + G_{;c'd'} G_{;ab} + G G_{;abc'd'} \\ & - 2 (G_{;b} G_{;c'ad'} + G_{;a} G_{;c'bd'} + G_{;d'} G_{;abc'} + G_{;c'} G_{;abd'}) \\ & + 2 (G_{;a} G_{;b} R_{c'd'} + G_{;c'} G_{;d'} R_{ab}) \\ & - (G_{;ab} R_{c'd'} + G_{;c'd'} R_{ab}) G + \frac{1}{2} R_{c'd'} R_{ab} G^2 \end{aligned} \quad (2.6a)$$

$$\begin{aligned} 288\bar{N}'_{ab} = & 8 \left(-G_{;p'b} G_{;p'a} + G_{;b} G_{;p'a}{}^{p'} + G_{;a} G_{;p'b}{}^{p'} \right) \\ & 4 \left(G_{;p'} G_{;abp'} - G_{;p'}{}^{p'} G_{;ab} - G G_{;abp'}{}^{p'} \right) \\ & - 2 R' (2 G_{;a} G_{;b} - G G_{;ab}) \\ & - 2 \left(G_{;p'} G_{;p'}{}^{p'} - 2 G G_{;p'}{}^{p'} \right) R_{ab} - R' R_{ab} G^2 \end{aligned} \quad (2.6b)$$

$$\begin{aligned} 288\bar{N} = & 2 G_{;p'q} G_{;p'q} + 4 \left(G_{;p'}{}^{p'} G_{;q}{}^q + G G_{;p}{}^p{}^{q'} \right) \\ & - 4 \left(G_{;p} G_{;q'}{}^{pq'} + G_{;p'} G_{;q}{}^q{}^{p'} \right) \\ & + R G_{;p'} G_{;p'}{}^{p'} + R' G_{;p} G_{;p}{}^p \\ & - 2 \left(R G_{;p'}{}^{p'} + R' G_{;p}{}^p \right) G + \frac{1}{2} R R' G^2 . \end{aligned} \quad (2.6c)$$

Note that the superscript $+$ on G^+ has been omitted in the above equations for notational simplicity. Primes on indices denote tensor indices at the point x' and unprimed ones denote indices at the point x . Also R_{ab} is the Ricci tensor evaluated at the point x , $R_{c'd'}$ is the Ricci tensor evaluated at the point x' , R is the scalar curvature evaluated at the point x , and R' is the scalar curvature evaluated at the point x' .

The definition (2.5) of the noise kernel immediately implies that it is symmetric under interchange of the two spacetime points and the corresponding pairs of indices so that

$$N_{abc'd'}(x, x') = N_{c'd'ab}(x', x) . \quad (2.7)$$

There are other important properties which the noise kernel has as well. These have been proven in Refs. [20, 21], so we just state them here. The first property, which is clear

¹ Notice that these equations have two slight but crucial differences with the equations of Ref. [48]. The sign of the last term of the equation for $N_{abc'd'}$ and also the sign of the term $G G_{;abp'}{}^{p'}$ have been corrected.

from (2.5), is that the following conservation laws must hold:

$$\nabla^a N_{abc'd'} = \nabla^b N_{abc'd'} = \nabla^{c'} N_{abc'd'} = \nabla^{d'} N_{abc'd'} = 0. \quad (2.8)$$

The second property which must be satisfied because the field is conformally invariant is that the partial traces must vanish, that is

$$N^a{}_{ac'd'} = N_{ab}{}^{c'}{}_{c'} = 0. \quad (2.9)$$

A third important property is that the noise kernel is positive semidefinite, namely

$$\int d^4x \sqrt{-g(x)} \int d^4x' \sqrt{-g(x')} f^{ab}(x) N_{abc'd'}(x, x') f^{c'd'}(x') \geq 0, \quad (2.10)$$

for any real tensor field $f^{ab}(x)$.

Finally the noise kernel for the conformally invariant field when the points are separated has a simple scaling behavior under conformal transformations. In the Appendix two proofs are given which show that under a conformal transformation between two conformally related D -dimensional spacetimes with metrics of the form $\tilde{g}_{ab} = \Omega^2(x) g_{ab}$, the noise kernel transforms as:

$$\tilde{N}_{abc'd'}(x, x') = \Omega^{2-D}(x) \Omega^{2-D}(x') N_{abc'd'}(x, x'). \quad (2.11)$$

III. GAUSSIAN APPROXIMATION IN THE OPTICAL SCHWARZSCHILD SPACETIME

As discussed in the introduction we want to compute the noise kernel in a background Schwarzschild spacetime for the conformally invariant scalar field when the points are separated. For an arbitrary separation it would be necessary to do this numerically. However, if the separation is small then it is possible to use approximation methods to compute the Wightman function analytically and from that the noise kernel. For a conformally invariant field a significant simplification is possible because the Green function and the resulting noise kernel can be computed in the optical Schwarzschild spacetime, which is conformal to Schwarzschild spacetime, and then conformally transformed to Schwarzschild spacetime. A similar calculation was done by Page [58] for the stress tensor for the conformally invariant scalar field. In his calculation first the Euclidean Green function for the field in a thermal state was computed using a Gaussian approximation. Then the stress tensor was computed

and conformally transformed to Schwarzschild spacetime. We shall use Page's approximation for the Euclidean Green function to obtain an approximation for the Wightman Green function and then compute the noise kernel using that approximation.

