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Review of AdS/CFT Integrability, Chapter II.2: Quantum Strings in $AdS_5 \times S^5$

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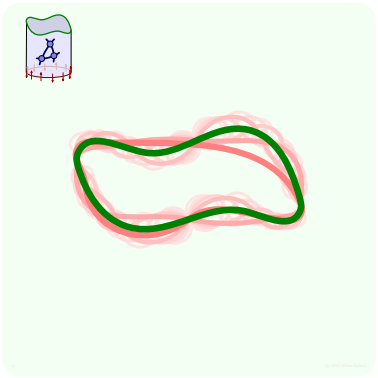


Abstract: We review the semiclassical analysis of strings in $AdS_5 \times S^5$ with a focus on the relationship to the underlying integrable structures. We discuss the perturbative calculation of energies for strings with large charges, using the folded string spinning in $AdS_3 \subset AdS_5$ as our main example. Furthermore, we review the perturbative light-cone quantization of the string theory and the calculation of the worldsheet S-matrix.

1 Introduction

The semiclassical study of strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ has played a key role in extending our understanding of the AdS/CFT correspondence beyond the supergravity approximation. The analysis of quantum corrections to the energies of strings with large charges has gone hand-in-hand with the discovery and application of the integrable structures present in the duality. In particular, it has been important for comparison with the Bethe ansatz predictions for the anomalous dimensions of long operators and to understand the finite size corrections of short operators.

Due to the presence of Ramond-Ramond fields one must make use of the Green-Schwarz formalism for the string action, adapted to the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry [1] (see [2] for a brief introduction), [1] which to quadratic order in fermionic fields is

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma h^{ab}\, G_{\mu\nu}\partial_a x^\mu \partial_b x^\nu - i\frac{\sqrt{\lambda}}{2\pi}\int d^2\sigma \left(h^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ}\right)\bar{\theta}^I \varrho_a D_b \theta^J. \tag{1.1}$$

Here we have used the rescaled worldsheet metric $h^{ab} = \sqrt{-g}g^{ab}$, the induced Dirac matrices $\varrho_a = \partial_a x^\mu E_\mu{}^A \Gamma_A$ and the covariant derivative

$$D_a\theta^I = \left(\partial_a + \frac{1}{4}\partial_a x^\mu \omega_\mu{}^{AB}\Gamma_{AB}\right)\theta^I + \frac{1}{2}\varrho_a \Gamma_{01234}\epsilon^{IJ}\theta^J \ . \tag{1.2}$$

Directly quantizing this action is beyond current methods and one must take a perturbative approach, expanding about a given classical solution in powers of the effective string tension, $\sqrt{\lambda}$. A classical solution is characterised by the conserved charges corresponding to the AdS energy, E, two AdS spins, S_i, and three angular momenta of the sphere, J_s, in addition to any parameters specifying further properties of the string such as non-trivial winding. The Virasoro conditions provide a constraint on these parameters and for the solutions we are interested in we can express the string energy as a function of the remaining charges: $E = E(S_i, J_s; k_r)$. In the semiclassical approach one takes a string solution where one or more of the rescaled charges are finite, $\mathrm{S}_i = \frac{S_i}{\sqrt{\lambda}}$ or $\mathrm{J}_s = \frac{J_s}{\sqrt{\lambda}}$, and computes the worldsheet loop corrections to the energy as an expansion in large tension,

$$E = \sqrt{\lambda}\Big[\mathrm{E}_0(\mathrm{S}_i, \mathrm{J}_s; k_r) + \frac{1}{\sqrt{\lambda}}\mathrm{E}_1(\mathrm{S}_i, \mathrm{J}_s; k_r) + \frac{1}{\lambda}\mathrm{E}_2(\mathrm{S}_i, \mathrm{J}_s; k_r) + \dots\Big] \ . \tag{1.3}$$

In general, calculating these corrections involves gauge-fixing the diffeomorphism and kappa gauge invariance, and studying the fluctuations of the fields – bosonic, fermionic and conformal ghosts from gauge fixing – about the classical solution. An important point is that all UV divergences of the worldsheet theory cancel and, relatedly, the conformal anomaly vanishes once the contribution from the path integral measure is accounted for; thus the semiclassical expansion is well defined. On general grounds this is expected as the string theory is of critical dimension and it was explicitly shown at

[1] One can also study strings in different backgrounds, $\mathrm{AdS}_4\times\mathbb{C}\mathrm{P}^3$ is of particular interest where many results parallel the $\mathrm{AdS}_5\times\mathrm{S}^5$ case. See [3].

one-loop in [4] and [5]. [2] A solution which has played a particularly important role in our quantitative understanding of the AdS/CFT duality is the spinning folded string in AdS_5, introduced in [6] and whose semiclassical analysis was initiated in [5]. In the large spin limit [6–8], the difference between its energy E and spin S scales like $\ln S$ with the coefficient being the universal scaling function, $f(\lambda)$. This function provided the first example of a result interpolating between weak and strong coupling which can be calculated from the all-order asymptotic Bethe ansatz (ABA) [9, 10] (see [11, 12] for a review of the all-order ABA). The one and two-loop semiclassical calculations [5, 13–15] have been shown to match the predictions of the string ABA [16–18] using the one-loop phase factor [19–21] and its all-order generalisation [22, 10] in a very non-trivial test of the duality and its quantum integrability (see [23] for a review of the ABA calculation and references). We will discuss this solution, its generalisations and related solutions in Sec. 2.3. While for the most part we focus on closed strings, similar semiclassical analysis has also been applied to open strings: duals to cuspy Wilson loops, to Wilson loops describing "quark–anti-quark" systems, [4, 24], to Wilson loops describing high energy scattering [25] and more recently, dimensionally reduced amplitudes [26].

