AEI-2010-172

Boundary conditions in Toda theories and minimal models

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Abstract

We show that the disc bulk one-point functions in a $sl(n)$ Toda conformal field theory have a well-defined limit for the central charge $c = n - 1$, and that their limiting values can be obtained from a limit of bulk one-point functions in the W_n minimal models. This comparison leads to a proposal for one-point functions for twisted boundary conditions in Toda theory.

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1 Introduction

Conformal field theories (CFTs) in two dimensions play an important role in string theory and statistical physics. Since the seminal paper by Belavin, Polyakov and Zamolodchikov [1], enormous progress has been made. In particular, for rational CFTs we have obtained a solid understanding, both in the mathematical structures and in the tools that allow to determine correlation functions (for a recent overview see [2]). The non-rational theories, in particular those with a continuous spectrum (usually called non-compact theories), are much less understood. This is unfortunate given that such theories are of prime importance when we discuss e.g. AdS/CFT correspondence, in which non-compact target spaces necessarily arise, or when we want to consider cosmological backgrounds in string theory.

Interestingly, rational CFTs and non-compact CFTs have some points of contact. Some families of rational CFTs have a non-rational limit theory with a continuous spectrum, and the properties of the limit theory can be understood from the rational CFT data. The first example of such a point of contact is the Runkel-Watts theory [3] that arises as the limit of unitary Virasoro minimal models at central charge $c = 1$. This theory can also be understood [4, 5] as a limit of the non-compact Liouville CFT. A similar story relates the $N = 1$ supersymmetric minimal models and the $N = 1$ super Liouville theory [6]. Note that the notion of a limit of CFTs is not unique, see [7] for a different approach.

Liouville theory is the prime example of a non-compact CFT, and by now we have achieved a very good understanding of this theory: the bulk three-point functions are known [8, 9], boundary conditions have been found [10, 11, 12], and the corresponding boundary structure constants have been determined [13, 14, 15]. Liouville theory can be seen as the $sl(2)$ case of the class of $sl(n)$ Toda CFTs. These theories have large chiral symmetry algebras, the W_n algebras, which for $n = 2$ is just the Virasoro algebra. Toda theories are interesting objects to study – on the one hand they are highly non-trivial examples for non-compact CFTs, on the other hand their large symmetry makes us hope that they are still tractable. CFTs with W_n algebras are likely to play a role for the duals of higher spin gauge theories on three-dimensional AdS backgrounds [16, 17, 18]. Furthermore, Toda theories appear in a recently proposed relation between four-dimensional $N = 2$ supersymmetric gauge theories and two-dimensional CFTs [19, 20].

Despite their importance and the recent interest, much less is known on Toda theories than on Liouville theory. The three-point correlators on the sphere are only known for a subset of the primary fields [21, 22]. Topological defects in these theories have been constructed from modular data in [23]. Recently, boundary conditions in $sl(n)$ Toda CFTs have been investigated [24], and bulk one-point functions in the presence of these boundary conditions have been determined.

In view of the relation between Liouville theory and the Virasoro minimal modes, it is natural to ask whether one can obtain Toda theories as a limit of a family of rational CFTs. As suggested in [6], one expects that the W_n minimal models approach a limit theory at central charge $c = n - 1$, which coincides with $sl(n)$ Toda CFT at this value of the central charge.

Boundary conditions in W_n minimal models are completely understood in terms of the Cardy construction [25] or twisted versions thereof. One can thus use the point of contact at $c = n - 1$ between the minimal models and Toda CFT to test and interpret the Toda boundary conditions of [24], and to improve our understanding of them. In particular, one might hope to better understand the divergences in the annulus partition functions from the limit of the boundary spectra of minimal model boundary conditions.

In this paper, the limits of $sl(n)$ Toda CFT and of W_n minimal models are analysed and compared. In section 2 we study the spectrum of Toda theory, the bulk two-point function, and the bulk one-point function in the boundary theories of [24] in the limit $c \to n - 1$. Section 3 reviews the W_n-minimal models and their untwisted and twisted boundary conditions. Then, in section 4, we define a limit theory for these models at $c = n - 1$, and obtain a continuous spectrum by averaging over the discrete minimal model fields. In the limit theory, we consider the bulk two-point function, as well as one-point functions for different classes of boundary conditions. Up to a change of normalisation of the fields we find complete agreement with the Toda analysis. We conclude in section 6 with the observation that the limit of twisted boundary conditions in the W_n minimal models leads to a precise proposal for the one-point function for twisted boundary conditions in Toda theory, which had not been determined in [24]. Two appendices contain details on the limit of the spectrum of the minimal models, and on untwisted and twisted modular S-matrices.

2 Boundary Toda conformal field theory

In this section, we shall first review the bulk Toda theory, and discuss the spectrum and the two-point functions in the limit $c \to n-1$. Then we shall analyse the limit of one-point functions in the boundary theories of [24].

2.1 Bulk Toda theory

Before we come to the boundary theory, we want to review a few facts about Toda conformal field theory as can be found e.g. in [21]. The two-dimensional $sl(n)$ Toda conformal field theory is described by the action

$$S = \int \left(\frac{1}{8\pi} (\partial_a \phi)^2 + \frac{(Q,\phi)}{4\pi} R + \mu \sum_{j=1}^{n-1} e^{b(e_j,\phi)} \right) \sqrt{g} d^2x \ , \tag{2.1}$$

where the scalar field $\phi = (\phi_1, \ldots, \phi_{n-1})$ lives in the Cartan subalgebra, the e_j are the simple roots of $sl(n)$, b is a dimensionless coupling constant, and μ is called the cosmological constant. R is the scalar curvature of the two-dimensional background metric g, and Q is a background charge that takes the value

$$Q = \left(b + b^{-1}\right) \rho \tag{2.2}$$

for a conformally invariant theory (here ρ denotes the Weyl vector of $sl(n)$). The central charge of this theory is

$$c = n - 1 + 12Q^2 = (n-1)(1 + n(n+1)(b + b^{-1})^2) \ . \tag{2.3}$$

In addition to the energy momentum tensor there are higher spin currents in the theory that form the W_n algebra. The spinless primary fields of Toda CFT are given by the exponentials

$$V_\alpha = e^{(\alpha,\phi)} \ , \tag{2.4}$$

they are labelled by a vector α. The conformal weight of V_α is given by

$$h(\alpha) = \frac{(\alpha, 2Q - \alpha)}{2} \ . \tag{2.5}$$

For the physical spectrum we have $\alpha = Q + ip$ with a real vector p, and the conformal weights are non-negative real numbers. The conformal weights and all representation properties of V_α and $V_{Q+w(\alpha-Q)}$ are the same for any Weyl transformation $w \in W$, and the corresponding fields coincide up to a factor,

$$V_\alpha(z) = R_w(\alpha) V_{Q+w(\alpha-Q)}(z) \ . \tag{2.6}$$

The reflection amplitude that occurs here is given [26] by

$$R_w(\alpha) = \frac{A(Q+w(\alpha-Q))}{A(\alpha)} \ , \tag{2.7}$$

with

$$A(\alpha) = \left(\pi\mu\gamma(b^2)\right)^{b^{-1}(\alpha-Q,\rho)} \prod_{e>0} \frac{2\pi b^{-1}}{\Gamma(b(\alpha-Q,e))\Gamma(1+b^{-1}(\alpha-Q,e))} \ , \tag{2.8}$$

where we take the product over all positive roots. We normalise the two-point correlation functions to be

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle = \sum_{w\in W} \delta(p_1 + w(p_2)) \frac{A(2Q-\alpha_1)}{A(\alpha_2)} |z_1 - z_2|^{-4h(\alpha_1)} \ , \tag{2.9}$$

where $\alpha_j = Q + ip_j$, and the delta distribution is defined with respect to the standard metric on the weight space. Note that the sum over the Weyl orbit is necessary to be consistent with the identifications (2.6) under Weyl transformations. If we choose to label fields only by their representatives V_α with $p = -i(\alpha - Q)$ being in the interior of the fundamental Weyl chamber, the two-point function reads

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle = \delta(p_1 - p_2^+) \frac{A(2Q-\alpha_1)}{A(\alpha_2)} |z_1 - z_2|^{-4h(\alpha_1)} \ , \tag{2.10}$$

where p^+ is the conjugate weight vector of p, $(p^+)_j = (p)_{n-j}$ for $j = 1, \ldots, n-1$. Here, the $(p)_j$ are the coefficients of p with respect to the basis of fundamental weights ω_j, $p = \sum_{j=1}^{n-1} (p)_j \omega_j$.

2.2 Taking the limit

We are interested in the connection between Toda CFTs and the corresponding minimal models. The minimal models of the W_n algebra all have central charge smaller than the rank $n-1$, with $n-1$ being the supremum of all central charges. To make contact to these models we therefore would like to take the limit of the $sl(n)$ Toda CFT to central charge $n-1$. From (2.3) we see that this value of the central charge is reached for $b = i$ (or $b = -i$). Now it is a priori not clear whether such a continuation of Toda CFT makes sense. However, it was shown in [4] that in Liouville theory the bulk correlation functions have a well-defined limit for $b \to i$. The same holds true for the correlators in $N = 1$ supersymmetric Liouville theory [6]. In [5] it was shown for Liouville theory that also the boundary conditions have a well-defined limit as $b \to i$. This suggests to evaluate this limit also for Toda CFTs.

We follow the strategy of [4, 5] and set

$$\alpha = Q + ip \ , \tag{2.11}$$

where we keep p constant in the limit. The factor $A(\alpha)$ defined in (2.8) assumes the limit

$$\tilde{A}(p) = (\pi\mu_{\text{ren}})^{(p,\rho)} \prod_{e>0} (2i \sin \pi(e,p)) \ . \tag{2.12}$$

Here we have introduced the renormalised cosmological constant μ_{ren}. The reflection amplitude (2.7) is then given by

$$R_w(p) = \frac{\tilde{A}(w(p))}{\tilde{A}(p)} \ . \tag{2.13}$$

The two-point function (2.10) of fields V_{ip} with p in the fundamental Weyl chamber reads in the limit

$$\langle V_{ip_1}(z) V_{ip_2}(w) \rangle = \delta(p_1 - p_2^+) \frac{\tilde{A}(-p_1)}{\tilde{A}(p_2)} |z-w|^{-4h_{p_1}} \tag{2.14}$$

$$= \delta(p_1 - p_2^+) \, (\pi\mu_{\text{ren}})^{-(p_1+p_2,\rho)} \, (-1)^{\frac{n(n-1)}{2}} |z-w|^{-4h_{p_1}} \ , \tag{2.15}$$

where the limit of the conformal weight (2.5) is

$$h_p = \frac{1}{2} p^2 \ . \tag{2.16}$$

The exponent of the sign in (2.15) is given by the number of positive roots in $sl(n)$.

The three-point correlation functions in $sl(n)$ Toda CFT are not known in general, therefore we do not know how to take the limit. On the other hand, they are known for a restricted set of fields [21, 22], and it would be interesting to evaluate the limit of those and compare it to the minimal model side.