A. Gaussian approximation for the Wightman Green function

In order to use Page's approximation we must relate the Euclidean Green function in a static spacetime to the Wightman function. To do so we begin by noting that the Wightman function can be written in terms of two other Green functions [61], the Hadamard function $G^{(1)}(x, x')$ and the Pauli-Jordan function $G(x, x')$ such that

$$G^+(x, x') = \frac{1}{2} [G^{(1)}(x, x') + iG(x, x')] \quad (3.1a)$$

$$G^{(1)}(x, x') \equiv \langle \{\phi(x), \phi(x')\} \rangle \quad (3.1b)$$

$$iG(x, x') \equiv \langle [\phi(x), \phi(x')] \rangle . \quad (3.1c)$$

As discussed in the Introduction, we restrict our attention in this paper to spacelike and timelike separations of the points. In general $G(x, x') = 0$ for spacelike separations of the points. In the optical Schwarzschild spacetime, $G(x, x') = O[(x - x')^4]$ for timelike separations of the points. To see this consider the general form of the Hadamard expansion for $G(x, x')$ which is [62]

$$G(x, x') = -\frac{u(x, x')}{4\pi} \delta(-\sigma) + \frac{v(x, x')}{8\pi} \theta(-\sigma) , \quad (3.2)$$

with $\sigma(x, x')$ defined to be one-half the square of the proper distance along the shortest geodesic connecting the two points. In [63] it was shown that in Schwarzschild spacetime $v(x, x') = O[(x - x')^4]$. Since the Green function in the optical spacetime can be obtained from that in Schwarzschild spacetime by a simple conformal transformation, the same must be true of the quantity $v(x, x')$. Thus so long as we work only to $O[(x - x')^2]$ and restrict our attention to points which are either spacelike or timelike separated, then in the optical Schwarzschild spacetime

$$G^+(x, x') = \frac{1}{2} G^{(1)}(x, x') + O[(x - x')^4] . \quad (3.3)$$

The Hadamard Green function can be computed using the Euclidean Green function in the following way. First define the Euclidean time as

$$\tau \equiv it . \quad (3.4)$$

Then the metric in a static spacetime takes the form

$$ds^2 = g_{\tau\tau}(\vec{x})d\tau^2 + g_{ij}(\vec{x})dx^i dx^j . \quad (3.5)$$

The Euclidean Green function obeys the equation

$$(\square - \frac{1}{6}R)G_E(x, x') = -\frac{\delta(x - x')}{\sqrt{g(x)}} . \quad (3.6)$$

Because the spacetime is static G_E will be a function of $(\Delta\tau)^2 = (\tau - \tau')^2$. It is possible to obtain the Feynman Green function $G_F(x, x')$ by making the transformation [64]

$$(\Delta\tau)^2 \rightarrow -(\Delta t)^2 + i\epsilon . \quad (3.7)$$

under which

$$G_E(x, x') \rightarrow iG_F(x, x') . \quad (3.8)$$

Using [61]

$$G_F(x, x') = -\frac{1}{2}iG^{(1)}(x, x') + \frac{1}{2}[\theta(t - t') - \theta(t' - t)]G(x, x') . \quad (3.9)$$

One then finds that Eq. (3.3) becomes

$$G^+(x, x') = -\text{Im} G_F(x, x') + O[(x - x')^4] . \quad (3.10)$$

As mentioned above, Page made use of the DeWitt-Schwinger expansion to obtain his approximation for the Euclidean Green function. Before displaying his approximation it is useful to discuss two quantities which appear in that expansion. For a more complete discussion see Ref. [59]. The fundamental quantity out of which everything is built is Synge's world function $\sigma(x, x')$ which is defined to be one-half the square of the proper distance between the two points x and x' along the shortest geodesic connecting them. It satisfies the relationship

$$\sigma(x, x') = \frac{1}{2}g_{ab}(x) \sigma^{;a}(x, x') \sigma^{;b}(x, x') . \quad (3.11)$$

It is traditional to use the notation

$$\sigma^a \equiv \sigma^{;a} . \quad (3.12)$$

As shown in [59] it is possible to expand biscalars, bivectors, and bitensors in powers of σ^a in an arbitrary spacetime. One chooses one of the points, usually x , and evaluates the coefficients in the expansion at that point. For example

$$\sigma_{;ab}(x, x') = g_{ab}(x) - \frac{1}{3}R_{abcd}(x) \sigma^c(x, x') \sigma^d(x, x') + \dots . \quad (3.13)$$

Examination of this expansion shows that to zeroth order in σ^a

$$\sigma_{;abc} = 0 . \quad (3.14)$$

The second quantity we shall need is

$$U(x, x') \equiv \Delta^{1/2}(x, x') \quad (3.15a)$$

$$\Delta(x, x') \equiv -\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}} \det(-\sigma_{;ab'}) . \quad (3.15b)$$

Note that covariant derivatives at the point x' commute with covariant derivatives at the point x . Two important properties of $U(x, x')$ are

$$U(x, x) = 1 \quad (3.16a)$$

$$(\ln U)_{;a} \sigma^a = 2 - \frac{1}{2} \square \sigma . \quad (3.16b)$$

One can also expand U in powers of σ^a with the result that [59]

$$\begin{aligned} U(x, x') = & 1 + \frac{1}{12} R_{ab} \sigma^a \sigma^b - \frac{1}{24} R_{ab;c} \sigma^a \sigma^b \sigma^c, \\ & + \frac{1}{1440} (18 R_{ab;cd} + 5 R_{ab} R_{cd} + 4 R_{paqb} R_c{}^q{}_d{}^p) \sigma^a \sigma^b \sigma^c \sigma^d. \end{aligned} \quad (3.17)$$

The above definitions, properties, and expansions apply to arbitrary spacetimes. Given any static metric, one can always transform it to an ultra-static one, called the optical metric, by a conformal transformation. This kind of metric is of the form

$$ds^2 = -dt^2 + g_{ij}(\vec{x}) dx^i dx^j \quad (3.18)$$

with the metric functions g_{ij} independent of the time t . In this case Synge's world function is

$$\sigma(x, x') = \frac{1}{2} (-(t - t')^2 + \mathbf{r}^2) \quad (3.19)$$

with

$$\mathbf{r} \equiv \sqrt{2} {}^{(3)}\sigma . \quad (3.20)$$

The quantity ${}^{(3)}\sigma$ is the world function for the three dimensional spatial part of the metric and thus depends only on the spatial coordinates. Note that we use \mathbf{r} (with bold roman

font) to denote the quantity in Eq.(3.20) while r (with normal italic font) denotes the radial coordinate. Some useful properties of \mathbf{r} are

$$\nabla_i \mathbf{r} = \frac{{}^3\sigma_i}{\mathbf{r}} \quad (3.21a)$$

$$\nabla^2 \mathbf{r} = \frac{{}^3\sigma_i^i - 1}{\mathbf{r}} \quad (3.21b)$$

$$\nabla_i \mathbf{r} \nabla^i \mathbf{r} = \frac{{}^3\sigma_i {}^3\sigma^i}{2 {}^3\sigma} = 1 \quad (3.21c)$$

$$2 (\Delta^{1/2})_{,i} \nabla^i \mathbf{r} = \left(\frac{3}{\mathbf{r}} - \nabla^2 \mathbf{r} \right) \Delta^{1/2} \quad (3.21d)$$

where $\nabla^2 = \nabla^i \nabla_i$. Note that from Eqs. (3.15b) and (3.19) one can easily see that for an ultra-static spacetime the Van-Vleck determinant ${}^{(3)}\Delta$ for the spatial metric g_{ij} coincides with the Van-Vleck determinant Δ for the full spacetime. (The advantage of using ${}^{(3)}\Delta$ rather than Δ is that, although noncovariant, it is valid for arbitrary time separations, and one only needs to expand in powers of \mathbf{r} .)