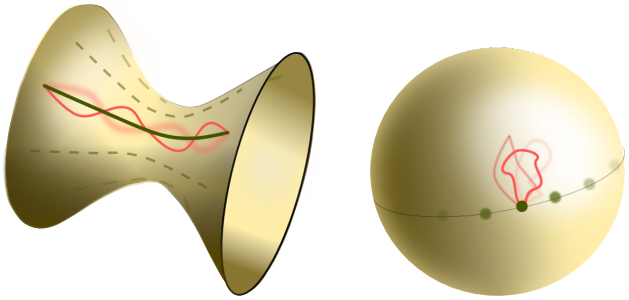
Another solution which has played a crucial role in our understanding of the quantum string in $AdS_5 \times S^5$ is the BMN string, [27] [6] see also [2], which is the BPS solution dual to the ferromagnetic vacuum of the spin chain description of the gauge theory. This solution is the natural vacuum state in the light-cone quantization of the worldsheet theory where the physical Hamiltonian, $H_{l.c.}$, is proportional to $P_- = E - J$, with J one of the sphere angular momenta. [3] Finding quantum string energies, E, corresponds to computing the spectrum of the $H_{l.c.}$. Unfortunately the exact light-cone Hamiltonian has a non-polynomial form [30, 34] and is not a suitable starting point for "first-principles" quantization. One can, however, solve for the spectrum perturbatively. At leading order the theory is simply that of free massive fields [27, 35] while at subleading orders [36, 29, 30, 37, 32] the interactions are somewhat more complicated and, due to the gauge fixing, do not respect worldsheet Lorentz invariance. Alternatively, as the worldsheet theory is integrable, it is possible to find the spectrum of the decompactified theory, via the ABA, by calculating the worldsheet S-matrix [17], [16, 18]. A review of the exact form of this S-matrix and its properties can be found [12], in this review we will restrict ourselves to briefly describing its perturbative calculation (for a more thorough review see [38]).

2 Quantum spinning strings

We will, as an illustrative example, consider the the folded spinning string [6], [5], see also [2]. This solution describes a string extended and rotating with spin, S, in an AdS_3

[2] Particular care must be taken with the fermionic fields. Importantly, they couple to the worldsheet metric rather than the zweibein and so contribute to the conformal anomaly four times the usual 2-d Majorana fermion amount.

[3] There are essentially two ways to fix the light-cone gauge in $AdS_5 \times S^5$, which differ by picking inequivalent light-cone geodesics. In one case, which is possible only in the Poincaré patch, the light-cone directions lie entirely in AdS_5 [28]. In our case the light-cone is shared between AdS_5 and S^5



(a) (b)

Figure 1: In (a) we show the classical folded spinning string moving in $AdS_3 \subset AdS_5$ at a certain time (dark solid line) and earlier/later times (dashed lines). The quantum fluctuations, corresponding to oscillations transverse (light wavy lines) to the classical solution, acquire mass due to the background curvature. In (b) we show the motion of the string on the sphere, essentially a point moving along a great circle, with its fluctuations again seeing more of the geometry.

subspace of AdS_5 while additionally moving along a great circle of the S^5 with angular momentum J (see Fig. 1). In terms of the global coordinates

$$ds^2_{AdS_5} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2\right) , \tag{2.1}$$

$$ds^2_{S^5} = +\cos^2\gamma\, d\varphi_3^2 + d\gamma^2 + \sin^2\gamma\left(d\psi^2 + \cos^2\psi\, d\varphi_1^2 + \sin^2\psi\, d\varphi_2^2\right) , \tag{2.2}$$

the string solution is given by $\theta = \gamma = \psi = \frac{\pi}{2}$,

$$t = \kappa\tau\ , \quad \phi_2 = \omega\tau\ , \quad \rho = \rho(\sigma) = \rho(\sigma + 2\pi)\ , \quad \varphi_2 = \nu\tau\ . \tag{2.3}$$

The equations of motion and the conformal constraints are satisfied provided

$$\rho'' = (\kappa^2 - \omega^2)\sinh\rho\cosh\rho\ , \qquad \rho'^2 = \kappa^2\cosh^2\rho - \omega^2\sinh^2\rho - \nu^2\ , \tag{2.4}$$

and the other fields are zero. This string can be thought of as four segments: the first, for $0 \le \sigma \le \frac{\pi}{2}$, extends from the origin of the AdS_5 space along the radial direction to a maximum $\rho(\frac{\pi}{2}) = \rho_0$ i.e. $\rho'(\frac{\pi}{2}) = 0$. The string then turns and runs back along itself to the origin, this then repeats before the string closes on itself. In fact, this solution is generically rather complicated however, in various limits it simplifies dramatically.