2.3 One-point functions

Due to the conformal symmetry, the one-point function of a bulk field Φ_α on the complex upper half plane with a conformal boundary condition at the real axis is given by

$$\langle \Phi_\alpha \rangle_s = U_s(\alpha) \frac{1}{|z - \bar{z}|^{2h(\alpha)}} \ , \tag{2.17}$$

where s labels the boundary condition, and $h(\alpha) = \bar{h}(\alpha)$ is the conformal weight of the bulk field Φ_α (only fields with $h = \bar{h}$ can couple to a conformal boundary condition). The coefficients $U_s(\alpha)$ characterise the boundary theory labelled by s.

Conformal boundary conditions and their one-point coefficients U^T for $sl(n)$ Toda CFTs have been determined in [24]. The computations are done explicitly for $sl(3)$, but it is suggested that similar formulae also hold for arbitrary n. The non-degenerate boundary conditions of [24] are labelled by a vector s, and the corresponding one-point coefficients are given by[1]

$$\begin{aligned} U_s^T(\alpha) &= A(\alpha)^{-1} \sum_{w\in W} e^{-2\pi(w(s),\alpha-Q)} \\ &= \big[\pi\mu\gamma(b^2)\big]^{\frac{(\rho,Q-\alpha)}{b}} \prod_{e>0} \frac{\Gamma\big(b(e,\alpha-Q)\big)\Gamma\big(1+b^{-1}(e,\alpha-Q)\big)}{2\pi b^{-1}} \sum_{w\in W} e^{-2\pi(w(s),\alpha-Q)} \ . \end{aligned} \tag{2.18}$$

We see immediately that the boundary condition only depends on the Weyl orbit of s. Also, the one-point function (2.18) has the expected reflection property (see (2.6)),

$$U_s^T(\alpha) = R_w(\alpha) U_s^T(Q + w(\alpha - Q)) \ . \tag{2.19}$$

To the boundary condition labelled by s, one can associate the so-called boundary cosmological constants [24],

$$\lambda_{i,\pm} = \chi_{\omega_i}(2\pi b^{\pm 1} s) \ , \tag{2.20}$$

where χ_{ω_i} is the character of the representation with highest weight vector the i^{th} fundamental weight ω_i of $sl(n)$, $i = 1, \ldots, n-1$.

In addition to the non-degenerate boundary conditions that correspond to $(n-1)$-dimensional branes, there are degenerate boundary conditions, which are associated to lower-dimensional branes.[2] In [24], these are described for $sl(3)$, but their results suggest a straightforward generalisation to arbitrary $sl(n)$. In general, these boundary conditions would then be labelled by a subgroup $W' \subset W$ of the Weyl group, a vector κ that is invariant under W', and two dominant integral weights Ω, Ω'. The coefficient of the bulk one-point function is given as a sum over the coefficients $U_s^T(\alpha)$ of the non-degenerate boundary conditions,

$$U^{T,W'}_{\kappa,\Omega,\Omega'}(\alpha) = \sum_{w\in W'} \epsilon(w) U^T_{s(\kappa,\Omega,\Omega',w)}(\alpha) \ , \tag{2.21}$$

where

$$s(\kappa,\Omega,\Omega',w) = \kappa - i(b(\Omega+\rho) + b^{-1} w(\Omega'+\rho)) \ . \tag{2.22}$$

Note that the boundary cosmological constants associated to the different $s(\kappa,\Omega,\Omega',w)$ are equal, they do not depend on w. We have

$$\chi_{\omega_j}(2\pi b s(\kappa,\Omega,\Omega',w)) = \chi_{\omega_j}(2\pi b\kappa - 2\pi i b^2(\Omega+\rho) - 2\pi i w(\Omega'+\rho)) \ , \tag{2.23}$$

[1] Note that our s is related to the parameter s_{FR} of [24] by $s_{\text{FR}} = -2\pi s$. Also we use a slightly different normalisation fo the fields.

[2] The dimensionality can be deduced on the one hand from the comparison with the classical analysis, on the other hand it is related to the infrared divergence of the one-point function; see [24] for details.

and the right hand side is independent of w, because $w(\Omega+\rho)$ differs from $\Omega+\rho$ by an element of the root lattice, leading to a trivial phase in the character. Similarly

$$\begin{aligned}\chi_{\omega_j}(2\pi b^{-1}s(\kappa,\Omega,\Omega',w)) &= \chi_{\omega_j}(2\pi b^{-1}\kappa - 2\pi i(\Omega+\rho) - 2\pi i b^{-2}w(\Omega'+\rho)) \\ &= \chi_{\omega_j}(2\pi b^{-1}w^{-1}(\kappa) - 2\pi i w^{-1}(\Omega+\rho) - 2\pi i b^{-2}(\Omega'+\rho)) \end{aligned} \quad (2.24)$$

is independent of w, because κ is invariant under $w\in W'$.

We call a boundary condition m-degenerate if the subspace of vectors invariant under the group W' is $(n-1-m)$-dimensional. Note that the labels κ,Ω,Ω' have in general a redundancy: the W'-invariant part of $2\pi i(b(\Omega+\rho)+b^{-1}w(\Omega'+\rho))$ can be absorbed into κ. Therefore we have $(n-m-1)$ continuous parameters to choose κ, and $2m$ discrete parameters labelling the components of Ω,Ω' orthogonal to the invariant subspace. If one requires that the W'-invariant part of $s(\kappa,\Omega,\Omega',w)$ is real, then an m-degenerate boundary condition can be labelled by the set $\mathbb{R}^{n-1-m}\times\mathbb{N}^{2m}$.

All the boundary conditions that we have described until now satisfy trivial gluing conditions for the currents of the W_n-algebra. For $n>2$, the algebra has an automorphism that is induced by an outer automorphism of $sl(n)$. One can then also study twisted boundary conditions, in which the currents satisfy gluing conditions that are twisted by the automorphism. These boundary conditions have been studied for $sl(3)$ in [24], but only in the light asymptotic limit (where $b\to 0$), for which bulk one-point and boundary two-point correlators have been determined.

We now take the limit $b\to i$. We keep $ip=\alpha-Q$ fixed, and for the non-degenerate boundary condition (2.18) the limit $\tilde{U}^T$ of the one-point coefficient is

$$\tilde{U}^T_s(p) = \left[\pi\mu_{\mathrm{ren}}\right]^{-(\rho,p)}\prod_{e>0}(2i\sin\pi(e,p))^{-1}\sum_{w\in W}e^{-2\pi i(w(s),p)}\ . \quad (2.25)$$

The degenerate ones are then described by

$$\tilde{U}^{T,W'}_{\kappa,\Omega,\Omega'}(p) = \sum_{w\in W'}\epsilon(w)\tilde{U}^T_{\kappa+(\Omega+\rho)-w(\Omega'+\rho)}(p)\ . \quad (2.26)$$

In the extreme case, $W'=W$, the only vector invariant under W' is the zero vector and we find the completely degenerate boundary conditions

$$\tilde{U}^T_{\Omega,\Omega'}(p) \equiv \tilde{U}^{T,W}_{0,\Omega,\Omega'}(p) = \sum_{w\in W}\epsilon(w)\tilde{U}^T_{(\Omega+\rho)-w(\Omega'+\rho)}(p)\ . \quad (2.27)$$

We will see later that not all of these are linearly independent (see the discussion at the end of section 4.2). For the twisted boundary conditions we cannot take the limit, as the results of [24] are only obtained in the limit $b\to 0$.

We are now going to analyse boundary conditions in minimal models, where we want to reproduce (2.25) and (2.26) in the corresponding limit.

3 Boundary conditions in minimal models

In this section we review the W_n minimal models and their untwisted and twisted boundary conditions.

3.1 Bulk theory

The W_n minimal models can be obtained [27] by a diagonal coset construction [28],

$$M_n(k) = \frac{sl(n)_k \oplus sl(n)_1}{sl(n)_{k+1}} \ . \tag{3.1}$$

Their central charge is given by

$$c_n(k) = (n-1)\left(1 - \frac{n(n+1)}{(k+n)(k+n+1)}\right) \ . \tag{3.2}$$

The sectors of the theory are labelled by three integral dominant weights $(\Lambda, \lambda; \Lambda')$ of the affine Lie algebras $sl(n)_k, sl(n)_1$ and $sl(n)_{k+1}$, respectively. When we write Λ (and similarly for λ, Λ'), we think of it as the finite part $\Lambda = (\Lambda_1, \ldots, \Lambda_{n-1})$ of an affine weight $(k - \Lambda_1 - \cdots - \Lambda_{n-1}, \Lambda_1, \ldots, \Lambda_{n-1})$. These labels are subject to the selection rule

$$\Lambda + \lambda - \Lambda' \in L_R \ , \tag{3.3}$$

where L_R is the root lattice of $sl(n)$. This selection rule determines λ completely in terms of Λ and Λ', and λ can thus be omitted. In addition, some sectors have to be identified according to the field identifications

$$(\Lambda; \Lambda') \sim (J\Lambda; J\Lambda') \ , \tag{3.4}$$

where J is the generator of the $\mathbb{Z}_n$ simple current groups (we denote the simple currents in $sl(n)_k$ and $sl(n)_{k+1}$ by the same symbol). The action of the simple current J on a weight Λ at level k is given by

$$J\Lambda = k\omega_1 + w_J\lambda \ , \tag{3.5}$$

with the Weyl group element $w_J = s_1 \cdots s_{n-1}$ acting as

$$w_J(\Lambda_1, \ldots, \Lambda_{n-1}) = (-\Lambda_1 - \cdots - \Lambda_{n-1}, \Lambda_1, \ldots, \Lambda_{n-2}) \ . \tag{3.6}$$

Here, s_i are the Weyl reflections for the simple roots α_i, and ω_i are the fundamental weights.