The optical Schwarzschild metric

$$ds^2 = -dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)^2} dr^2 + \frac{r^2}{1 - \frac{2M}{r}} (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (3.22)$$

is of the form (3.18) and is conformally related to normal Schwarzschild metric with conformal factor

$$\Omega^2 = \left(1 - \frac{2M}{r}\right) . \quad (3.23)$$

For this metric Page [58] used a Gaussian approximation to obtain an expression for the Euclidean Green function in a thermal state at the temperature

$$T = \frac{\kappa}{2\pi} . \quad (3.24)$$

The expression is valid for any temperature but if

$$\kappa = \frac{1}{4M} , \quad (3.25)$$

then the temperature is that of the black hole in the Schwarzschild spacetime which is conformal to the optical metric (3.22). In this case the field is in the Hartle-Hawking state, which is regular on the horizon. For an arbitrary temperature Page found that [58]

$$G_E(\Delta\tau, \vec{x}, \vec{x}') = \frac{\kappa \sinh(\kappa \mathbf{r})}{8\pi^2 \mathbf{r} [\cosh(\kappa \mathbf{r}) - \cos(\kappa \Delta\tau)]} U(\Delta\tau, \vec{x}, \vec{x}') . \quad (3.26)$$

Analytically continuing to the Lorentzian sector using the prescriptions (3.7) and (3.8), and substituting the result into Eq. (3.10) gives

$$G^+(\Delta t, \vec{x}, \vec{x}') = \frac{\kappa \sinh \kappa \mathbf{r}}{8\pi^2 \mathbf{r} [\cosh(\kappa \mathbf{r}) - \cosh(\kappa \Delta t)]} U(\Delta t, \vec{x}, \vec{x}') . \quad (3.27)$$

To determine the accuracy of this approximation we can substitute the above expression into the equation satisfied by G^+ which is

$$\square G^+(x, x') - \frac{R}{6} G^+(x, x') = 0 . \quad (3.28)$$

The accuracy of the Gaussian approximation in the optical Schwarzschild metric will be determined by the lowest order in $(x - x')$ at which Eq. (3.28) is not satisfied.

Applying the differential operator for the metric (3.22) and using (3.21c) and (3.21d) one finds after some calculation that

$$\left(\square - \frac{1}{6} R \right) G^+(x, x') = \frac{\kappa \sinh(\kappa \mathbf{r})}{\mathbf{r} [\cosh(\kappa \mathbf{r}) - \cosh(\kappa \Delta t)]} \left(\square - \frac{1}{6} R \right) U(x, x') \quad (3.29)$$

If Eq. (3.17) is substituted into Eq. (3.29) and Eqs. (3.13) and (3.14) are used then one finds

$$\left(\square - \frac{R}{6} \right) U(x, x') = Q_0 + Q_p \sigma^p + Q_{pq} \sigma^p \sigma^q + \dots \quad (3.30)$$

with

$$Q_0 = 0 \quad (3.31a)$$

$$Q_a \sigma^a = \sigma^a G_a{}^b{}_{;b} = 0 \quad (3.31b)$$

$$Q_{ab} \sigma^a \sigma^b = \frac{1}{360} (9R_{;ab} + 9R_{ab;c}{}^c - 24R_{ac; b}{}^c - 12R_{ac}R_b{}^c + 6R^{cd}R_{cadb} + 4R_{acde}R_b{}^{edc} + 4R_{acde}R_b{}^{cde}) \sigma^a \sigma^b . \quad (3.31c)$$

Here G_{ab} is the Einstein tensor. For the optical Schwarzschild metric (3.22), $Q_{ab} \sigma^a \sigma^b = 0$. Thus Eq. (3.30) is zero to $O[(x - x')^2]$. Since \square is a second order derivative operator, this means that the Gaussian approximation for $G^+(x, x')$ is accurate up to and including $O[(x - x')^2]$. Note that to leading order $G^+(x, x') \sim (x - x')^{-2}$.

It is important to emphasize that the order of accuracy obtained here is for the Schwarzschild optical metric (3.22). Because the Gaussian approximation is equivalent to the lowest order term in the DeWitt-Schwinger expansion, it is only guaranteed to be accurate to leading order in $x - x'$ in a general spacetime.

B. Order of validity of the noise kernel

In Sec. II an expression for the noise kernel is given in terms of covariant derivatives of the Wightman function. In each term there is a product of Wightman functions and varying numbers of covariant derivatives. The accuracy of the Gaussian approximation for the Wightman function can be used to estimate the order of accuracy of the noise kernel. First recall that the leading order of the Wightman function goes like $(x - x')^{-2}$. Since there is a maximum of four derivatives acting on a product of Wightman functions one therefore expects that at leading order the noise kernel will go like $(x - x')^{-8}$. Since the Gaussian approximation to the Wightman function in the optical Schwarzschild spacetime is accurate through terms of order $(x - x')^2$, it is clear from Eq. (2.6) that our expression for the noise kernel should be accurate up to and including terms of order $(x - x')^{-4}$.

IV. COMPUTATION OF THE NOISE KERNEL

In this section we give two methods which can be used to compute the Noise Kernel for the conformally invariant scalar field in Schwarzschild spacetime using a Gaussian approximation. Using the second method, we have computed several components of the noise kernel and one of them is displayed below.