2.1 Quantum corrections

It is possible to extract the one-loop correction to the energy by various means though, of course, all give identical results. The most direct method is to fix a physical gauge, such as light-cone, solve the resulting constraints and quantise the remaining degrees of freedom; the correction to the AdS energy of the string is the correction to the two-dimensional energy of the vacuum state. However, for many purposes, and particularly for more

e.g. [29–33].

complicated solutions at higher orders, the most convenient method, introduced in this context by [14,26,13] and most completely described in [39,40], is to relate the correction to the energy to the calculation of the worldsheet effective action. [4]As in standard QFT, and in analogy with the thermodynamic Gibbs free energy, in the presence of a non-trivial background solution, $\varphi_c(x)$, the expectation value of the conjugate source, $J(x)$, is given by the functional derivative of the effective action, $\Gamma[\varphi_c(x)]$, which is simply the Legendre transform of the vacuum energy functional. For the theory we are interested in the sources are simply the conserved charge densities, such as E, S and J. These are conjugate to time derivatives of the fields and so the background is specified by the constant parameters e.g. κ, ω, and ν. Thus

$$\frac{1}{T}\Gamma(\kappa,\omega,\nu) = -\frac{i}{T}\ln\,\langle e^{iH_{2d}T}\rangle + \kappa\langle E\rangle - \omega\langle S\rangle - \nu\langle J\rangle \tag{2.5}$$

where $T\to\infty$ is the worldsheet time interval. Due to the classical Virasoro constraints not all parameters are independent e.g. $\kappa = \kappa(\omega,\nu)$. Furthermore, the energy functional vanishes as $\langle H_{2d}\rangle = 0$ due to the quantum conformal constraint. The charges are thus found from the effective action by e.g.

$$\frac{1}{T}\frac{\partial\Gamma(\omega,\nu)}{\partial\nu} = \frac{\partial\kappa(\omega,\nu)}{\partial\nu}\langle E\rangle - \langle J\rangle\ . \tag{2.6}$$

Hence, we need only calculate the worldsheet effective action to determine the corrections to the string charges. In general, the leading quantum correction to the effective action, Γ_1, is found by expanding the Lagrangian, L, about a classical solution, $\varphi = \varphi_c + \tilde{\varphi}$, and performing the Gaussian integral

$$\Gamma_1 = \frac{i}{2}\log\ \det\Big[-\frac{\delta^2 L}{\delta\tilde{\varphi}\delta\tilde{\varphi}}\Big] = \frac{i}{2}\mathrm{Tr}\ \log\Big[-\frac{\delta^2 L}{\delta\tilde{\varphi}\delta\tilde{\varphi}}\Big]\ . \tag{2.7}$$

For the string theory we must include not only the bosonic fluctuations but also those of the fermionic and the ghost fields which give inverses of determinants.

In general the effective action is an extrinsic quantity. [5] This can be seen by considering the simple case where the quadratic fluctuation operator is given by $K = -\partial^2 + m^2$ with constant masses, m. Fourier transformed this is $\tilde{K} = -\omega^2 + n^2 + m^2$, and so

$$\Gamma_1 = \frac{iT}{2}\int\frac{d\omega}{2\pi}\sum_n \log\left(-\omega^2 + n^2 + m^2\right) = \frac{lT}{2}\int\frac{d^2p_E}{(2\pi)^2}\log\left(p_E^2 + m^2\right) \tag{2.8}$$

where in the last identity we have Wick rotated to Euclidean signature and taken the extent of the spatial direction, l, to also be large. Note that by performing the integration over ω in this constant mass case, or in fact for any stationary solution, one recovers the sum over fluctuation frequencies which gives the more common expression for the correction to the string energy c.f. appendix A [5]. [6]

[4]There is yet another method, essentially a generalisation of the WKB formula, for finding the leading quantum correction to periodic solutions due to Daschen, Hasslacher and Neveu [41]. Such methods were applied to the semiclassical quantization of the giant magnon [42] in [43]

[5]Strictly speaking all our considerations are only valid in the large volume limit and under the assumption that interactions are local.

[6]It is also possible to make use of the integrable structure and extract the fluctuation frequencies

2.2 Point-like BMN string

If we consider the case $\omega = 0$, $\kappa = \nu$, for (2.3), this forces $\rho_0 = 0$ and so corresponds to the point-like BMN string rotating only in the S^5 (see Fig. 1 (b)). As mentioned in the introduction, this solution plays a fundamental role in our understanding the quantum string. Here we merely calculate the one-loop correction to its classical AdS energy $E_0 = J = \sqrt{\lambda}\kappa$.

It is convenient to switch to Cartesian coordinates: $(\rho, \theta, \phi_1, \phi_2) \to z_k$, $k = 1, ..., 4$ and $(\gamma, \psi, \varphi_1, \varphi_3) \to y_s$, $s = 1, ..., 4$ such that

$$ds^2 = -\frac{(1+\frac{1}{4}z^2)^2}{(1-\frac{1}{4}z^2)^2}dt^2 + \frac{dz_k dz_k}{(1-\frac{1}{4}z^2)^2} + \frac{(1-\frac{1}{4}y^2)^2}{(1+\frac{1}{4}y^2)^2}d\varphi_3^2 + \frac{dy_s dy_s}{(1+\frac{1}{4}y^2)^2} \ . \tag{2.9}$$

Now, expanding near $z_k = y_s = 0$,

$$t = \nu\tau + \frac{\tilde{t}}{\lambda^{1/4}} \ , \qquad z_k = \frac{\tilde{z}_k}{\lambda^{1/4}} \ , \qquad \varphi_2 = \nu\tau + \frac{\tilde{\varphi}}{\lambda^{1/4}} \ , \qquad y_s = \frac{\tilde{y}_s}{\lambda^{1/4}} \ , \tag{2.10}$$

the bosonic terms of the action (1.1), in conformal gauge, give the quadratic term [7]

$$I_B = -\frac{1}{4\pi}\int d^2\sigma \Big[-\partial_a\tilde{t}\partial^a\tilde{t} + \partial\tilde{\varphi}\partial^a\tilde{\varphi} + \nu^2(\tilde{z}^2 + \tilde{y}^2) + \partial_a\tilde{z}_k\partial^a\tilde{z}_k + \partial_a\tilde{y}_s\partial^a\tilde{y}_s \Big] \ . \tag{2.11}$$