The conformal weight $h_{\Lambda,\Lambda'}$ of a primary state with label $(\Lambda; \Lambda')$ is given by

$$h_{\Lambda,\Lambda'} = \frac{1}{2t}\left(d^2_{\Lambda+\rho,\Lambda'+\rho} - d^2_{\rho,\rho}\right) \ , \tag{3.7}$$

where

$$d_{v,v'} := v - tv' \quad , \quad t = \frac{k+n}{k+n+1} \ , \tag{3.8}$$

and ρ is the Weyl vector. Under a field identification, the vector $d_{v,v'}$ transforms by a Weyl transformation,

$$\begin{aligned} d_{Jv,Jv'} &= Jv - tJv' \\ &= k\omega_1 + w_J v - t\big((k+1)\omega_1 + w_J v'\big) \\ &= w_J\big(v - tv'\big) = w_J d_{v,v'} \ . \end{aligned} \tag{3.9}$$

We normalise the fields such that the two-point correlator is given by

$$\langle \phi_{(\Lambda_1;\Lambda_1')}(z)\phi_{(\Lambda_2;\Lambda_2')}(w)\rangle = \delta_{(\Lambda_1;\Lambda_1')(\Lambda_2^+;\Lambda_2'^+)}|z-w|^{-4h_{\Lambda_1,\Lambda_1'}} \ . \tag{3.10}$$

3.2 One-point functions for untwisted boundary conditions

The maximally symmetric, untwisted boundary conditions in the W_n minimal models are labelled by the same labels as the bulk fields, we denote them by $(L;L')$. The one-point functions are then given by the Cardy construction,

$$U^M_{(L;L')}(\Lambda;\Lambda') = \frac{S_{(\Lambda;\Lambda')(L;L')}}{\sqrt{S_{(\Lambda;\Lambda')(0;0)}}} \ , \tag{3.11}$$

where S is the modular S-matrix of the minimal model. It can be expressed in terms of the modular S-matrix $S^{(n,k)}$ of the $sl(n)_k$ affine Lie algebra,

$$S_{(\Lambda,\lambda;\Lambda')(L,l;L')} = nS^{(n,k)}_{\Lambda L}\overline{S^{(n,k+1)}_{\Lambda' L'}}S^{(n,1)}_{\lambda l} \ . \tag{3.12}$$

The S-matrix depends on Λ and Λ' only via the combination $d_{\Lambda+\rho,\Lambda'+\rho}$, and can be expressed as (see (B.10))

$$S_{(\Lambda;\Lambda')(L;L')} = \mathcal{N}\sum_{w,w'\in W}\epsilon(ww')e^{-2\pi i t^{-1}\left(w(d_{\Lambda+\rho,\Lambda'+\rho}),L+\rho\right)}e^{2\pi i\left(w'(d_{\Lambda+\rho,\Lambda'+\rho}),L'+\rho\right)} \ , \tag{3.13}$$

where

$$\mathcal{N} = n^{-1/2}\left((k+n)(k+n+1)\right)^{-(n-1)/2} \ . \tag{3.14}$$

In total we arrive at the bulk one-point coefficient

$$\begin{aligned} U^M_{(L;L')}(\Lambda;\Lambda') = A_M(d_{\Lambda+\rho,\Lambda'+\rho}) &\sum_{w\in W}\epsilon(w)e^{-2\pi i t^{-1}\left(w(d_{\Lambda+\rho,\Lambda'+\rho}),L+\rho\right)} \\ &\times\sum_{w'\in W}\epsilon(w')e^{2\pi i\left(w'(d_{\Lambda+\rho,\Lambda'+\rho}),L'+\rho\right)} \ , \end{aligned} \tag{3.15}$$

with $A_M(d)$ given by

$$
\begin{aligned}
A_M(d) &= \mathcal{N}^{1/2}\left(\sum_{w\in W}\epsilon(w)e^{-2\pi i t^{-1}(w(d),\rho)}\right)^{-1/2} \\
&\quad\times\left(\sum_{w'\in W}\epsilon(w)e^{-2\pi i(w'(d),\rho)}\right)^{-1/2} &(3.16)\\
&= \mathcal{N}^{1/2}\prod_{e>0}\left(4\sin\left(\pi t^{-1}(e,d)\right)\sin\left(\pi(e,d)\right)\right)^{-1/2}\ , &(3.17)
\end{aligned}
$$

where we used the Weyl denominator formula (see e.g. [29]). The product runs over the positive roots of $sl(n)$.

3.3 One-point functions for twisted boundary conditions

For $n \geq 3$, the $sl(n)$ algebra has an outer automorphism ω coming from the reflection symmetry of the Dynkin diagram. Correspondingly, the coset algebra also has an outer automorphism, and we can look for boundary conditions that glue the right-moving and left-moving currents with a twist given by this automorphism.

The twisted boundary states for $SU(n)$ WZW models have been constructed in [30]. From this construction it is straightforward to obtain the twisted boundary states in the associated coset models [31, 32] (see also [33] for a discussion of boundary conditions in W_n minimal models).

The details of the construction depend on n being even or odd. The case of even n is technically more complicated, because in the standard construction of twisted coset boundary states one has to do a fixed-point resolution. Although this can be solved in a straightforward way, we shall concentrate here on the case of odd $n = 2m+1$, where these technical problems are absent.

The twisted boundary states of the $sl(2m+1)$ theories at level k can be labelled by symmetric $sl(2m+1)$-weights $L = (L_1, \ldots, L_m, L_m, \ldots, L_1)$ with $2\sum_{i=1}^m L_i \leq k$ (one should think of them as labels of representations of the twisted affine Lie algebra $A_{2m}^{(2)}$). The coefficient of the bulk one-point functions in the $SU(2m+1)$ WZW models for a twisted boundary condition L is

$$U^M_{\omega,L}(\Lambda) = \frac{\psi^{(n,k)}_{L\Lambda}}{\sqrt{S^{(n,k)}_{0\Lambda}}}\delta_{\Lambda,\Lambda^+}\ . \tag{3.18}$$

Here, ψ is the twisted S-matrix (given in eq. (B.18) in the appendix B.2). Only those bulk-fields ϕ_Λ can couple that are invariant under the automorphism, i.e. which are labelled by self-conjugate representations $\Lambda = \Lambda^+$.

The twisted boundary conditions in the coset theory are then given by three symmetric labels L, ℓ, L' at levels k, 1 and $k+1$, respectively. Note that ℓ can only take the value

$\ell = (0, \ldots, 0)$, so that we can label the boundary states just by $(L; L')$. There are no selection or identification rules for the twisted coset boundary states.
The one-point coefficients are given by

$$U^M_{\omega,(L;L')}(\Lambda, \lambda; \Lambda') = \frac{\psi^{(n,k)}_{L\Lambda} \psi^{(n,1)}_{0\lambda} \overline{\psi}^{(n,k+1)}_{L'\Lambda'}}{\sqrt{n S^{(n,k)}_{0\Lambda} S^{(n,1)}_{0\lambda} S^{(n,k+1)}_{0\Lambda'}}} \delta_{(\Lambda,\lambda;\Lambda'),(\Lambda^+,\lambda^+;\Lambda'^+)} . \tag{3.19}$$

Only those bulk fields couple that are labelled by self-conjugate representations. In the above formula it is understood that in the field identification orbit of a self-conjugate coset representation one chooses the unique representative that consists itself of self-conjugate labels $\Lambda = \Lambda^+$, $\lambda = \lambda^+$ and $\Lambda' = \Lambda'^+$. Note that for odd n the only self-conjugate label λ at level 1 is $\lambda = (0, \ldots, 0)$.

We want to rewrite the one-point functions in a way that is more useful when we take the limit $k \to \infty$, namely we want to express it such that the field labels $(\Lambda; \Lambda')$ only enter in the combination $d_{\Lambda+\rho,\Lambda'+\rho}$. To rewrite the product of the twisted S-matrices we use eq. (B.25) from the appendix, and we arrive at the following formula for the one-point coefficient for self-conjugate bulk labels $\Lambda = \Lambda^+$, $\Lambda' = \Lambda'^+$,

$$\begin{aligned} U^M_{\omega,(L;L')}(\Lambda; \Lambda') = n^{1/2} \left((k+n)(k+n+1)\right)^{m/2} A_M(d_{\Lambda+\rho,\Lambda'+\rho}) \\ \times \sum_{w,w' \in W^\omega} \epsilon(w)\epsilon(w') e^{-2\pi i \left(t^{-1} w(L+\rho) - w'(L'+\rho), d_{\Lambda+\rho,\Lambda'+\rho}\right)} , \end{aligned} \tag{3.20}$$

where $A_M(d)$ was given in (3.17). The subgroup $W^\omega \subset W$ consists of all those Weyl transformations that leave the subspace of symmetric weights invariant.

4 The limit of the minimal models

4.1 Bulk spectrum

As we have seen, the spectrum of a minimal model is labelled by two integral, dominant weights Λ, Λ' of an affine $sl(n)$ algebra at level k and $k+1$, respectively. Their conformal weight (see eq. (3.7)), and actually all their higher W-charges are determined [34] by the combination $d_{\Lambda+\rho,\Lambda'+\rho}$, or better by its Weyl orbit. After rotating $d_{\Lambda+\rho,\Lambda'+\rho}$ to the fundamental Weyl chamber, these vectors approach a uniform distribution in the whole fundamental Weyl chamber in the limit $k \to \infty$. The spectrum hence becomes continuous in this limit, and the primary states are labelled by vectors d in the fundamental Weyl chamber. Following the strategy of [3] for the $sl(2)$ case, we want to define fields ϕ_d in the limit theory as an average over fields $\phi_{(\Lambda;\Lambda')}$ whose values $d_{\Lambda+\rho,\Lambda'+\rho}$ are close to d in the limit (modulo a Weyl transformation). As an approximation to the fields ϕ_d, we introduce the averaged fields

$$\phi^{(\epsilon,k)}_d = \frac{1}{|N(d,\epsilon,k)|} \sum_{(\Lambda;\Lambda') \in N(d,\epsilon,k)} \phi_{(\Lambda;\Lambda')} , \tag{4.1}$$

where

$$N(d,\epsilon,k) = \{(\Lambda;\Lambda') : \exists\ w \in W \text{ s.t. } |(w(d_{\Lambda+\rho,\Lambda'+\rho}))_i - d_i| < \epsilon/2 \ \text{ for } i=1,\ldots,n-1\}\ . \tag{4.2}$$

In appendix A we analyse the structure of the sets $N(d,\epsilon,k)$. In particular, we show that for any d in the interior of the fundamental Weyl chamber, there is an ϵ_d such that for $\epsilon < \epsilon_d$ the cardinality $|N(d,\epsilon,k)|$ behaves for large k as

$$|N(d,\epsilon,k)| = (\epsilon(k+n+1))^{n-1} + \mathcal{O}\left((k+n+1)^{n-2}\right)\ . \tag{4.3}$$

The leading term is independent of d; therefore the set $d_{\Lambda+\rho,\Lambda'+\rho}$ (better: the set of their representatives in the fundamental chamber) assumes a uniform distribution in the limit. Although we have chosen here a very specific "ϵ-box" of d to define the average, the results will be independent of the shape of the neighbourhood of d that is used.