A. Partially Covariant Expansion

To calculate the noise kernel we begin with the Gaussian approximation for the Wightman function (3.27). This expression is actually valid in any spacetime with metric of the form (3.18), although it will not in general be as accurate as it is in the optical Schwarzschild spacetime with metric (3.22). Examination shows that this expression is of the form

$$G^+(x, x') = P(\mathbf{r}, \Delta t) U(x, x'). \quad (4.1)$$

The first factor is of course noncovariant, but a covariant expansion for $U(x, x')$ has been given in Eq. (3.17). Thus it is possible to compute a partially covariant expansion for the noise kernel that is valid in any spacetime with metric of the form (3.18). One simply substitutes Eq. (3.17) into Eq. (3.27) and then substitutes the result into Eqs. (2.6).

To go further one must choose a specific metric, expand σ and its derivatives in powers of $x - x'$ and thus obtain an expansion for the noise kernel in powers of $x - x'$. Such an expansion can be obtained using the method outlined in Ref. [65] and also in Appendix B of Ref. [66].

B. Direct Calculation

The method outlined in the previous subsection would work for any spacetime with a metric of the form (3.18). However, for a spacetime such as the optical Schwarzschild spacetime which is also spherically symmetric there is an alternative approach which allows one to compute expansions for $\sigma(x, x')$ and $U(x, x')$ more directly. This is the method which we have used to compute explicitly some components of the noise kernel.

Consider first the quantity $\sigma(x, x')$ which is one-half the square of the proper distance between x and x' along the shortest geodesic that connects them. Here we assume that the two points are very close together. It has already been shown in Eq. (3.19) that in a spacetime with metric (3.18) σ can be written as a sum of one term proportional to $(t - t')^2$ and another with no time dependence. Further since the metric (3.22) is also spherically symmetric, σ can only depend on the angular quantity

$$\cos(\gamma) \equiv \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'), \quad (4.2)$$

where γ is the angle between \vec{x} and \vec{x}' . It turns out to be convenient to write σ in terms of the quantity

$$\eta \equiv \cos \gamma - 1. \quad (4.3)$$

Then for points that are sufficiently close together one can use the expansion

$$\sigma(x, x') = \sum_{i,j,k} s_{ijk}(r) (t - t')^{2i} \eta^j (r - r')^k, \quad (4.4)$$

with the sum over i being from 0 to 1 and the other sums starting at $j = 0$ and $k = 0$ respectively.

For the metric (3.22), Eq. (3.11) has the explicit form

$$\sigma = \frac{1}{2} \left[- \left(\frac{\partial \sigma}{\partial t} \right)^2 + \left(1 - \frac{2M}{r} \right)^2 \left(\frac{\partial \sigma}{\partial r} \right)^2 - \frac{1}{r^2} \left(1 - \frac{2M}{r} \right) \left(\frac{\partial \sigma}{\partial \eta} \right)^2 (2\eta + \eta^2) \right]. \quad (4.5)$$

Substituting the expansion (4.4) into Eq. (4.5) and equating powers of $(x^a - x'^a)$ one finds that

$$\begin{aligned} \sigma(x, x') = & -\frac{(\Delta t)^2}{2} + \frac{(\Delta r)^2}{2f^2} - \frac{r^2\eta}{f} + (\Delta r)^3 \left(\frac{1}{2rf^3} - \frac{1}{2rf^2} \right) \\ & + \eta\Delta r \left(\frac{3r}{2f} - \frac{r}{2f^2} \right) + O[(x - x')^4]. \end{aligned} \quad (4.6)$$

Similarly one can write

$$\ln U(x, x') = \sum_{j,k} u_{jk}(r) \eta^j (r - r')^k. \quad (4.7)$$

It can be seen from Eqs. (3.19) and (3.15) that there is no time dependence in U in a spacetime with metric (3.22). If Eq. (4.7) is substituted into Eq. (3.16b), Eqs. (3.16a) and (4.4) are used, and the result is expanded in powers of $x - x'$ then one finds that

$$U(x, x') = 1 + \frac{(\Delta r)^2}{8r^2} \left(1 - \frac{2}{3f} - \frac{1}{3f^2} \right) + \eta \left(\frac{f}{4} - \frac{1}{3} + \frac{1}{12f} \right) + O[(x - x')^3]. \quad (4.8)$$

To compute the noise kernel we need an expansion for the Wightman function. The general expression (3.26) for $G^+(x, x')$ in the Gaussian approximation can be expanded as [49]

$$G^+(x, x') = \frac{1}{8\pi^2} \left[\frac{1}{\sigma} + \frac{\kappa^2}{6} - \frac{\kappa^4}{180} (2(\Delta t)^2 + \sigma) + O[(x - x')^4] \right] U(x, x') \quad (4.9)$$

with $\Delta r = r - r'$ and

$$f \equiv 1 - \frac{2M}{r}. \quad (4.10)$$

Since to lowest order $U(x, x') = 1$, terms in the above expansion have been kept through $O[(x - x')^2]$ which is consistent with the order to which the Gaussian approximation was shown to be valid in Sec. III. Next one must substitute the expansions (4.6) for σ and (4.8) for U into Eq. (4.9). To obtain a final expression that is accurate up to and including $O[(x - x')^2]$, it is necessary to have the expansion for U contain terms through $O[(x - x')^4]$. To obtain these it is in turn necessary to have the expansion for σ contain terms through $O[(x - x')^6]$. The result for G^+ can be substituted into the expressions (2.6) for the noise kernel and the derivatives can be computed. As discussed in Sec. III B one should keep terms through $O[(x - x')^{-4}]$ since this is the highest order for which the Gaussian approximation for the noise kernel is valid.