This action corresponds to two massless longitudinal fluctuations $\tilde{t}$ and $\tilde{\varphi}$, plus eight free, massive scalars, with mass $m = \nu$. For the fermions we find for the induced Dirac matrices $\varrho_0 = \kappa\ \Gamma^-$ and $\varrho_1 = 0$ so that the action becomes

$$I_F = \frac{i\nu}{2\pi}\int d^2\sigma \left[\bar{\theta}^1\Gamma^-\partial_+\theta^1 + \bar{\theta}^2\Gamma^-\partial_-\theta^2 - 2\nu\bar{\theta}^1\Gamma^-\Pi\theta^2\right] \tag{2.12}$$

where we have defined $\partial_\pm = \partial_0 \pm \partial_1$, $\Gamma^\pm = \mp\Gamma_0 + \Gamma_9$ and $\Pi = \Gamma_{1234}$. Furthermore because of the form of the fermionic kinetic operator it was natural to choose the kappa-gauge fixing $\Gamma^+\theta^I = 0$ which simplified the mass term. This action corresponds to eight free, massive fermionic excitations, with $m = \pm\nu$. Finally, one must include contributions from the conformal bosonic ghosts, however for the cases in which we are interested, as was shown in [4, 5], the ghost contribution is essentially trivial. Their only effect is to cancel the two massless longitudinal bosonic fluctuations.

As the masses of the transverse bosons and physical fermions are equal one immediately sees that the ratio of fluctuation determinants cancels and the one-loop effective action is zero. Thus the correction to the AdS energy, (2.5), $\langle E - J\rangle = \frac{1}{\kappa T}\Gamma$ is zero which is exactly as expected as this state is BPS. As we will see later, it provides a sensible vacuum about which to study fluctuation interactions.

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from the string algebraic curve. While this powerful method is widely used in the calculation of quantum corrections we will not discuss it here, but simply refer the reader to [44] for a review and references.

[7] We note that this is essentially the same action as that found by expanding the action for a string in the plane-wave geometry, [35], $ds^2 = dx^+dx^- + \frac{1}{4}x^2dx^+dx^+ + dx^idx^i$ about the solution $x^+ = 2\nu\tau$ [27,35].

2.3 Spinning folded string

While for the BPS solution we find zero correction to the string energy, a generic spinning string solution spontaneously breaks supersymmetry and we expect to find a non-trivial correction at one-loop. We will consider the so-called "semi-classical scaling" or long-string limit of the spinning string solutions, see [7, 8] and also [39],

$$\mathrm{S} \gg \mathrm{J} \gg 1, \qquad \text{with} \qquad \ell \equiv \frac{\mathrm{J}}{2\,\ln \mathrm{S}}\ . \tag{2.13}$$

As discussed at length in [8, 39], upon taking $\omega = \kappa$ the solution simplifies dramatically becoming homogeneous so that $\rho(\sigma) = \mu\sigma$. The conformal gauge condition becomes $\kappa = \sqrt{\mu^2 + \nu^2}$ and in this limit of large spin, $\mu = \frac{1}{\pi}\ln \mathrm{S}$ and $\ell = \frac{\nu}{\mu}$.

As μ is thus very large, by rescaling the worldsheet coordinate σ such that $\rho = \sigma$, we find the string length $l = 2\pi\mu$ becomes infinite. The folded string becomes two overlapping, infinite, open strings. One can further expand in small ℓ, the so called "slow long string limit", [8, 39]. In this further limit the quantum string energy is given by

$$E - S = \frac{\sqrt{\lambda}}{\pi} f(\lambda) \ln S\ , \tag{2.14}$$

where $f(\lambda)$ is the universal scaling function. At leading order this can be checked by expanding the classical energy which is given by $\mathrm{E}_0 - \mathrm{S} = \mu\sqrt{1+\ell^2}$. We will see this form persists at subleading orders in the semiclassical expansion, i.e. there are no $\ln^k S$ terms, and furthermore we can calculate the numerical coefficients [5, 8, 13, 39]

$$f(\sqrt{\lambda}) = 1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{\lambda} + \dots \tag{2.15}$$

where K is the Catalan constant.

To calculate these coefficients we expand about the homogeneous, $J = 0$ solution, $\hat{t} = \kappa\tau$, $\hat{\rho} = \kappa\sigma$, $\hat{\theta} = \frac{\pi}{2}$, $\hat{\phi}_2 = \kappa\tau$, and (following [5] closely, where full details can be found) we again consider the conformal gauge action.

Bosons The bosonic action (1.1) to quadratic order in fluctuations (using coordinates (2.1) for the AdS_5 space but (2.9) for the sphere) is

$$\begin{aligned} I_B \ = \ -\frac{1}{4\pi}\int d^2\sigma\ \Big[& -\cosh^2\hat{\rho}(\partial\tilde{t})^2 + \sinh^2\hat{\rho}(\partial\tilde{\phi}_2)^2 + 2\kappa\sinh\hat{\rho}\tilde{\rho}(\partial_0\tilde{t} - \partial_0\tilde{\phi}_2) \\ & + (\partial\tilde{\rho})^2 + \sinh^2\hat{\rho}((\partial\tilde{\theta})^2 + \tilde{\theta}^2(\partial\phi_1)^2 + \kappa^2\tilde{\theta}^2) + (\partial\tilde{\phi}_3)^2 + \sum_s (\partial\tilde{y}_s)^2\Big] \end{aligned} \tag{2.16}$$

where e.g. $(\partial t)^2 = \partial_a t \partial^a t$. In this expression the coefficients depend on the worldsheet coordinates, however by making the field redefinitions