The correlators of the averaged fields $\phi_d^{(\epsilon,k)}$ do not have a well behaved limit. On the other hand, we have the freedom to change the normalisation of the fields, as well as to rescale the correlators (corresponding to a rescaling of the vacuum state). Let us denote the field rescaling by a factor α, and the vacuum rescaling by a factor β. A bulk two-point function on the sphere is then rescaled by $\alpha^2\beta^2$, and a bulk one-point function on the disk by $\alpha\beta$ (for a more detailed discussion of these rescalings see [3, 6]). As these are the only two correlators we are discussing in this work, we cannot disentangle the contribution of the different rescalings, and we will just consider the rescaling of these correlators by the combination $\gamma = \alpha\beta$.
The bulk two-point function in the limit theory is then given by

$$\langle\phi_{d_1}(z)\phi_{d_2}(w)\rangle = \lim_{\epsilon\to 0}\lim_{k\to\infty}(\gamma(k,n))^2\langle\phi_{d_1}^{(\epsilon,k)}(z)\phi_{d_2}^{(\epsilon,k)}(w)\rangle\ . \tag{4.4}$$

Here, we chose the normalisation factor $\gamma(k,n)$ to be independent of d. We will now determine γ by the requirement that the bulk two-point function in the limit should be given by

$$\langle\phi_{d_1}(z)\phi_{d_2}(w)\rangle = \delta(d_1 - d_2^+)|z-w|^{-4h_{d_1}}\ , \tag{4.5}$$

where d_2^+ is the label conjugate to d_2, i.e. $(d_2^+)_i = (d_2)_{n-i}$. The conformal weight h_d is obtained as the $k\to\infty$ limit of (3.7),

$$h_d = \frac{1}{2}d^2\ . \tag{4.6}$$

When we evaluate (4.4) by using the expression (3.10) for the two-point function in the minimal models, we obtain

$$\begin{aligned}
\langle \phi_{d_1}(z)\phi_{d_2}(w)\rangle &= \lim_{\epsilon\to 0}\lim_{k\to\infty}(\gamma(k,n))^2(\epsilon(k+n+1))^{-2(n-1)} \\
&\quad\times \sum_{(\Lambda_1;\Lambda_1')\in N(d_1,\epsilon,k)}\ \sum_{(\Lambda_2;\Lambda_2')\in N(d_2,\epsilon,k)} \langle \phi_{(\Lambda_1;\Lambda_1')}(z)\phi_{(\Lambda_2;\Lambda_2')}(w)\rangle \qquad (4.7)\\
&= \lim_{\epsilon\to 0}\lim_{k\to\infty}(\gamma(k,n))^2(\epsilon(k+n+1))^{-2(n-1)} \\
&\quad\times \sum_{(\Lambda_1;\Lambda_1')\in N(d_1,\epsilon,k)\cap N(d_2,\epsilon,k)^+} |z-w|^{-4h_{\Lambda_1,\Lambda_1'}} \qquad (4.8)\\
&= \lim_{\epsilon\to 0}\lim_{k\to\infty}(\gamma(k,n))^2(k+n+1)^{-(n-1)} \\
&\quad\times \prod_{i=1}^{n-1}\left(\epsilon^{-2}(\epsilon-|d_{1,i}-d_{2,i}|)\Theta(\epsilon-|d_{1,i}-d_{2,i}|)\right)|z-w|^{-4h_{d_1}}\ . \qquad (4.9)
\end{aligned}$$

In the last step we used the result (A.12) for the intersection of the two sets $N(d_i,\epsilon,k)$. The Heaviside function $\Theta(x)$ is defined to be 1 for $x>0$ and 0 otherwise. The ϵ-dependent term leads to a delta distribution in the limit,

$$\lim_{\epsilon\to 0}\epsilon^{-2}(\epsilon-|x|)\Theta(\epsilon-|x|)=\delta(x)\ . \qquad (4.10)$$

The coefficients d_i, $i=1,\ldots,n-1$, are the coordinates of d with respect to the fundamental weights ω_i, which do not form an orthonormal basis. The standard inner product on the weight space,

$$(d,d')=\sum_{i=1}^{n-1}d_iQ_{ij}d_j'\ , \qquad (4.11)$$

is given by the quadratic form matrix Q with $\det Q=n^{-1}$, so that the integration measure is

$$d^{n-1}d=\frac{1}{\sqrt{n}}\prod_{i=1}^{n-1}dd_i\ . \qquad (4.12)$$

The delta distribution on the weight space is therefore given by

$$\delta(d_1-d_2)=\sqrt{n}\prod_{i=1}^{n}\delta(d_{1,i}-d_{2,i})\ . \qquad (4.13)$$

If we choose

$$\gamma(k,n)=n^{1/4}(k+n+1)^{(n-1)/2}\ , \qquad (4.14)$$

we obtain the canonically normalised two-point function (4.5) in the limit.

4.2 Untwisted boundary conditions

Let $s = (s_1, \ldots, s_{n-1}) = \sum_i s_i \omega_i$ be a vector in weight space in the fundamental Weyl chamber, so that the coefficients s_i of the fundamental weights are real non-negative numbers. We decompose the vector s into its integer $\lfloor s \rfloor$ part and its fractional part $\{s\}$ (meaning just to take integer and fractional parts of the coefficients s_i). Then we consider the boundary conditions

$$(L_1; L_2)(s, k) = (\lfloor s \rfloor + \lfloor k\{s\} \rfloor, \lfloor k\{s\} \rfloor) \ . \tag{4.15}$$

Notice that the labels L_1, L_2 can lie outside of the fundamental affine Weyl chambers at level k and $k+1$, respectively. In that case we reflect the label back to the fundamental affine Weyl chamber by some affine Weyl transformation, and consider the corresponding boundary condition. To find the appropriate bulk one-point function, we observe that the coefficient (3.15) of the one-point function can be evaluated for arbitrary elements in the fundamental Weyl chamber of the finite dimensional algebra $sl(n)$. It coincides with the coefficient for the reflected labels up to a sign, which is determined by the affine Weyl transformation that is necessary to bring the label to the fundamental affine chamber. For large level k, the necessary Weyl elements for the numerator label L and the denominator label L' will coincide, so that their signs cancel. Therefore, for large levels k we can directly use eq. (3.15) for the one-point functions of the boundary conditions (4.15).

When we take the limit $k \to \infty$, we will keep s fixed and scale the boundary labels $(L; L')$ according to (4.15). In the bulk one-point function, we also keep the bulk label combination $d = d_{\Lambda+\rho,\Lambda'+\rho}$ fixed. The exponential term in the bulk one-point function (3.15) then reads

$$e^{-2\pi i t^{-1}\left(w(d_{\Lambda+\rho,\Lambda'+\rho}),L+\rho\right)+2\pi i\left(w'(d_{\Lambda+\rho,\Lambda'+\rho}),L'+\rho\right)} = e^{-2\pi i\left(d,w^{-1}(\lfloor k\{s\}\rfloor)-w'^{-1}(\lfloor k\{s\}\rfloor)\right)+\cdots} \ . \tag{4.16}$$

In the limit $k \to \infty$ we find strongly oscillating terms in the bulk one-point functions. On the other hand, a field ϕ_d in the limit theory is obtained from the average (4.1) over bulk fields $\phi_{(\Lambda;\Lambda')}$ with $d_{\Lambda+\rho,\Lambda'+\rho} \to d$. In averaging the strongly oscillating terms are suppressed, and we only get contributions from the terms with $w_1 = w_2$ for which the oscillating terms cancel (if we assume generic $\{s\}$ – we shall comment on the degenerate case below). The prefactor $\mathcal{N}^{1/2}$ in (3.15) behaves like $k^{-(n-1)/2}$ for large k. Similarly to our Ansatz for the limit of the bulk two-point function in (4.4), we have to rescale the one-point function by the factor $\gamma(k,n)$ (see (4.14)) to obtain the one-point function in the limit theory,

$$\tilde{U}^M_s(d) := \lim_{k\to\infty} \gamma(k,n) A_M(d) \sum_{w\in W} e^{-2\pi i\left(w(d),\lfloor s\rfloor+(t^{-1}-1)\lfloor k\{s\}\rfloor\right)} \tag{4.17}$$

$$= \prod_{e>0} |2\sin(\pi(e,d))|^{-1} \sum_{w\in W} e^{-2\pi i\left(w(s),d\right)} \ . \tag{4.18}$$

Note that the so obtained one-point coefficient is the same on the whole Weyl orbit of d, so it is independent of which representative we choose. When we compare to the one-point functions (2.25) that we obtained from the non-degenerate boundary conditions in Toda theory, we find coincident results if we identify the bulk fields V_{ip} from Toda theory and ϕ_d from the minimal models by

$$\phi_d \leftrightarrow \pm i^{n(n-1)/2} (\pi \mu_{\text{ren}})^{(d,\rho)} V_{id} \ . \tag{4.19}$$

This identification is also consistent with the two-point functions (2.15) and (4.5).

The above result (4.18) was derived for generic s. If $\{s\}$ sits on a boundary of the fundamental Weyl chamber, there are some Weyl reflections that leave it invariant, in other words, the stabiliser group W_s,

$$W_s = \big\{ w \in W \mid w(\{s\}) = \{s\} \big\} \ , \tag{4.20}$$

is non-trivial. In that case, requiring the strongly oscillating terms to cancel leads to the condition $w_2 w_1^{-1} \in W_s$ (instead of $w_1 = w_2$ for generic s). The limit then becomes

$$\tilde{U}^M_s(d) = \prod_{e>0} |2 \sin(\pi(e,d))|^{-1} \sum_{w \in W, w' \in W_s} \epsilon(w') e^{-2\pi i \left(w(s+\rho-w'(\rho)), d \right)} \ . \tag{4.21}$$

This reproduces the degenerate one-point functions $\tilde{U}^{T,W'}_{\kappa,\Omega,\Omega'}$ of Toda theory (see eq. (2.26)) with $W' = W_s$, $\kappa + \Omega = s$ and $\Omega' = 0$.

In the completely degenerate case ($\{s\} = 0$), the boundary labels $(L; L') = (s; 0)$ are kept fixed in the limit. One might expect that one could get more boundary states by choosing arbitrary $(L; L')$ with non-trivial L' and keeping the labels fixed in the limit. These, however, do not lead to new boundary conditions, as we shall see now.
Keeping the labels fixed, we obtain the one-point function

$$\begin{aligned} \tilde{U}^M_{(L;L')}(d) &= \prod_{e>0} |2 \sin(\pi(e,d))|^{-1} \sum_{w \in W} \epsilon(w) e^{-2\pi i \left(w(d), L+\rho \right)} \\ &\quad \times \sum_{w' \in W} \epsilon(w') e^{2\pi i \left(w'(d), L'+\rho \right)} \qquad (4.22) \\ &= \prod_{e>0} |2 \sin(\pi(e,d))| \, \chi_L(-2\pi i d) \chi_{L'}(2\pi i d) \ , \qquad (4.23) \end{aligned}$$

where χ_L is the finite $sl(n)$ character of the representation with highest weight L. The product of the characters is just given by the tensor product rules,

$$\chi_L(-2\pi i d) \chi_{L'}(2\pi i d) = \sum_{L''} N_{L L'^+}{}^{L''} \chi_{L''}(-2\pi i d) \ , \tag{4.24}$$

where L'^+ labels the representation conjugate to L'. For the one-point function coefficients this means

$$\tilde{U}^M_{(L;L')}(d) = \sum_{L''} N_{L L'^+}{}^{L''} \tilde{U}^M_{(L'';0)}(d) \ , \tag{4.25}$$

i.e. the boundary condition $(L;L')$ can be decomposed into a superposition of boundary conditions of the form $(L'';0)$ in the limit.[3]

We have seen that we can reproduce both the generic and the various degenerate boundary conditions that we obtained from Toda theory. The only mismatch seems to be that we do not get the generic Toda boundary condition $\tilde{U}^T_s$ for a vector s whose fractional part is degenerate (and similar situations for partially degenerate boundary conditions). Is there something special about those boundary conditions? Let us take $s = \Omega$ as an integral weight, so its fractional part is completely degenerate. We then claim that the non-degenerate boundary condition $\tilde{U}^T_s$ (given in (2.25)) for this value of s decomposes into an infinite collection of completely degenerate boundary conditions (2.27),

$$\tilde{U}^T_{s=\Omega}(p) = \sum_{\{m_e\}\in\mathbb{N}_0^{n(n-1)/2}} \tilde{U}^T_{\Omega+\sum_{e>0} m_e e,0}(p)\ . \tag{4.26}$$

We rewrite this equation by inserting the expressions (2.25) and (2.27),

$$\sum_{w\in W} e^{-2\pi i(\Omega,w(p))} = \sum_{\{m_e\}\in\mathbb{N}_0^{n(n-1)/2}} \sum_{w'\in W} \epsilon(w') \sum_{w\in W} e^{-2\pi i(\Omega+\sum_{e>0} m_e e+\rho-w'(\rho),w(p))}\ . \tag{4.27}$$

This equality then follows from

$$\sum_{\{m_e\}\in\mathbb{N}_0^{n(n-1)/2}} e^{-2\pi i(\sum_{e>0} m_e e,p)} = \prod_{e>0}\left(\sum_{m_e\geq 0} e^{-2\pi i(m_e e,p)}\right) \tag{4.28}$$

$$= \prod_{e>0} e^{\pi i(e,p)}\left(e^{\pi i(e,p)} - e^{-\pi i(e,p)}\right)^{-1} \tag{4.29}$$

$$= e^{2\pi i(\rho,p)}\left(\sum_{w'\in W}\epsilon(w')e^{2\pi i(w'(\rho),p)}\right)^{-1}\ . \tag{4.30}$$

Here we used the Weyl denominator formula (see e.g. [29]).