We have computed several components of the noise kernel through $O[(x - x')^{-4}]$. The resulting expressions are too long to display in full here for an arbitrary separation of the points. Because the Gaussian approximation for $G^+(x, x')$ has terms proportional to κ^0 , κ^2 , and κ^4 , the noise kernel in the Gaussian approximation will have terms proportional to these factors of κ as well. To give the reader some sense as to what the noise kernel looks like when the points are separated in an arbitrary direction we give the leading order terms in $(x - x')$ which are proportional to each factor of κ for one component of the noise kernel. For an arbitrary separation of the points it is useful to write the noise kernel in terms of the quantity

$$\epsilon^2 \equiv -(\Delta t)^2 + \frac{(\Delta r)^2}{f^2} - \frac{r^2 \eta}{f} . \quad (4.11)$$

Then we find

$$\begin{aligned} N_{tt't'v} = & \frac{1}{32\pi^4(\epsilon^2)^4} + \frac{2fr^2\eta - (\Delta r)^2}{6\pi^4 f^2(\epsilon^2)^5} \\ & + \frac{8f^2r^4\eta^2 - 10fr^2(\Delta r)^2\eta + 3(\Delta r)^4}{18\pi^4 f^4(\epsilon^2)^6} + O[(x - x')^{-7}] \\ & + \kappa^2 \left\{ \frac{5}{576\pi^4(\epsilon^2)^3} - \frac{4fr^2\eta + 5(\Delta r)^2}{144\pi^4 f^2(\epsilon^2)^4} + \frac{(\Delta r)^4 - 4f^2r^4\eta^2}{36\pi^4 f^4(\epsilon^2)^5} + O[(x - x')^{-5}] \right\} \\ & + \kappa^4 \left\{ \frac{17}{8640\pi^4(\epsilon^2)^2} + \frac{46fr^2\eta - 153(\Delta r)^2}{25920\pi^4 f^2(\epsilon^2)^3} - \frac{26f^2r^4\eta^2 + 11fr^2(\Delta r)^2\eta - 12(\Delta r)^4}{2160\pi^4 f^4(\epsilon^2)^4} \right. \\ & \left. - \frac{8f^3r^6\eta^3 - 4f^2r^4(\Delta r)^2\eta^2 - 2fr^2(\Delta r)^4\eta + (\Delta r)^6}{540\pi^4 f^6(\epsilon^2)^5} + O[(x - x')^{-3}] \right\} . \quad (4.12) \end{aligned}$$

If we set $\Delta r = \eta = 0$ then it is possible to display the full Gaussian approximation for this component. It is

$$\begin{aligned} N_{tt't'v} = & \frac{1}{32\pi^4(\Delta t)^8} - \frac{f^2 - 2f + 1}{2304\pi^4 r^2(\Delta t)^6} + \frac{3f^4 - 8f^3 + 8f^2 - 4f + 1}{27648\pi^4 r^4(\Delta t)^4} \\ & + \kappa^2 \left[-\frac{5}{576\pi^4(\Delta t)^6} - \frac{5f^2 - 10f + 5}{27648\pi^4 r^2(\Delta t)^4} \right] \\ & + \kappa^4 \left[\frac{17}{8640\pi^4(\Delta t)^4} \right] + O[(x - x')^{-3}] . \quad (4.13) \end{aligned}$$

If we set $\Delta t = \eta = 0$ then it is also possible to display the full Gaussian approximation for

this component. It is

$$\begin{aligned}
N_{ttt't'} &= \frac{f^8}{32\pi^4(\Delta r)^8} + \frac{f^8 - f^7}{8\pi^4 r(\Delta r)^7} + \frac{673f^8 - 1122f^7 + 449f^6}{2304\pi^4 r^2(\Delta r)^6} \\
&+ \frac{1205f^8 - 2557f^7 + 1691f^6 - 339f^5}{2304\pi^4 r^3(\Delta r)^5} \\
&+ \frac{36775f^8 - 89540f^7 + 74680f^6 - 24384f^5 + 2469f^4}{46080\pi^4 r^4(\Delta r)^4} \\
&+ \kappa^2 \left[\frac{f^6}{576\pi^4(\Delta r)^6} + \frac{f^6 - \kappa^2 f^5}{192\pi^4 r(\Delta r)^5} + \frac{279f^6 - 430f^5 + 151f^4}{27648\pi^4 r^2(\Delta r)^4} \right] \\
&+ \kappa^4 \left[-\frac{\kappa^4 f^4}{4320\pi^4(\Delta r)^4} \right] + O[(x - x')^{-3}] . \tag{4.14}
\end{aligned}$$

The above results are for the optical Schwarzschild metric (3.22). They can be transformed to Schwarzschild spacetime trivially by using the transformation (2.11) and expanding $\Omega(r')$ about $r' = r$.

As discussed in Sec. II, the noise kernel has two properties which can be used to check our calculations. One of these is the vanishing of the partial traces so that

$$N^a{}_{ac'd'} = N_{abc'e'} = 0 . \tag{4.15}$$

The other property is that the noise kernel should be separately conserved at the points x and x' , so that

$$g^{ae} N_{abc'd';e} = g^{be} N_{abc'd';e} = g^{c'e'} N_{abc'd';e'} = g^{d'e'} N_{abc'd';e'} = 0 . \tag{4.16}$$

V. DISCUSSION

Using Page's approximation for the Euclidean Green function of a conformally invariant scalar field in the optical Schwarzschild spacetime, which is conformal to the static region of Schwarzschild spacetime, we have computed two expressions for the Wightman function associated with the Hartle-Hawking state in Schwarzschild. One is a partially covariant expression and the other is a completely noncovariant expression. The noncovariant expression has been used to explicitly compute some components of the noise kernel in the optical Schwarzschild spacetime for separate points which are either spacelike or timelike related,

and one component has been explicitly displayed. The transformation to Schwarzschild spacetime of the noise kernel has been shown to be trivial for the conformally invariant scalar field so long as the points are separated.

There are several more or less immediate generalizations of our work. First, although the Hartle-Hawking state corresponds to a specific temperature, given by Eqs. (3.24)-(3.25), our results also apply to any other temperature since we kept κ arbitrary in all our expressions. The states for those other values of the temperature are singular on the horizon (e.g. the expectation value of the stress tensor diverges there), but can sometimes be of interest (e. g. the Boulware vacuum corresponds to the particular case of $T = 0$). Second, the noise kernel corresponds to the expectation value of the anticommutator of the stress tensor. However, our results are also valid for other orderings of the stress tensor operator (in fact for any 2-point function of the stress tensor). In general, that is always true for spacelike separated points because the commutator of any local operator (such as the stress tensor) vanishes as a consequence of the microcausality condition. Moreover, since the commutator of the field, $iG(x, x')$, also vanishes for timelike separated points up to the order to which we are working, the previous statement also holds for timelike separations in our case. (In general one would need to use the appropriate prescription when analytically continuing the Euclidean Green function to obtain the Wightman function for timelike separated points in the Lorentzian case, and use expressions analogous to Eqs. (3.24)-(3.25) but without symmetrizing with respect to the two points. One can explicitly see how this is done in Refs. [15, 56].) Third, since the Gaussian approximation is valid for any ultra-static spacetime which is conformal to an Einstein metric (a solution of the Einstein equation in vacuum, with or without cosmological constant) [58], one can directly extend our calculation to all those cases by taking the general expression for the Wightman function under the Gaussian approximation, given by Eq. (3.27), and substituting it into the general expression for the noise kernel as described in Sec. IV A.