$$\begin{aligned} \bar{\chi} = \tfrac{1}{2}\sinh 2\hat{\rho}\ (\tilde{\phi}_2 - \tilde{t})\ , \qquad \bar{\xi} = -\sinh^2\hat{\rho}\ \tilde{\phi}_2 + \cosh^2\hat{\rho}\ \tilde{t}\ , \qquad \bar{\theta} = \sinh\hat{\rho}\ \tilde{\theta}\ , \\ \bar{\rho} = \tilde{\rho}\ , \qquad \bar{x}_1 = \tilde{\theta}\cos\phi_1\ , \qquad \bar{x}_2 = \tilde{\theta}\sin\phi_1\ , \end{aligned} \tag{2.17}$$

this can be put in the form

$$I_B = -\frac{1}{4\pi}\int d^2\sigma \Big[(\partial\bar{\chi})^2 - (\partial\bar{\xi})^2 + (\partial\bar{\rho})^2 + 4\kappa(\partial_1\bar{\chi})\bar{\xi} - 4\kappa(\partial_0\bar{\chi})\bar{\rho} + \sum_i \big((\partial\bar{x}_i)^2 + 2\kappa^2 x_i^2\big) + (\partial\tilde{\phi}_3)^2 + \sum_s (\partial\tilde{y}_s)^2 \Big] . \quad (2.18)$$

It is now straightforward to calculate the determinant of the fluctuation operator

$$\det K_B = -(\partial^2)^7(\partial^2 + 2\kappa^2)^2(\partial + 4\kappa^2) \quad (2.19)$$

corresponding to two scalars with mass $\sqrt{2}\kappa$, one with mass 2κ and seven massless scalars – two from the AdS space, five from the sphere.

Fermions Substituting the classical solution in the expressions for the induced Dirac matrices we find (where the flat index 0 is the homologue of t, 1 corresponds to ρ, and 2 to ϕ_2)

$$\varrho_0 = \kappa\, \Gamma_0 (\cosh\hat{\rho} - \sinh\hat{\rho}\, \Gamma_{02}) \ , \qquad \varrho_1 = \kappa\, \Gamma_1 \ . \quad (2.20)$$

Using the expression for the quadratic action (1.1), we again find that the dependence on the worldsheet coordinates can be removed by a field redefinition

$$\theta^I = S\Psi^I, \qquad \text{with} \qquad S = \exp\left(\tfrac{\kappa\sigma}{2}\Gamma_{02}\right) \ , \quad (2.21)$$

such that the corresponding transformations of the induced Dirac matrices are

$$\tau_0 = S^{-1}\varrho_0 S = \kappa\, \Gamma_0 \ , \quad \text{and} \quad \tau_1 = S^{-1}\varrho_1 S = \kappa\, \Gamma_1 \ . \quad (2.22)$$

Making use of the relevant terms of the spin connection, $\omega_t{}^{01} = \sinh\rho$ and $\omega_{\phi_2}{}^{41} = \cosh\rho\cos\theta$, one can show that the portion of the covariant derivative that couples to the background curvature, $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, essentially becomes trivial: $S^{-1}\mathrm{D}_a S = \partial_a + B_a$ where $\eta^{ab}\tau_a B_b = \epsilon^{ab}\tau_a B_b = 0$. Thus the fermionic action can be written as

$$I_F = \frac{i\sqrt{\lambda}}{2\pi}\int d^2\sigma\ (\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})(\bar{\Psi}^I\tau_a\partial_b\Psi^J + \tfrac{1}{2}\epsilon^{JK}\bar{\Psi}^I\tau_a\Gamma_{01234}\tau_b\Psi^K) \ . \quad (2.23)$$

As can be seen from the form of the kinetic operator one can fix the fermionic kappa-symmetry by imposing $\Psi^1 = \Psi^2 = \Psi$ resulting in the fermion action [8]

$$I_F = \frac{i\sqrt{\lambda}}{\pi}\int d^2\sigma\ \bar{\Psi}^I(\tau^a\partial_a + iM)\Psi \ , \quad \text{where} \quad M = i\kappa^2\Gamma_{234} \ . \quad (2.24)$$

Of the eight physical fermions four have mass κ and four have $-\kappa$, thus

$$\det K_F = (\partial^2 + \kappa^2)^8 \ . \quad (2.25)$$

[8] While it is not relevant for the case at hand in general one must be careful with the boundary conditions imposed on the fermions which can be subtle. See [45] for a discussion.

Energy Correction To determine the correction to the energy we must evaluate the sum over momenta. As we are interested in the leading term in the large κ expansion we can treat the worldsheet, after rescaling by κ, as having infinite extent and so the worldsheet momenta are continuous. In momentum space the one-loop effective action is (having taken into account the conformal ghosts which cancel two massless bosons)

$$\Gamma_1 = \frac{1}{2} V_2 \int \frac{d^2p}{(2\pi)^2} \Big[\ln(p^2+4) + 2\ln(p^2+2) + 5\ln p^2 - 8\ln(p^2+1) \Big] \tag{2.26}$$

where we recall that two-dimensional volume is given by $V_2 = 2\pi\kappa^2 T$. While the complete expression is finite the individual terms are divergent so we introduce a cut-off at intermediate stages to perform the integration. The quadratic and logarithmic divergences cancel and the finite result is

$$\langle E - S \rangle \,|_{\text{one-loop}} = \frac{1}{\kappa T}\Gamma_1 = -\frac{3\ln 2}{\pi} \ln S \tag{2.27}$$

which is the leading correction to the universal scaling function. We note that the $\ln S$ dependence arises from the fact that the effective action is proportional to the worldsheet volume as, in the scaling limit, we can completely remove κ from the action. This remains true at all orders.