When the fractional part of s degenerates, the generic boundary condition turns into a superposition of degenerate ones. This phenomenon is known [5] from the $sl(2)$ case, where in the limit $c \to 1$, the (generic) FZZT boundary condition of Liouville theory decomposes into an infinite array of (degenerate) ZZ boundary conditions for a discrete set of boundary parameters.

4.3 Twisted boundary conditions

We have seen in section 3.3 that twisted boundary conditions in the $sl(2m+1)$ cosets can be labelled by self-conjugate weights of $sl(2m+1)$. To describe twisted boundary

[3] It is known that in a minimal model the boundary state labelled by $(L;L')$ can flow to a superposition of $(L'';0)$ boundary states, and that this boundary renormalisation group flow can be described in perturbation theory in $1/k$ for large levels, becoming shorter and shorter for higher levels [35] (see [36, 37] for the Virasoro case). Also from this perspective it can be expected that these boundary configurations are identified in the limit.

conditions in the limit $k \to \infty$, we now choose a self-conjugate vector $s = s^+$ in the fundamental Weyl chamber of the weight space of $sl(2m+1)$. With this we associate the boundary condition

$$(L; L') = (\lfloor s \rfloor + \lfloor k\{s\} \rfloor; \lfloor k\{s\} \rfloor) \ . \tag{4.31}$$

The weights L and L' are not necessarily in the fundamental affine Weyl chamber of $sl(2m+1)$ at the levels k and $k+1$, respectively. They can be reflected back by the use of the affine Weyl group, or better by the subgroup that maps self-conjugate labels to self-conjugate ones (for details see the discussion at the end of appendix B.2). The coefficients (3.20) of the boundary state are invariant under these reflections up to signs. For large enough level k, the Weyl reflections used for L and L' will coincide so that the signs cancel.

We then define the one-point functions for twisted boundary conditions in the limit theory as the limit of the one-point functions (3.20) of averaged fields (4.1) with boundary conditions (4.31) where we take s fixed. We obtain

$$\tilde{U}^M_{\omega,s}(d) = \lim_{\epsilon\to 0} \lim_{k\to\infty} \gamma(k,n) \frac{1}{|N(d,\epsilon,k)|} \sum_{(\Lambda;\Lambda')\in N(d,\epsilon,k)} U^M_{\omega,(\lfloor s\rfloor+\lfloor k\{s\}\rfloor,\lfloor k\{s\}\rfloor)}(\Lambda;\Lambda') \tag{4.32}$$

$$\begin{aligned} &= n^{1/2} \prod_{e>0} |2\sin(\pi(e,d))|^{-1} \lim_{\epsilon\to 0} \lim_{k\to\infty} k^m \frac{1}{|N(d,\epsilon,k)|} \sum_{(\Lambda;\Lambda')\in N(d,\epsilon,k)} \delta_{(\Lambda;\Lambda')(\Lambda^+;\Lambda'^+)} \\ &\quad \times \sum_{w,w'\in W^\omega} e^{-2\pi i\left(t^{-1}w(\lfloor s\rfloor+\lfloor k\{s\}\rfloor+\rho)-w'(\lfloor k\{s\}\rfloor+\rho),d_{\Lambda+\rho,\Lambda'+\rho}\right)} \ , \end{aligned} \tag{4.33}$$

where the normalisation factor γ was given in (4.14). Similarly to the arguments for the limit of untwisted boundary conditions, we observe a strongly oscillating behaviour if $w \neq w'$ and generic s. The averaging over d suppresses these terms, so that in the limit we are left with the contributions from $w = w'$. On the other hand, the restriction on self-conjugate $(\Lambda; \Lambda')$ restricts the sum over $N(d,\epsilon,k)$ to a subset of size

$$\left|\{(\Lambda;\Lambda') \in N(d,\epsilon,k) \,|\, (\Lambda;\Lambda') = (\Lambda^+;\Lambda'^+)\}\right| = k^m \prod_{j=1}^{m} (\epsilon - |d_i - d_{n-i}|)\, \Theta(\epsilon - |d_i - d_{n-i}|) + \cdots \tag{4.34}$$

where we left out subleading contributions in k (see eq. (A.15)). Upon sending $\epsilon \to 0$, we obtain a product of delta distributions (see (4.10)), and the one-point coefficient is given by

$$\tilde{U}^M_{\omega,s}(d) = n^{1/2} \prod_{e>0} |2\sin(\pi(e,d))|^{-1} \prod_{i=1}^{m} \delta(d_i - d_{n-i}) \sum_{w\in W^\omega} e^{-2\pi i(w(s),d)} \ . \tag{4.35}$$

We would like to rewrite the delta distributions. The weight space V can be decomposed as an orthogonal sum of self-conjugate and anti self-conjugate vectors,

$$V = V_S \oplus V_A \ . \tag{4.36}$$

The measure $d^{2m}v = (d^m v_S)(d^m v_A)$ factorises, and on the symmetric vectors parameterised by $(v_1, \ldots, v_m, v_m, \ldots, v_1)$ the measure takes the form

$$d^m v_S = \prod_{j=1}^{m} dv_j \ , \tag{4.37}$$

because the quadratic form matrix $\tilde{Q}$, $(v_S, v'_S) = \sum_{i,j=1}^{m} v_i \tilde{Q}_{ij} v_j$, has determinant 1 (it is related to the quadratic form matrix $\hat{Q}$ of $sp(2m)$ by $\tilde{Q}_{ij} = 2\hat{Q}_{ij}$, and $\det \hat{Q} = 2^{-m}$). The measure on the full space is given by (4.12). We then have

$$\begin{aligned}\int d^{2m}v \left(n^{1/2} \prod_{j=1}^{m} \delta(v_i - v_{n-i}) \right) f(v) &= \int \prod_{i=1}^{2m} dv_i \prod_{j=1}^{m} \delta(v_i - v_{n-i}) f(v) &(4.38)\\ &= \int \prod_{i=1}^{m} dv_i f(v_S) &(4.39)\\ &= \int d^m v_S f(v_S) &(4.40)\end{aligned}$$

for an arbitrary function f, so that

$$\delta^{(m)}(v_A) = n^{1/2} \prod_{j=1}^{m} \delta(v_i - v_{n-i}) \ . \tag{4.41}$$

The one-point coefficient (4.35) hence becomes

$$\tilde{U}^M_{\omega,s}(d) = \delta^{(m)}(d_A) \prod_{e>0} |2 \sin(\pi(e,d))|^{-1} \sum_{w \in W^\omega} e^{-2\pi i (w(s),d)} \ . \tag{4.42}$$

This is the answer for generic s. If the stabiliser subgroup $W^\omega_s \subset W^\omega$,

$$W^\omega_s = \{ w \in W^\omega \,|\, w(\{s\}) = \{s\} \} \ , \tag{4.43}$$

is non-trivial, we obtain degenerate boundary conditions with one-point functions determined by

$$\tilde{U}^M_{\omega,s}(d) = \delta^{(m)}(d_A) \prod_{e>0} |2 \sin(\pi(e,d))|^{-1} \sum_{w \in W^\omega, w' \in W^\omega_s} \epsilon(w') e^{-2\pi i (w(s+\rho-w'(\rho)),d)} \ . \tag{4.44}$$

In the completely degenerate case ($\{s\} = 0$), the boundary labels $(L; L') = (s; 0)$ are kept fixed in the limit. As in the case of untwisted boundary conditions, one might ask the question whether one obtains further boundary conditions by taking $(L; L')$ fixed in the limit with a non-trivial label L'. These lead to a one-point coefficient

$$\begin{aligned}\tilde{U}^M_{\omega,(L;L')} = \delta^{(m)}(d_A) \prod_{e>0} |2 \sin(\pi(e,d))|^{-1} &\sum_{w \in W^\omega} \epsilon(w) e^{-2\pi i (w(d), L+\rho)}\\ \times &\sum_{w' \in W^\omega} \epsilon(w') e^{2\pi i (w'(d), L'+\rho)} \ . \qquad (4.45)\end{aligned}$$

The symmetric subgroup $W^\omega \subset W$ is isomorphic to the Weyl group $\hat{W}$ of the finite dimensional algebra $sp(2m)$. The sums in (4.45) can therefore rewritten in terms of characters of $sp(2m)$,

$$\tilde{U}^M_{\omega,(L;L')} = \delta^{(m)}(d_A) \prod_{e>0} |2\sin(\pi(e,d))|^{-1} \prod_{\hat{e}>0} \left(2\sin\left(2\pi\left(\hat{e},\hat{d}\right)\right)\right)^2 \\ \times \chi_{\hat{L}}\left(-4\pi i\hat{d}\right)\chi_{\hat{L}'}\left(4\pi i\hat{d}\right) \ , \tag{4.46}$$

where $\hat{L}$, $\hat{L}'$ and $\hat{d}$ are vectors in the weight space of $sp(2m)$ such that $\hat{L}_i = L_i$ for $i = 1,\dots,m$ and so on. By $\hat{e}$ we denote the roots of $sp(2m)$. The product of characters can be decomposed into a sum of characters using the tensor product rules[4] of $sp(2m)$,

$$\chi_{\hat{L}}(-4\pi i\hat{d})\chi_{\hat{L}'}(4\pi i\hat{d}) = \sum_{\hat{L}''} N_{\hat{L}\hat{L}'}{}^{\hat{L}''}\chi_{\hat{L}''}(-4\pi i\hat{d}) \ . \tag{4.47}$$

Notice that the $sp(2m)$ representations are self-conjugate, so that $\chi_{\hat{L}}(\xi) = \chi_{\hat{L}}(-\xi)$. We conclude that in the limit theory the twisted boundary condition labelled by $(L;L')$ can be identified[5] with a superposition of boundary conditions $(L'';0)$,

$$\tilde{U}^M_{\omega,(L;L')}(d) = \sum_{L''=L''+} N_{\hat{L}\hat{L}'}{}^{\hat{L}''}\tilde{U}^M_{\omega,(L'';0)}(d) \ . \tag{4.48}$$

5 Conclusion

We have analysed untwisted boundary conditions in $sl(n)$ Toda CFTs and in W_n minimal models in the limit $c \to n-1$. The expressions for the one-point function in the presence of these boundary conditions agree. Furthermore, we have studied the limit of twisted boundary conditions in W_n minimal models for odd n. The results (4.42) and (4.44) for the twisted one-point functions in the limit theory suggest a generalisation to Toda theory. An obvious guess for the non-degenerate twisted one-point coefficients in Toda theory would be

$$U^T_{\omega,s}(\alpha) = \delta^{(m)}(p_A)A(\alpha)^{-1}\sum_{w\in W^\omega} e^{-2\pi(w(s),\alpha-Q)} \ . \tag{5.1}$$

Here, the boundary parameter s is a symmetric weight vector, $\alpha = Q + ip$, and p_A is the anti-symmetric part of p under conjugation. Similarly one is led to proposals for the degenerate boundary conditions. In particular, the completely degenerate twisted boundary condition is expected to be given by

$$U^T_{\omega,(L,L')}(\alpha) = \delta^{(m)}(p_A)A(\alpha)^{-1}\sum_{w,w'\in W^\omega} \epsilon(w')e^{2\pi i\left(w\left(b(L+\rho)+b^{-1}w'(L'+\rho)\right),\alpha-Q\right)} \ , \tag{5.2}$$

[4] The appearance of the $sp(2m)$ tensor product rules might be expected from the analysis of twisted D-brane charges in $SU(2m+1)$ WZW models [38, 39].