As discussed in the Introduction, one of the most interesting uses of the noise kernel is to investigate the effects of quantum fluctuations near the horizon of the black hole. Whereas having an explicit expression for the noise kernel is an essential ingredient in this respect, there is an important limitation to our result. In the optical spacetime, where the original computation was made, our expression for the noise kernel is valid through order $(x - x')^{-4}$, with the leading order terms being of order $(x - x')^{-8}$. However, this expression

is then conformally transformed to Schwarzschild spacetime. Far from the event horizon the accuracy is the same as in the optical spacetime. Nevertheless, the conformal factor that relates the Schwarzschild metric to the optical metric diverges as one approaches the event horizon. This means that in the Schwarzschild spacetime the noise kernel near the horizon is only accurate if the points are extremely close together. The situation can be slightly ameliorated if one takes into account the fact that Page's approximation should actually be valid for arbitrary time separations in the optical metric while keeping the spatial separations small [58]. This would imply that taking Eq. (3.26), introducing the prescription $\Delta\tau \rightarrow i(\Delta t - i\epsilon)$, using the Van-Vleck determinant ${}^{(3)}\Delta$ for the spatial metric g_{ij} (which for the optical metric is equivalent to the four-dimensional one, Δ), and expanding everything in terms of powers of \mathbf{r} , gives a Wightman function which is valid for arbitrary time separations. Using such an expression for the Wightman function in the optical metric, one would not be restricted to small time separations in Schwarzschild spacetime even when considering points arbitrarily close to the horizon. This does not, however, solve completely the problem because one is still restricted to considering very small spatial separations when getting sufficiently close to the horizon.

It is worthwhile to discuss briefly how the present paper is related to an earlier study of the noise kernel in Schwarzschild spacetime [49], which also considered a conformal scalar field and made use of Page's Gaussian approximation. The main interest there was evaluating the noise kernel in the coincidence limit. In order to get a finite result the Hadamard elementary solution was subtracted from the Wightman function before evaluating the noise kernel. Since the Hadamard elementary solution coincides with the $\kappa = 0$ expression for the Gaussian approximation through order $(x - x')^2$, which is the order through which the approximation is valid for the optical Schwarzschild spacetime, their subtracted Wightman function will also be valid through that order. The fact that they found a non-vanishing trace for their noise kernel is also compatible with our results because, as we have reasoned, the noise kernel should only be valid through order $(x - x')^{-4}$ when the Gaussian approximation for the Wightman function is employed. Instead, one would need an expression for the noise kernel accurate through order $(x - x')^0$ or higher to get a vanishing trace in the coincidence limit. In contrast, for the reasons given in the introduction, here we consider the unsubtracted noise kernel, which we believe to be a more interesting object (for instance, the subtracted one would lead to a vanishing result –and no fluctuations– for the Minkowski

vacuum). Furthermore, in this way one can still get useful and accurate information for the terms of order $(x - x')^{-8}$ through $(x - x')^{-4}$, which dominate at small separations.

From the noise kernel one can immediately obtain the symmetrized quantum 2-point function for the Einstein tensor (or, equivalently, the Ricci tensor) including the one-loop correction from the matter fields (see discussion in Sec. 8 of Ref. [15]). Strictly speaking, however, one should not employ the Schwarzschild background, but a slightly corrected one which takes into account the back-reaction of the quantum matter fields on the mean geometry via the semiclassical Einstein equation [52]. Nevertheless, one can consider an expansion in powers of m_p^2/M^2 and the previous statement will still be true to zeroth order in that expansion. (Incidentally, at that order the 2-point function of the Ricci tensor is also gauge-invariant, as a consequence of the Ricci tensor vanishing for the background Schwarzschild geometry.) The Ricci tensor, however, does not fully characterize the local geometry. In order to do so, one needs to solve the Einstein-Langevin equation and obtain the metric perturbations around the background in terms of the stochastic source. One can then calculate the correlation function for some appropriate gauge-invariant quantity linear in the metric perturbations.

The Einstein-Langevin equation is an integro-differential equation, which is in general difficult to solve exactly. A simpler possibility is to solve it perturbatively in powers of the gravitational coupling constant (or, equivalently, in powers of the Planck length squared, l_p^2). To quadratic order (i.e. order l_p^4) the result for the 2-point stochastic correlation function of the metric perturbations or related quantities corresponds to calculating the 2-point quantum correlation function including the one-loop correction from the matter fields. (It is at this order that the 2-point function of the Ricci tensor is directly related to the noise kernel in a simple way as mentioned above.) This kind of perturbative treatment can typically give rise to spurious secular effects for sufficiently long times, for which the perturbative expansion breaks down. For a given static background geometry and static unperturbed quantum state (like the Hartle-Hawking state), if one introduces the initial conditions appropriately at an asymptotic initial time, the correlation functions will only depend on the time difference. In that case, the breakdown of the perturbative expansion will limit this time difference but not the absolute time values. Moreover, for macroscopic black holes one has $M \gg m_p$ (which is anyway a necessary condition for such a semiclassical treatment to be valid) and one can introduce an additional perturbative expansion in powers of m_p^2/M^2 . This means

that at zeroth order, the objects appearing in typical computations can be evaluated on a Schwarzschild background rather than the semiclassically corrected one (e.g. both the retarded propagator G_{ret} associated with the Einstein-Langevin equation and the noise kernel appearing in convolutions of the form $G_{\text{ret}} \cdot N \cdot G_{\text{ret}}^T$ [10]).

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Appendix: Noise kernel and conformal transformations

In this appendix we derive the result for the rescaling of the noise kernel under conformal transformations. We provide two alternative proofs based respectively on the use of quantum operators and on functional methods.