Generalisations The two-loop calculation of the universal scaling function was carried out in [13–15]. The equivalence [26] of the spinning folded string, in the $l \to \infty$ limit, to the null cusp Wilson loop solution [46] plays a key role in these calculations; as does a form of the action with particularly simple fermions [47]. One can obviously include the effects of non-zero J by keeping finite ν, or equivalently ℓ, dependence. The generalised one-loop calculation in the "long string" limit was performed in [8] and the two-loop analysis in [39,40,48]. Here, it is necessary to take into account the quantum corrections to the Virasoro condition and to the relations between solution parameters and charges as described in Sec. 2.1. Furthermore, the calculation is simplified by using a light-cone gauge [28] adapted to a geodesic entirely in the AdS_5 space. These results match those found from the ABA [49]. These calculations thus provide vigorous checks of the two-loop finiteness of the worldsheet theory and the underlying quantum integrability.

2.4 Circular spinning strings

While the energies of spinning folded strings have provided stringent checks of ABA the relationship is slightly complicated. It is a separate class of solutions, rigid circular spinning strings (see [2] for a review and further references), whose energies are most transparently related to the strong coupling expression for the S-matrix entering the ABA. The simplest circular strings come in two types: the so-called $\mathfrak{su}(2)$ circular strings moving on a $S^3 \subset S^5$, [50], and the $\mathfrak{sl}(2)$ circular strings lying in $AdS_3 \times S^1 \subset AdS_5 \times S^5$ [51].

The computation of the one-loop correction to the energies of the $\mathfrak{su}(2)$ [52,53] and $\mathfrak{sl}(2)$ [54,19,55,56] strings [9] played a key part in discovering the presence of the one-loop

[9] An early semiclassical analysis of circular strings in AdS was performed in [57].

term [20] in the phase in the strong-coupling (or "string") form of the Bethe Ansatz [16–18].

The (S, J) string solution of [51] has a spiral-like shape, with projection to AdS_3 being a constant radius circle (with winding number k), and projection to S^5 – a big circle (with winding number m). The corresponding spins are, respectively, S and J with the Virasoro condition implying that $u \equiv \frac{S}{J} = -\frac{m}{k}$. Expanding the classical energy in large semiclassical parameters S and J with fixed k and u [51, 54] we have

$$E_0 = S + J + \frac{\lambda}{J} e_1(u,k) + \frac{\lambda^2}{J^3} e_3(u,k) + \frac{\lambda^2}{J^5} e_5(u,k) + \dots \,. \tag{2.28}$$

For circular strings the expressions for the fluctuation frequencies are sufficiently complicated that they must be expanded in J to be evaluated and subsequently summing over modes becomes slightly subtle [54, 58, 53, 59, 19, 55, 60, 56]. The correct procedure, given in [19] for the $\mathfrak{sl}(2)$ case (see also [56] for the $\mathfrak{su}(2)$ case), gives two types of terms for the one-loop correction, $E_1 = E_1^{\text{even}} + E_1^{\text{odd}}$, where

$$E_1^{\text{even}} = \frac{\lambda}{J^2} g_2(u,k) + \frac{\lambda^2}{J^4} g_4(u,k) + \dots \,, \qquad E_1^{\text{odd}} = \frac{\lambda^{5/2}}{J^5} g_5(u,k) + \dots \,. \tag{2.29}$$

The absence of the $\frac{1}{J}$ and $\frac{1}{J^3}$ terms suggests that the two leading $\frac{\lambda}{J}$ and $\frac{\lambda^2}{J^3}$ terms receive no quantum corrections and their coefficients should directly match weak coupling gauge theory results. Indeed, the coefficient g_2 of the "even" $\frac{1}{J^2}$ term in (2.29) can be reproduced as a leading $\frac{1}{J}$ (finite spin chain length) correction from the one-loop gauge theory Bethe Ansatz [53, 58]. At the same time, the presence of the non-analytic term $\frac{\lambda^{5/2}}{J^5}$ in (2.29) implies that a similar $\frac{1}{J^5}$ term in the classical energy (2.28) is not protected so that its coefficient cannot be directly compared to three-loop result on the gauge theory side which implies [19] that the corresponding "string" Bethe Ansatz [16] should be modified to contain a non-trivial one-loop correction to the phase. This phase was determined by directly matching to higher orders in this expansion [20, 21].

2.5 Finite size effects and short operators

Semiclassical analysis can also be applied to strings of finite length and even, to a certain degree, short strings. For the folded spinning string, Sec. 2.3, the large S corrections to the one-loop calculation were analysed in [61] and the exact one-loop expression for the fluctuation determinants was found in [62] (for two-loop results see [48]). The one-loop correction to the small spin or short string limit of the string were calculated in [63] and the generalisation with non-zero J in [64]. Short, excited strings dual to operators in the Konishi multiplet are particularly important in testing the conjectured exact results for the spectrum at finite volume. The correction to their energies at strong coupling was calculated semiclassically, with caveats regarding the validity of these methods in this regime, in [65]. For the circular spinning strings, in addition to the energy correction (2.29), a careful analysis shows the presence of exponential corrections, $\mathcal{O}(e^{-\mathrm{J}})$ [55,56,66]. Similar exponential corrections are found for quantum corrections to finite-sized giant-magnons calculated using algebraic curve methods (see [44]). Such corrections cannot

be accounted for by modifying the phase in the BA but rather arise from finite volume effects. See [67] for reviews and references.