[5] Similarly to the discussion in footnote 3 on page 17, this identification is expected from the work of [35, 38, 40], from which one can show that a twisted boundary state in a minimal model labelled by $(L;L')$ flows to a superposition of boundary states $(L'';0)$, and that this flow is perturbative in $1/k$.

where L and L' are self-conjugate, dominant, integral weights of $sl(n)$. It is not hard to see that these proposals are consistent with the analysis of the light asymptotic limit of such one-point functions in [24].

We have not discussed the boundary spectrum in this work. In the Virasoro case ($n = 2$), it turns out that the boundary spectrum has an interesting band structure [5] that varies with the parameter s. A similar story is expected for higher n. We saw a glimpse of it at the end of section 4.2 when we observed the decomposition (4.26) of generic boundary conditions into infinite collections of degenerate ones when the fractional part $\{s\}$ of the boundary label degenerates. For $\{s\} = 0$ the boundary spectrum therefore becomes discrete, and for general parameters s one expects that the spectrum is contained in some bands which degenerate for integral values of s. The complete analysis will be more complicated than in the $sl(2)$ case, because the fusion rules are more complicated, and in particular non-trivial multiplicities appear that might diverge in the limit. Still, this analysis could be helpful to understand the divergences of the annulus partition functions in [24] also from the minimal model side.

There are several ways to generalise and extend our analysis. First of all, the results should have a straightforward generalisation to minimal models and Toda theories based on other simply-laced Lie algebras. Another extension would be to study defects in minimal models and Toda theories and in their common limit. All maximally symmetric, topological defects in the minimal models can be obtained by the standard constructions, and their limit can be taken following the steps that we used for the boundary conditions. This limit can be compared to the Toda theories. There, untwisted maximally symmetric defects in Toda theories have been described in [23]. The twisted ones should have a very similar form, and should also resemble the twisted boundary conditions (5.1). Indeed, the above form (5.1) suggests the generic twisted topological defect to be

$$\mathcal{O}_s = \int dp_S \frac{\sum_{w\in W^\omega} e^{-2\pi(w(s),\alpha-Q)}}{\prod_{e>0}\left(-4\sin\pi b(\alpha-Q,e)\sin\pi b^{-1}(\alpha-Q,e)\right)} \tag{5.3}$$

$$\times \sum_{\{k\},\{l\}} |\alpha,k;\alpha,l\rangle \otimes \langle\alpha,k;\alpha,l^+| \;, \tag{5.4}$$

where the integral goes over self-conjugate labels $p = i(Q-\alpha)$. The states $|\alpha,k\rangle$ form an orthonormal basis in the W_n representation based on a ground state $|\alpha\rangle$, so that $\sum_k |\alpha,k\rangle\langle\alpha,k|$ is a projector on this representation. Similarly $\sum_l |\alpha,l\rangle\langle\alpha,l^+|$ is a "twisted" projector that implements the charge conjugation twist. This defect would then correspond to having trivial gluing conditions for the holomorphic currents and twisted gluing conditions for the anti-holomorphic ones. These twisted defects could also be of interest for the relation between Toda theories and four-dimensional supersymmetric gauge theories [23, 41].

Another way of extending the analysis of the limit theory would be to consider bulk three-point correlators. In Toda theories they are only known for a subset of primary fields [21, 22]. For minimal models, the structure constants can be obtained from a free field construction [42], but the expressions contain integrals over the screening charges.

It is not known how to explicitly evaluate these integrals for arbitrary fields, but if one of the field labels takes a special form, the integrals can be evaluated following the work of [21]. It would be interesting to compare at least these accessible data in Toda theory and minimal models.

Recently, limits of W_n minimal models have been investigated in the context of AdS/CFT duality [18] (see also [43]). There, in addition to sending the level k to infinity, also the rank of the algebra grows, while the ratio $\lambda = \frac{n}{k+n}$ is kept fixed (λ takes the role of a 't Hooft coupling). Sending first $k \to \infty$ while keeping n fixed, as we did in our analysis, and then sending $n \to \infty$ would correspond to zero 't Hooft coupling $\lambda = 0$. However, the analysis of [18] shows that the $\lambda \to 0$ limit of their theories corresponds to a limit theory that is different from ours, in particular it should have discrete spectrum. The relevant limiting procedure for the $\lambda = 0$ case of [18] therefore seems to be the $k \to \infty$ limit in the sense of [7], followed by the $n \to \infty$ limit.

Acknowledgements

I would like to thank Matthias Gaberdiel and Volker Schomerus for useful comments on the draft of this paper.

A The spectrum of minimal models in the limit

In section 4.1 we introduced the set $N(d, \epsilon, k)$ (see eq. (4.2)) of minimal model labels $(\Lambda; \Lambda')$ whose associated Weyl orbit of the weight vector $d_{\Lambda+\rho,\Lambda'+\rho}$ has a representative $w(d_{\Lambda+\rho,\Lambda'+\rho})$ close to d in the sense that it is contained in an "ϵ-box" around d,

$$|(w(d_{\Lambda+\rho,\Lambda'+\rho}))_i - (d)_i| < \frac{\epsilon}{2} \quad \text{for } j = 1, \ldots, n-1 \ . \tag{A.1}$$

We here want to analyse the structure of these sets, and in particular determine their cardinalities and the cardinalities of their intersections.
First let us look at the structure of $d_{\Lambda+\rho,\Lambda'+\rho}$. We have

$$d_{\Lambda+\rho,\Lambda'+\rho} = (\Lambda - \Lambda') + \frac{1}{k+n+1}(\Lambda' + \rho) \ , \tag{A.2}$$

so that the integer and fractional parts of $d_{\Lambda+\rho,\Lambda'+\rho}$ are given by

$$\lfloor d_{\Lambda+\rho,\Lambda'+\rho} \rfloor = \Lambda - \Lambda' \qquad , \qquad \{d_{\Lambda+\rho,\Lambda'+\rho}\} = \frac{1}{k+n+1}(\Lambda' + \rho) \ . \tag{A.3}$$

We observe that for an integral dominant weight Λ' of $sl(n)$ at level $k+1$, we have the restriction

$$\sum_{i=1}^{n-1} \{d_{\Lambda+\rho,\Lambda'+\rho}\}_i \leq 1 \ . \tag{A.4}$$

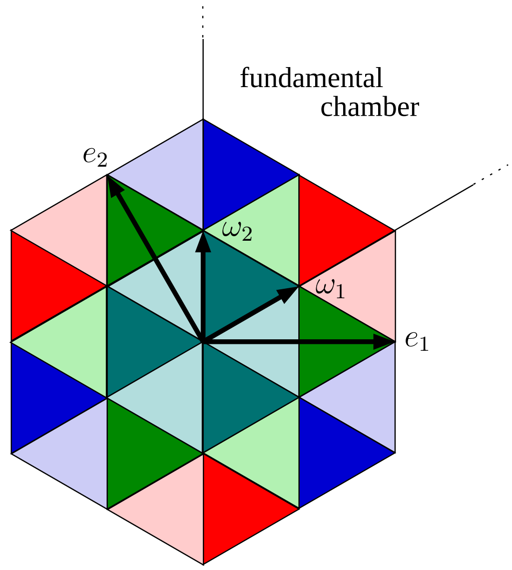

Figure 1: This is the weight space of $sl(3)$ with fundamental weights ω_1, ω_2 and simple roots e_1, e_2. The dark regions contain those vectors d satisfying $\sum_{i=1}^{n-1}\{d_i\} \leq 1$. The Weyl orbit of one triangular region contains three dark triangles with vectors satisying the condition, and three light triangles. For any vector d there are therefore Weyl images $w(d)$ satisfying the condition, and these images are related by the group I_W given in (A.10), which is the $\mathbb{Z}_3$ rotation group in the case of $sl(3)$.

A generic weight vector d does not satisfy this condition, but for any vector d there is a Weyl transformation $w \in W$ such that $w(d)$ is in accord with the restriction,

$$\sum_{i=1}^{n-1}\{w(d)\}_i \leq 1 \ . \tag{A.5}$$

To see this, think of the action of the affine Weyl group at level 1. Its action on the weights is generated by translations by vectors in the (co-)root lattice, which leave the fractional part invariant, and by ordinary Weyl transformations. For any vector d there is an affine Weyl transformation sending d to a vector d' in the fundamental affine chamber C_0 at level 1 such that $d'_i \geq 0$ and $\sum d'_i \leq 1$. We have illustrated this for the case of $sl(3)$ in figure 1.

Assume now that $d' = w(d)$ is such that $\sum_i\{d'\}_i \leq 1$. We further assume that $\{d'\}$ does not sit at the boundary of the affine Weyl chamber C_0, i.e. that $\{d'\}_i > 0$ and $\sum_i\{d'\}_i < 1$; we will comment on the general case later. The labels Λ, Λ' with $d_{\Lambda+\rho,\Lambda'+\rho}$ close to d' in the sense of (A.1) then satisfy

$$\Lambda = \Lambda' + \lfloor d' \rfloor \quad , \quad \{d'\}_i - \frac{\epsilon}{2} < \frac{\Lambda'_i}{k+n+1} < \{d'\}_i + \frac{\epsilon}{2} \ . \tag{A.6}$$

Here we want to assume that ϵ is small enough so that the ϵ-box around $\{d'\}$ is still contained in the interior of C_0. Then the labels Λ' that are allowed by the inequality in (A.6) are dominant weights at level $k+1$. Furthermore we assume that k is large enough so that Λ is a dominant weight at level k.
The number of labels Λ' that satisfy (A.6) is then given by

$$|N(d,\epsilon,k)| = (\epsilon(k+n+1))^{n-1} + \mathcal{O}(k^{n-2}) \ . \tag{A.7}$$

This describes the behaviour of the cardinality at small but fixed ϵ and k going to infinity. Note that the ϵ-box around d' that we used to derive (A.6) is not precisely the Weyl image of the ϵ-box around d, but for the counting only the volume of the box matters.