First, we start by showing how the classical stress tensor of a conformally invariant scalar field rescales under a conformal transformation $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2(x) g_{ab}$. The key point is that the classical action of the field, $S[\phi, g]$, remains invariant (up to surface terms) if one rescales appropriately the field: $\phi \rightarrow \tilde{\phi} = \Omega^{(2-D)/2} \phi$. Taking that into account, one easily gets the result from the definition of the stress tensor as a functional derivative of the classical action:

$$\tilde{T}_{ab} = \frac{2\tilde{g}_{ac}\tilde{g}_{bd}}{\sqrt{-\tilde{g}}} \frac{\delta S[\tilde{\phi}, \tilde{g}]}{\delta \tilde{g}_{cd}} = \frac{2\tilde{g}_{ac}\tilde{g}_{bd}}{\sqrt{-\tilde{g}}} \frac{\delta S[\phi, g]}{\delta \tilde{g}_{cd}} = \Omega^{2-D} \frac{2g_{ac}g_{bd}}{\sqrt{-g}} \frac{\delta S[\phi, g]}{\delta g_{cd}} = \Omega^{2-D} T_{ab}. \quad (\text{A.1})$$

1. Proof based on quantum operators

A possible way of proving Eq. (2.11) is by promoting the classical field ϕ in Eq. (A.1) to an operator in the Heisenberg picture. The operator $\hat{T}_{ab}(x)$ would be divergent because it involves products of the field operator at the same point. However, in order to calculate the noise kernel what one actually needs to consider is $\hat{t}_{ab}(x) = \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle$ and this object is UV finite, i.e., its matrix elements $\langle \Phi | \hat{t}_{ab}(x) | \Psi \rangle$ for two arbitrary states $|\Psi\rangle$

and $|\Phi\rangle$ (not necessarily orthogonal) are UV finite because Wald's axioms [61] guarantee that $\langle\Phi|\hat{T}_{ab}(x)|\Psi\rangle$ and $\langle\Phi|\Psi\rangle\langle\hat{T}_{ab}(x)\rangle$ have the same UV divergences and they cancel out. Therefore, one can proceed as follows. One starts by introducing a UV regulator (it is useful to consider dimensional regularization since it is compatible with the conformal symmetry for scalar and fermionic fields, but this is not indispensable since we will remove the regulator at the end without having performed any subtraction of non-invariant counterterms). One can next apply the operator version of Eq. (A.1) to the operators $\hat{t}_{ab}(x)$ appearing in Eq. (2.1) defining the noise kernel. Since all UV divergences cancel out, as argued above, we can then safely remove the regulator and are finally left with Eq. (2.11).

2. Proof based on functional methods

An alternative way of proving Eq. (2.11) is by analyzing how the closed-time-path (CTP) effective action $\Gamma[g, g']$ changes under conformal transformations. This effective action results from treating g_{ab} and g'_{ab} as external background metrics and integrating out the quantum scalar field within the CTP formalism [69]:

$$e^{i\Gamma[g, g']} = \int \mathcal{D}\varphi_f \mathcal{D}\varphi_i \mathcal{D}\varphi'_i \rho[\varphi_i, \varphi'_i] \int_{\varphi_i}^{\varphi_f} \mathcal{D}\phi e^{iS_g[g] + iS[\phi, g]} \int_{\varphi'_i}^{\varphi_f} \mathcal{D}\phi' e^{-iS_g[g'] - iS[\phi', g']}, \quad (\text{A.2})$$

where $\rho[\varphi_i, \varphi'_i]$ is the density matrix functional for the initial state of the field ² (in particular one has $\rho[\varphi_i, \varphi'_i] = \Psi[\varphi_i]\Psi^*[\varphi'_i]$ for a pure initial state with wave functional $\Psi[\varphi_i] = \langle\varphi_i|\Psi\rangle$ in the Schrödinger picture), $S_g[g]$ is the gravitational action including local counterterms, $S[\phi, g]$ is the action for the scalar field, and the two background metrics are also taken to coincide at the same final time at which the final configuration of the scalar field for the two branches are identified and integrated over. The fields φ_f on the one hand and $\{\varphi_i, \varphi'_i\}$ on the other, correspond to the values of the field restricted respectively to the final and initial Cauchy surfaces, and their functional integrals are over all possible configurations of the field on those surfaces. Integrating out the scalar field gives rise to UV divergences, but they can be dealt with by renormalizing the cosmological constant and the gravitational

² Under appropriate conditions it is also possible to consider asymptotic initial states. For instance, given a static spacetime, a generalization to the CTP case of the usual $i\epsilon$ prescription involving a small Wick rotation in time selects the ground state of the Hamiltonian associated with the time-translation invariance as the initial state.

coupling constant as well as introducing local counterterms quadratic in the curvature in the bare gravitational action $S_g[g]$, so that the total CTP effective action is finite. Functionally differentiating and identifying the two background metrics after that, one gets the renormalized expectation value of the stress tensor operator together with the contributions from the gravitational action [21, 70]:

$$g_{ac}g_{bd} \frac{2}{\sqrt{-g}} \frac{\delta\Gamma[g, g']}{\delta g_{cd}} \Big|_{g'=g} = -\frac{1}{8\pi G} (G_{ab} + \Lambda g_{ab}) + \langle \hat{T}_{ab} \rangle_{\text{ren}}, \quad (\text{A.3})$$

where the contribution from the counterterms quadratic in the curvature has been absorbed in $\langle \hat{T}_{ab} \rangle_{\text{ren}}$. The renormalized gravitational coupling and cosmological constants, G and Λ , depend on the renormalization scale, but the expectation value also depends on it in such a way that the total expression is renormalization-group invariant since that is the case for the effective action. The equation that one obtains by equating the right-hand side of Eq. (A.3) to zero governs the dynamics of the mean field geometry in semiclassical gravity, including the back-reaction effects of the quantum matter fields.