3 Perturbative light-cone quantization

As we saw in Sec. 2.2, the string action expanded about the BMN string is particularly simple and is exactly solvable to quadratic order in fluctuations. This string solution provides a sensible vacuum about which to perturbatively quantize the $\mathrm{AdS}_5\times\mathrm{S}^5$ Green-Schwarz string [36, 29, 30, 32, 68]. In this context it is natural to make use of light-cone gauge, introducing the coordinates and momenta, $p_\mu = h^{0a}G_{\mu\nu}\partial_a x^\nu$,

$$x^+ = \frac{1}{2}(t+\phi)\ , \quad x^- = \phi - t\ , \quad p_- = \frac{1}{2}(p_\phi - p_t)\ , \quad p_+ = p_\phi + p_t \tag{3.1}$$

where we focus on the bosonic fields for simplicity. The Hamiltonian density $\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L}$ is given by

$$\mathcal{H} = -\frac{h^{\tau\sigma}}{h^{\tau\tau}}(x'^\mu p_\mu) + \frac{1}{2h^{\tau\tau}}(p_\mu G^{\mu\nu} p_\nu + x'^\mu G_{\mu\nu} x'^\nu)\ , \tag{3.2}$$

with the notation $x' = \partial_\sigma x$ and $\dot{x} = \partial_\tau x$. As is usual in theories with general coordinate invariance, the Hamiltonian is a sum of constraints times Lagrange multipliers.

To impose light-cone gauge one sets $x^+ = \tau$ and $p_- = \text{const}$. The metric coefficients $1/h^{\tau\tau}$ and $h^{\tau\sigma}/h^{\tau\tau}$ act as Lagrange multipliers, generating delta functions that impose two constraints which determine x^- and p_+ in terms of the transverse variables (and the constant p_-). [10] The transverse coordinates and momenta $x^A,\ p_A\ A = 1,\ldots,8$ will then have dynamics which follow from the light-cone Hamiltonian $-p_+ = \mathcal{H}_{lc}$. The first constraint, or level-matching constraint, yields $x'^- = -x'^A p_A/p_-$. While solving the quadratic constraint equation for p_+ we obtain the somewhat dispiriting result

$$-\mathcal{H}_{\rm lc} = \frac{p_- G_{+-}}{G_{--}} + \frac{p_-\sqrt{G}}{G_{--}}\sqrt{1 + \frac{G_{--}}{p_-^2}(p_A G^{AB} p_B + x'^A G_{AB} x'^B) + \frac{G_{--}^2}{p_-^4}(x'^A p_A)^2}\ , \tag{3.3}$$

with $G \equiv G_{+-}^2 - G_{++}G_{--}$. [11] Using the relation between the canonical momenta and the target space charges we have

$$E - J = -P_+ = \tfrac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\ \mathcal{H}_{\rm lc}\ , \qquad \tfrac{1}{2}(E+J) = P_- = \tfrac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\ p_-\ . \tag{3.4}$$

Perturbative expansion To make progress we perform the large tension expansion: rescaling the transverse fields by $\lambda^{-1/4}$ and expanding in large $\sqrt{\lambda}$, or equivalently $P_- =$

[10] In fact, the constraints determine the derivatives of x^- and so x^- itself is non-local in this gauge. This has important consequences for the "off-shell" symmetry algebra.

[11] We have made use of the fact that the $AdS_5 \times S^5$ metric, (2.9), rewritten in light-cone coordinates, (3.1), has no G_{+A} or G_{-A} components.

$\sqrt{\lambda}p_- \sim J$, while keeping $-P_+ = E - J$ fixed. Being careful with the expansion of the G_{--} terms, see e.g. [30], one finds the first two orders,

$$\begin{aligned}\mathcal{H}^{pp}_{\rm lc} &= \frac{1}{2p_-}\left[(\dot{x}^A)^2 + (x'^A)^2 + p_-^2(x^A)^2\right] \\ &\quad + \frac{1}{4\sqrt{\lambda}p_-}\left(z^2(p_y^2 + y'^2) - y^2(p_z^2 + z'^2) + 2z^2z'^2 - 2y^2y'^2\right) \ , \end{aligned} \tag{3.5}$$

where beyond leading order the eight transverse fields split into two sets of four, $x^A = (z^i, y^s)$. One can remove the dependence on the density p_- by rescaling the worldsheet coordinates, and thus we see that we are taking the large charge limit but keeping the worldsheet compact.

The leading order term is simply the plane-wave Hamiltonian whose spectrum consists of an infinite tower of non-interacting massive oscillators,

$$x^A(\sigma,\tau) = \sum_{n=-\infty}^{\infty} x_n^A(\tau)e^{-in\sigma} \ , \qquad x_n^A(\tau) = \frac{i}{\sqrt{2\omega_n}}(a_n^A e^{-i\omega_n\tau} - a_{-n}^{A\dagger}e^{i\omega_n\tau}) \ , \tag{3.6}$$

where $n \in \mathbb{Z}$, $\omega_n = \sqrt{p_-^2 + n^2}$, and the raising and lowering operators obey the usual commutation relations. One can straightforwardly include the fermions, though the subleading interaction terms become somewhat involved [29,30,32]. At leading order one again gets massive oscillators, b_n^α, $\alpha = 1, \ldots, 8$ and thus the full plane-wave Hamiltonian, $H_{\rm pp}$, is

$$H_{\rm pp} = \frac{1}{p_-}\sum_{n=-\infty}^{\infty}\omega_n\left(a_n^{A\dagger}a_n^A + b_n^{\alpha\dagger}b_n^\alpha\right) \ , \tag{3.7}$$

where one can immediately see that the energy of the vacuum state, $|\text{Vac}\rangle$, corresponding to a string with charge P_- vanishes.