The result (A.7) was derived for a generic d such that $\{w(d)\}$ is contained in the interior of the fundamental affine Weyl chamber C_0 at level 1 for some $w \in W$. There are two issues where we should have been more careful. In the derivation we only considered one Weyl image $w(d)$, and we have to make sure that we do not get further contributions from other images. On the other hand, we have not considered the identification rules (3.4) for the coset labels, which might lead to an overcounting.

Let us therefore analyse which $w' \in W$ map a vector d' with $\{d'\}$ inside the fundamental chamber C_0 at level 1 to a vector $w'(d')$ with $\{w'(d')\}$ also lying in C_0. We denote the group consisting of such w' by I_W. Given a generic vector d there is always a unique affine Weyl transformation consisting of a Weyl rotation w and a translation by an element α of the (co-)root lattice that maps it to the fundamental affine chamber C_0,

$$w(d) + \alpha \in C_0 \ \Rightarrow \ \{w(d)\} \in C_0 \ . \tag{A.8}$$

The affine Weyl transformations only include translations by vectors of the root lattice. The fractional part $\{d\}$ is also invariant under translations by vectors from the weight lattice. To find another $w' \in W$ such that $\{w'(d)\} \in C_0$ we add any integral weight vector v to d, and then look for the unique affine Weyl transformation (w', α') mapping it to the fundamental chamber,

$$w'(d+v) + \alpha' \in C_0 \ \Rightarrow \ \{w'(d)\} \in C_0 \ . \tag{A.9}$$

In that way we have found another w' that does the job. The number of such Weyl group elements is given by the number of different integral weight vectors $v \in L_W$ modulo the root lattice L_R. For $sl(n)$ this number is given by $|L_R/L_W| = n$. The group I_W then consists of those Weyl elements that map the fundamental affine chamber C_0 back to itself modulo the weight lattice L_W. From (3.5) we see that the action of the simple currents at level 1 precisely has this property. Therefore the group I_W is generated by the Weyl group element w_J given in (3.6),

$$I_W = \{w_J^j \,|\, j = 0, \ldots, n-1\} \ . \tag{A.10}$$

For any vector d we thus find n different Weyl transformations w_i with $\{w_i(d)\} \in C_0$, and they are related by the action of the group I_W (this is illustrated in figure 1). On the other hand, transformations of the vector $d_{\Lambda+\rho,\Lambda'+\rho}$ by a Weyl element $w_J^j \in I_W$ corresponds to considering the field identification of the label pair $(\Lambda;\Lambda')$ by the simple current J^j (see eq. (3.9)). The additional contributions due to the appearance of I_W are precisely cancelled by dividing out the field identifications, and the formula (A.7) is therefore correct.

What does happen if d is non-generic such that $\{d\}$ sits at the boundary of C_0? Let us assume that d itself is in the interior of the fundamental Weyl chamber, and also its ϵ-box. If now $\{d\}$ (or one of its Weyl images) is on the boundary of C_0, this just means that for different parts of the ϵ-box around d we must use different Weyl transformations to bring their fractional parts to C_0. The overall counting does not change.

For vectors d that sit at the boundary of the fundamental chamber, the situation is different because for any arbitrarily small ϵ, different parts of the ϵ-box around d have to be identified and the counting changes. We do not investigate here what this implies for the spectrum of the limit theory.

Let us now discuss the intersection of two sets,

$$N(d_1,\epsilon,k)\cap N(d_2,\epsilon,k)\ , \tag{A.11}$$

where d_1 and d_2 are in the interior of the fundamental Weyl chamber, and ϵ is small enough such that the counting we have done above works both for d_1 and d_2.

Assume that d_1 is such that $\{d_1\}$ is in the interior of C_0. For d_2 we consider the vectors $w(d_2)$ with $\{w(d_2)\}\in C_0$, and all the label pairs Λ,Λ' such that $d_{\Lambda+\rho,\Lambda'+\rho}$ is in the ϵ-box around $w(d_2)$. Now the question is: which of them coincide with labels Λ,Λ' with $d_{\Lambda+\rho,\Lambda'+\rho}$ being contained in the ϵ-box around d_1? To have coincident labels Λ' we need that $\{d_1\}$ and $\{w(d_2)\}$ are close, i.e. their ϵ-boxes must have a non-empty intersection. To also have coincident labels Λ, we need that $\lfloor d_1\rfloor$ and $\lfloor w(d_2)\rfloor$ coincide. Therefore the ϵ-boxes around d_1 and $w(d_2)$ have to overlap, which can only occur for $w=1$.
The counting of coincident labels in the intersection then results in

$$|N(d_1,\epsilon,k)\cap N(d_2,\epsilon,k)| = \prod_{i=1}^{n-1}\left((k+n+1)(\epsilon-|(d_1-d_2)_i|)\Theta(\epsilon-|(d_1-d_2)_i|)\right)+\cdots\ . \tag{A.12}$$

Here, the Heaviside function Θ is defined such that $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ otherwise, and it encodes the condition that the two ϵ-boxes intersect.

When we discuss twisted boundary conditions, we also have to analyse how many self-conjugate labels are contained in $N(d,\epsilon,k)$, i.e. the cardinality of the set

$$N^{\omega}(d,\epsilon,k)=\{(\Lambda;\Lambda')\in N(d,\epsilon,k)\mid (\Lambda;\Lambda')=(\Lambda^+;\Lambda'^+)\}\ . \tag{A.13}$$

A self-conjugate coset label has a representative $(\Lambda;\Lambda')$ with self-conjugate weights $\Lambda=\Lambda^+$, $\Lambda'=\Lambda'^+$. The corresponding vector $d_{\Lambda+\rho,\Lambda'+\rho}$ is then also self-conjugate, and it therefore has to lie in the fundamental Weyl chamber or in one if its images under the action of the invariant subgroup $W^{\omega}\subset W$. For a vector d' and small enough ϵ, there are no self-conjugate $d_{\Lambda+\rho,\Lambda'+\rho}$ in its ϵ-box, unless d' is in one of those Weyl chambers.

Assume therefore that d' is in such a Weyl chamber, and additionally that $\{d'\}$ (and also the fractional parts of vectors in its ϵ-box) is contained in the fundamental affine chamber C_0 at level 1. Then the number of self-conjugate label pairs is given by

$$\begin{aligned}&\big|\{\Lambda=\Lambda^+,\ \Lambda'=\Lambda'^+ \mid |(d'-d_{\Lambda+\rho,\Lambda'+\rho})_i|<\epsilon/2\}\big| \\ &\qquad=\prod_{i=1}^{m}((k+n+1)(\epsilon-|(d')_i-(d')_{n-i}|)\Theta(\epsilon-|(d')_i-(d')_{n-i}|))+\cdots\ , \quad \text{(A.14)}\end{aligned}$$

where $n=2m+1$. We again have to ask whether for a given d there are several such representatives d' on its Weyl orbit. If we adapt the arguments around eqs. (A.8) and (A.9) to our situation, we see that this is not the case, because the self-conjugate sublattice of the weight lattice agrees with the self-conjugate sublattice of the root lattice. The cardinality $|N^\omega(d,\epsilon,k)|$ is therefore determined by the volume of the self-conjugate part of the ϵ-box around d,

$$|N^\omega(d,\epsilon,k)|=\prod_{i=1}^{m}((k+n+1)(\epsilon-|(d)_i-(d)_{n-i}|)\Theta(\epsilon-|(d)_i-(d)_{n-i}|))+\cdots\ . \quad \text{(A.15)}$$

B Modular S-matrices

B.1 Untwisted coset S-matrix

The modular S-matrix for the diagonal coset model $\frac{SU(n)_k\times SU(n)_1}{SU(n)_{k+1}}$ is given by

$$S_{(\Lambda,\lambda;\Lambda')(L,l;L')}=nS^{(n,k)}_{\Lambda L}\overline{S^{(n,k+1)}_{\Lambda'L'}}S^{(n,1)}_{\lambda l}\ . \quad \text{(B.1)}$$

Here, $S^{(n,k)}$ denotes the S-matrix of the affine Lie algebra $\widehat{sl(n)}_k$, which can be written as (see e.g. [29])

$$S^{(n,k)}_{\Lambda L}=i^{n(n-1)/2}n^{-1/2}(k+n)^{-(n-1)/2}\sum_{w\in W}\epsilon(w)e^{-2\pi i\frac{(w(\Lambda+\rho),L+\rho)}{k+n}}\ , \quad \text{(B.2)}$$

with W being the Weyl group and ρ the Weyl vector of $sl(n)$. Therefore the coset S-matrix can be expressed as

$$\begin{aligned}S_{(\Lambda,\lambda;\Lambda')(L,l;L')}&=((k+n)(k+n+1))^{-(n-1)/2}\,S^{(n,1)}_{\lambda l}\\ &\quad\times\sum_{w,w'\in W}\epsilon(w)\epsilon(w')e^{-2\pi i\frac{(w(\Lambda+\rho),L+\rho)}{k+n}}e^{2\pi i\frac{(w'(\Lambda'+\rho),L'+\rho)}{k+n+1}}\ . \quad \text{(B.3)}\end{aligned}$$

The coset S-matrix only depends on Λ and Λ' via their combination

$$d_{\Lambda+\rho,\Lambda'+\rho}=(\Lambda+\rho)-t(\Lambda'+\rho)\ , \quad \text{(B.4)}$$

where $t = \frac{k+n}{k+n+1}$. To see this we rewrite

$$\frac{\Lambda+\rho}{k+n} = t^{-1} d_{\Lambda+\rho,\Lambda'+\rho} - (\Lambda - \Lambda') \tag{B.5}$$

and

$$\frac{\Lambda'+\rho}{k+n+1} = d_{\Lambda+\rho,\Lambda'+\rho} - (\Lambda - \Lambda') \ . \tag{B.6}$$

Inserting this into (B.3), we obtain

$$\begin{aligned} S_{(\Lambda,\lambda;\Lambda')(L,l;L')} = ((k+n)(k+n+1))^{-(n-1)/2}\, S^{(n,1)}_{\lambda l} \sum_{w,w'\in W} \epsilon(ww') e^{-2\pi i t^{-1}\left(w(d_{\Lambda+\rho,\Lambda'+\rho}),L+\rho\right)} \\ \times\, e^{2\pi i\left(w'(d_{\Lambda+\rho,\Lambda'+\rho}),L'+\rho\right)} e^{2\pi i(w(\Lambda-\Lambda'),L+\rho)} e^{-2\pi i(w'(\Lambda-\Lambda'),L'+\rho)} \ . \end{aligned} \tag{B.7}$$

In the last two exponentials we can replace $w(\Lambda - \Lambda')$ by $\Lambda - \Lambda'$ (and similarly for w'): $w(\Lambda - \Lambda')$ differs from $\Lambda - \Lambda'$ by an element of the root lattice L_R, whose scalar product with an integral weight gives an integer leading to a trivial phase in the exponential. We thus arrive at

$$\begin{aligned} S_{(\Lambda,\lambda;\Lambda')(L,l;L')} = ((k+n)(k+n+1))^{-(n-1)/2}\, e^{2\pi i(\Lambda-\Lambda',L-L')} S^{(n,1)}_{\lambda l} \\ \times \sum_{w,w'\in W} \epsilon(ww') e^{-2\pi i t^{-1}\left(w(d_{\Lambda+\rho,\Lambda'+\rho}),L+\rho\right)} e^{2\pi i\left(w'(d_{\Lambda+\rho,\Lambda'+\rho}),L'+\rho\right)} \ . \end{aligned} \tag{B.8}$$