On the other hand, the noise kernel can be obtained by functionally differentiating twice the imaginary part of the CTP effective action:

$$N_{abc'd'}(x, x') = g_{ae}(x)g_{bf}(x)g_{c'g'}(x')g_{d'h'}(x') \frac{4}{\sqrt{g(x)g(x')}} \frac{\delta^2 \text{Im} \Gamma[g, g']}{\delta g_{e'f'}(x)\delta g_{g'h'}(x')} \Big|_{g'=g}. \quad (\text{A.4})$$

It is well-known that the imaginary part of the effective action does not contribute to the equations of motion for expectation values derived within the CTP formalism, like Eq. (A.3), which are real. Furthermore, one can easily see from Eq. (A.2) that, being real, the gravitational action (whose contribution can be factored out of the path integral) only contributes to the real part of the effective action. In particular this means that the counterterms and the renormalization process have no effect on the noise kernel, which will be a key observation in order to prove Eq. (2.11). Indeed, let us start with Eq. (A.2) for the conformally related metric and scalar field, \tilde{g}_{ab} and $\tilde{\phi}$, and assume that we use dimensional regularization:

$$\begin{aligned} e^{i\Gamma[\tilde{g}, \tilde{g}']} &= e^{iS_g[\tilde{g}] - iS_g[\tilde{g}']} \int \mathcal{D}\tilde{\varphi}_f \mathcal{D}\tilde{\varphi}_i \mathcal{D}\tilde{\varphi}'_i \tilde{\rho}[\tilde{\varphi}_i, \tilde{\varphi}'_i] \int_{\tilde{\varphi}_i}^{\tilde{\varphi}_f} \mathcal{D}\tilde{\phi} e^{iS[\tilde{\phi}, \tilde{g}]} \int_{\tilde{\phi}'_i}^{\tilde{\phi}'_f} \mathcal{D}\tilde{\phi}' e^{-iS[\tilde{\phi}', \tilde{g}']} \\ &= e^{iS_g[\tilde{g}] - iS_g[\tilde{g}']} \int \mathcal{D}\varphi_f \mathcal{D}\varphi_i \mathcal{D}\varphi'_i \rho[\varphi_i, \varphi'_i] \int_{\varphi_i}^{\varphi_f} \mathcal{D}\phi \left| \frac{\mathcal{D}\tilde{\phi}}{\mathcal{D}\phi} \right| e^{iS_g[g] + iS[\phi, g]} \\ &\quad \times \int_{\varphi'_i}^{\varphi'_f} \mathcal{D}\phi' \left| \frac{\mathcal{D}\tilde{\phi}'}{\mathcal{D}\phi'} \right| e^{-iS_g[g'] - iS[\phi', g']}, \end{aligned} \quad (\text{A.5})$$

where we took into account in the second equality that dimensional regularization is compatible with the invariance of the classical action $S[\tilde{\phi}, \tilde{g}]$ under conformal transformations (since it is invariant in arbitrary dimensions). We also considered that the initial states of the scalar field are related by

$$\begin{aligned} \tilde{\rho}[\tilde{\varphi}_i(x), \tilde{\varphi}'_i(x')] &= \Omega_i^{(D-2)/4}(x) \Omega_i^{(D-2)/4}(x') \rho[\varphi_i(x), \varphi'_i(x')] \\ &= \Omega_i^{(D-2)/4}(x) \Omega_i^{(D-2)/4}(x') \rho\left[\Omega_i^{-1}(x)\tilde{\varphi}_i(x), \Omega_i^{-1}(x)\tilde{\varphi}'_i(x')\right], \end{aligned} \quad (\text{A.6})$$

where Ω_i^2 is the conformal factor restricted to the initial Cauchy surface and so are the points $\{x, x'\}$ in this equation. (This relation between the initial states is the choice compatible with conformal invariance after one takes into account the relation between $\tilde{\phi}$ and ϕ , and the prefactor is determined by requiring that the state remains normalized.) The logarithm of the functional Jacobian $|\mathcal{D}\tilde{\phi}/\mathcal{D}\phi|$ is divergent but formally zero in dimensional regularization³, so that we can take $|\mathcal{D}\tilde{\phi}/\mathcal{D}\phi| = 1$ in both path integrals on the right-hand side of Eq. (A.5). Taking all this into account, we are left with

$$\Gamma[\tilde{g}, \tilde{g}'] = \Gamma[g, g'] + (S_g[\tilde{g}] - S_g[g]) - (S_g[\tilde{g}'] - S_g[g']), \quad (\text{A.7})$$

where the last two pairs of terms on the right-hand side correspond to the difference between the bare gravitational actions of the two conformally related metrics in dimensional regularization; note that whereas each bare action is separately divergent, the difference $S_g[\tilde{g}] - S_g[g]$ is finite. When working in dimensional regularization, conformally invariant fields only exhibit divergences associated with counterterms quadratic in the curvature. These terms lead to the standard result for the trace anomaly of the stress tensor when one takes the functional derivative of Eq. (A.7) with respect to the conformal factor, which can be shown to be equivalent to the trace of Eq. (A.3).

The key aspect for our purposes is that the extra terms on the right-hand side of Eq. (A.7) only change the real part of the CTP effective action, as already mentioned above, so that the imaginary part remains invariant under conformal transformations. Starting with Eq. (A.4)

³ This can be seen by taking Eq. (18) in Ref. [71] and using dimensional regularization [61] to evaluate the trace of the heat kernel appearing there. Any possible dependence left on the conformal factor evaluated at the initial or final Cauchy surfaces would correspond to a prefactor on the right-hand side of Eq. (A.5), and would not contribute to the noise kernel (or the expectation value of the stress tensor) at any intermediate time since it involves functionally differentiating the logarithm of that expression with respect to the metric at such intermediate times.

for the metric \tilde{g}_{ab} and taking into account the invariance of the imaginary part of the CTP effective action under conformal transformations, one immediately obtains

$$\tilde{N}_{abc'd'}(x, x') = \Omega^{2-D}(x) \Omega^{2-D}(x') N_{abc'd'}(x, x'), \quad (\text{A.8})$$

in agreement with Eq. (2.11). Note that we have employed dimensional regularization in our argument for simplicity, but one would reach the same conclusion if other regularization schemes had been used. In those cases one would get in general a contribution to the analog of Eq. (A.7) from the change of the functional measure, but it would only affect the real part of the effective action (see Ref. [71], where the calculations are performed in Euclidean time, and analytically continue the result to Lorentzian time) and one could still apply exactly the same argument as before.

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