The S-matrix $S^{(n,1)}$ at level 1 is very simple, because for $sl(n)$ all dominant integral weights at level 1 correspond to simple currents [44]. We find

$$S^{(n,1)}_{\lambda l} = e^{-2\pi i(\lambda,l)} S^{(n,1)}_{00} = e^{-2\pi i(\lambda,l)} n^{-1/2} \ . \tag{B.9}$$

From the selection rules (3.3) we know that $\Lambda - \Lambda' + \lambda$ and $L - L' + l$ are in the root lattice L_R. Therefore the phases in front of the sum in (B.8) cancel and we find

$$\begin{aligned} S_{(\Lambda,\lambda;\Lambda')(L,l;L')} = n^{-1/2}\, ((k+n)(k+n+1))^{-(n-1)/2} \\ \times \sum_{w,w'\in W} \epsilon(ww') e^{-2\pi i t^{-1}\left(w(d_{\Lambda+\rho,\Lambda'+\rho}),L+\rho\right)} e^{2\pi i\left(w'(d_{\Lambda+\rho,\Lambda'+\rho}),L'+\rho\right)} \ . \end{aligned} \tag{B.10}$$

B.2 Twisted S-matrix

Twisted S-matrices describe the behaviour of characters of twisted affine Lie algebras under modular transformation [45]. We are here interested in the case of $sl(2m+1)$. In [46] (see also [47]) it was observed that the twisted S-matrix of $sl(2m+1)$ is related to the untwisted S-matrix of $so(2m+1)$ at level $k+2$, and to the untwisted S-matrix of $sp(2m)$ at level $(k-1)/2$ (for odd level k). Here we want to express the twisted S-matrix in terms of the untwisted S-matrix of $sp(2m)$ at level $k+m$. Our starting point is the determinant formula that can be found e.g. in [46]. We label the twisted representations by a symmetric label $L = (L_1, \ldots, L_m, L_m, \ldots, L_1)$. For a twisted label L and a symmetric weight $\Lambda = (\Lambda_1, \ldots, \Lambda_m, \Lambda_m, \ldots, \Lambda_1)$ the entry of the twisted S-matrix is given by

$$\psi^{(2m+1,k)}_{L\Lambda} = (-1)^{\frac{m(m-1)}{2}} \frac{2^m}{(k+2m+1)^{m/2}} \det \left[\sin\left(\frac{2\pi L[i]\Lambda[j]}{k+2m+1} \right) \right]_{1 \leq i,j \leq m} , \tag{B.11}$$

where

$$L[i] = m + 1 - i + \sum_{j=i}^{m} L_j , \tag{B.12}$$

and similarly for $\Lambda[j]$. For the S-matrix of $sp(2m)$ at level $k+m$, the determinant formula is [48]

$$\hat{S}^{(2m,k+m)}_{\hat{L}\hat{\Lambda}} = (-1)^{\frac{m(m-1)}{2}} \frac{2^{m/2}}{(k+2m+1)^{m/2}} \det \left[\sin\left(\frac{\pi \hat{L}[i]\hat{\Lambda}[j]}{k+2m+1} \right) \right]_{1 \leq i,j \leq m} , \tag{B.13}$$

where $\hat{L}$ and $\hat{\Lambda}$ are m-tuples labelling $sp(2m)$ weights. The two determinants in (B.11) and (B.13) are very similar, and we find

$$\psi^{(2m+1,k)}_{L\Lambda} = 2^{m/2} \hat{S}^{(2m,k+m)}_{\hat{L},2\hat{\Lambda}+\hat{\rho}} . \tag{B.14}$$

Here, the hat ($\hat{\ }$) denotes the map that sends a symmetric $sl(2m+1)$ weight to a $sp(2m)$-weight,

$$\hat{\ } : \Lambda = (\Lambda_1, \ldots, \Lambda_m, \Lambda_m, \ldots, \Lambda_1) \mapsto \hat{\Lambda} = (\Lambda_1, \ldots, \Lambda_m) . \tag{B.15}$$

Under this map the Weyl vector ρ of $sl(2m+1)$ is mapped to the Weyl vector $\hat{\rho}$ of $sp(2m)$.

Using standard expressions for untwisted modular S-matrices (see e.g. [29]), we can rewrite (B.14) as

$$\psi^{(2m+1,k)}_{L\Lambda} = i^{m^2} (k+2m+1)^{-m/2} \sum_{w \in \hat{W}} \epsilon(w) e^{-2\pi i \frac{(w(\hat{L}+\hat{\rho}), 2\hat{\Lambda}+2\hat{\rho})}{k+2m+1}} , \tag{B.16}$$

where $\hat{W}$ is the Weyl group of $sp(2m)$. The scalar product appearing in the exponential is the standard quadratic form of the $sp(2m)$ algebra. It is related to the quadratic form on the $sl(2m+1)$ weight space by [47]

$$(L, \Lambda)_{sl(2m+1)} = 2(\hat{L}, \hat{\Lambda})_{sp(2m)} . \tag{B.17}$$

The action of the Weyl group $\hat{W}$ on $sp(2m)$ weights induces an action on symmetric $sl(2m+1)$ weights which corresponds precisely to the action of the subgroup $W^\omega \subset W$ of all $sl(2m+1)$ Weyl transformations that map symmetric weights to symmetric weights. Therefore we can rewrite the twisted S-matrix as

$$\psi^{(2m+1,k)}_{L\Lambda} = i^{m^2} (k+2m+1)^{-m/2} \sum_{w \in W^\omega} \epsilon(w) e^{-2\pi i \frac{(w(L+\rho), \Lambda+\rho)}{k+2m+1}} , \tag{B.18}$$

which coincides with the expression given in [30].

When we discuss boundary conditions in the limit of minimal models, we want to make sense of symmetric boundary labels L that are outside of the usual range and do not satisfy $\sum_{i=1}^{n-1} L_i \leq k$. The formulae (B.16) and (B.18) can also be applied for those labels L. In particular, in the $sp(2m)$ language, the formula (B.16) giving the twisted S-matrix in terms of $\hat{L}$ is invariant under a shifted Weyl reflection by $w \in \hat{W}$ (up to a sign), and under translations by $(k+n)/2$-multiples of co-root vectors. These transformations can be interpreted as the shifted action of the affine Weyl group at level $(k-1)/2$ (this action also makes sense for even k), and these transformations can be used to map any label to some $\hat{L}$ satisfying $\sum_{i=1}^{m} \hat{L}_i \leq k/2$. In the $sl(2m+1)$ language this also has a natural interpretation. The lattice spanned by half the coroot vectors of $sp(2m)$ coincides with the weight lattice of $sp(2m)$. Translations of $\hat{L}$ by the weight lattice of $sp(2m)$ correspond to translations of L by the symmetric (self-conjugate) part of the weight lattice of $sl(2m+1)$. This in turn coincides with the symmetric part of the root lattice of $sl(2m+1)$. Therefore we can use the symmetric part of the affine Weyl group to bring any symmetric label to some L lying in the usual range.

B.3 Twisted coset S-matrix

The twisted coset S-matrix for the $sl(2m+1)$ diagonal coset model is given by

$$\psi_{(L,L')(\Lambda,\Lambda')} = \psi^{(2m+1,k)}_{L\Lambda} \psi^{(2m+1,1)}_{00} \overline{\psi}^{(2m+1,k+1)}_{L'\Lambda'} \quad , \tag{B.19}$$

where $\Lambda = \Lambda^+$ and $\Lambda' = \Lambda'^+$ are self-conjugate labels. The twisted S-matrix for the level 1 part is trivial, $\psi^{(2m+1,1)}_{00} = 1$, and can be omitted.
Using (B.16) the twisted S-matrix takes the form

$$\begin{aligned}\psi_{(L,L')(\Lambda,\Lambda')} &= \left((k+2m+1)(k+2m+2)\right)^{-m/2} \\ &\quad \times \sum_{w,w' \in \hat{W}} \epsilon(ww') e^{-2\pi i \frac{(w(\hat{L}+\hat{\rho}),2\hat{\Lambda}+2\hat{\rho})}{k+2m+1}} e^{2\pi i \frac{(w'(\hat{L}'+\hat{\rho}),2\hat{\Lambda}'+2\hat{\rho})}{k+2m+2}} \quad .\end{aligned} \tag{B.20}$$

We rewrite

$$\frac{\hat{\Lambda}+\hat{\rho}}{k+2m+1} = t^{-1}\left((\hat{\Lambda}+\hat{\rho}) - t(\hat{\Lambda}'+\hat{\rho})\right) - (\hat{\Lambda}-\hat{\Lambda}') \tag{B.21}$$

and

$$\frac{\hat{\Lambda}'+\rho}{k+2m+2} = (\hat{\Lambda}+\rho) - t(\hat{\Lambda}'+\rho) - (\hat{\Lambda}-\hat{\Lambda}') \ , \tag{B.22}$$

where $t = \frac{k+2m+1}{k+2m+2}$. For the combination of $\hat{\Lambda}$ and $\hat{\Lambda}'$ we introduce the notation

$$\hat{d}_{\Lambda+\rho,\Lambda'+\rho'} = (\hat{\Lambda}+\hat{\rho}) - t(\hat{\Lambda}'+\hat{\rho}) \ . \tag{B.23}$$

This allows us to express the S-matrix as

$$\begin{aligned}\psi_{(L,L')(\Lambda,\Lambda')} &= ((k+2m+1)(k+2m+2))^{-m/2} \\ &\quad \times \sum_{w,w'\in\hat{W}} \epsilon(ww')e^{-2\pi i\left(t^{-1}w(\hat{L}+\hat{\rho})-w'(\hat{L}'+\hat{\rho}),2\hat{d}_{\Lambda+\rho,\Lambda'+\rho'}\right)} \\ &\quad \times e^{2\pi i\left(w(\hat{L}+\hat{\rho})-w'(\hat{L}'+\hat{\rho}),2(\hat{\Lambda}-\hat{\Lambda}')\right)} \ .\end{aligned} \tag{B.24}$$

The phase in the last line is trivial: the quadratic form on the weight lattice of $sp(2m)$ takes values in $\frac{1}{2}\mathbb{Z}$, and $2(\hat{\Lambda}-\hat{\Lambda}')$ is an even weight vector, therefore it has integer scalar product with any vector in the weight lattice. Finally we express everything in terms of symmetric $sl(2m+1)$ labels (similar to (B.18)) and we obtain

$$\begin{aligned}\psi_{(L,L')(\Lambda,\Lambda')} &= ((k+2m+1)(k+2m+2))^{-m/2} \\ &\quad \times \sum_{w,w'\in W^{\omega}} \epsilon(ww')e^{-2\pi i\left(t^{-1}w(L+\rho)-w'(L'+\rho),d_{\Lambda+\rho,\Lambda'+\rho'}\right)} \ .\end{aligned} \tag{B.25}$$

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