Near-BMN energy spectrum The quartic terms give rise to corrections of order $\mathcal{O}(1/J)$, the effects of which can be perturbatively included in the spectrum. In the simple case where we consider a single complex boson from the sphere $y = y^1 + iy^2$, the leading correction to the two excitation state $a_n^\dagger a_{-n}^\dagger|P_-\rangle$ is

$$E - J = 2\sqrt{1 + \lambda' n^2} - 2\frac{\lambda' n^2}{J} + \frac{N_B(n^2)}{J} \tag{3.8}$$

with $\lambda' = \lambda/J^2$ an effective coupling. Due to the form of the interactions there is a normal ordering ambiguity, here characterised by the arbitrary function $N_B(n^2)$. There are related functions in the correction to all energies and they are fixed by demanding that the full spectrum possess the underlying global $\mathfrak{psu}(2,2|4)$ symmetry. This implies, for example, $N_B = 0$. Equivalently, they could be fixed by demanding that the algebra of generators, including the Hamiltonian, is satisfied at this order. These expressions for string energies can be compared to the string ABA [37, 31, 32, 68] and were one of the first pieces of evidence for a non-trivial dressing phase interpolating between strong and weak coupling.

3.1 Worldsheet S-matrix

As the theory in light-cone gauge has only massive particles, we can study the interactions by calculating the worldsheet S-matrix. Modulo issues of gauge dependence [12] this object should match the spin chain S-matrix introduced in [17], see [12] for reviews. The perturbative study of the worldsheet S-matrix was initiated in [70] while its symmetries and many properties were analysed in [71,72] (see [38] for an extensive review). To define the S-matrix one must consider the theory on the plane: this corresponds to scaling p_- out of the action and taking the decompactification limit $p_- \to \infty$. In order to define free, asymptotic states for generic momentum one relaxes the level matching condition and then studies the interactions in powers of $\sqrt{\lambda}$ or equivalently in a small (worldsheet) momentum expansion.

Asymptotic states Of the global group, the light-cone gauge preserves a subset $\mathrm{PSU}(2|2)_L \times \mathrm{PSU}(2|2)_R \subset \mathrm{PSU}(2,2|4)$. The bosonic subgroup of each PSU(2|2) factor consists of two SU(2) groups and it is useful to introduce a bispinor notation for the physical bosons $Z_{\alpha\dot\alpha} = (\sigma_i)_{\alpha\dot\alpha} z^i$, $Y_{a\dot a} = (\sigma_s)_{a\dot a} y^s$ and fermions, $\Psi_{a\dot\alpha}, \Upsilon_{\alpha\dot a}$, which are charged under different combinations of the SU(2)'s. One may define superindices $A = (a|\alpha)$ and $\dot A = (\dot a|\dot\alpha)$ combining all asymptotic fields creating incoming or outgoing particles into a single bi-fundamental supermultiplet of which we will denote by $\Phi^{(in/out)}_{A\dot A}$.

The S-matrix. The two-particle S-matrix is a unitary operator relating *in*- and *out*-states. In the basis $\Phi_{A\dot A}(p)$, so that $|\Phi_{A\dot A}(p)\Phi_{B\dot B}(p')\rangle^{(in)} = \Phi^{(in)}_{A\dot A}(p)\Phi^{(in)}_{B\dot B}(p')|\mathrm{Vac}\rangle$, its matrix representation is

$$\mathbb{S}\,|\Phi_{A\dot A}(p)\Phi_{B\dot B}(p')\rangle^{(in)} = |\Phi_{C\dot C}(p)\Phi_{D\dot D}(p')\rangle^{(out)}\,\mathbb{S}^{C\dot C D\dot D}_{A\dot A B\dot B}(p,p')\ . \tag{3.9}$$

Before gauge fixing the worldsheet theory is classically integrable [73]; since fixing light-cone may be interpreted as expanding about the BMN solution and solving some of the equations of motion, the gauge-fixed theory is also expected to be integrable at the classical level. In such an integrable theory, the S-matrix, invariant under a non-simple product group, must be a tensor product of S-matrices for each of the factors (see e.g. [74]) [13]

$$\mathbb{S} = \mathbf{S}\otimes\mathbf{S} \quad , \quad \mathbb{S}^{C\dot C D\dot D}_{A\dot A B\dot B}(p,p') = \mathbf{S}^{CD}_{AB}(p,p')\mathbf{S}^{\dot C\dot D}_{\dot A\dot B}(p,p')\ . \tag{3.10}$$

It is important to note that a factorised tensor structure does not follow solely from the PSU(2|2) × PSU(2|2) symmetry considerations; confirming group factorisation is thus an important test of integrability.

[12] The S-matrix is gauge-dependent, since unlike the spectrum it is not a physical object with a clear target-space interpretation. The differences between gauges can be attributed to the definition of the string length [17]. The difference in the definition of length and the gauge-dependence of the S-matrix, mutually cancel in the Bethe equations [32,69].

[13] This can be understood as a requirement that the Faddeev-Zamolodchikov algebra is also a direct product: the field $\Phi_{A\dot A}$ is represented by a bilinear in oscillators: $\Phi_{A\dot A} \sim z_A z_{\dot A}$ each transforming under one of the PSU(2|2) factors [72]. The two sets of oscillators mutually commute. The braiding relations for each of these sets are determined by an PSU(2|2)-invariant S-matrix $\mathbf{S}$ consistent with the Lagrangian of the theory.

The first nontrivial order in the expansion of the S-matrix in the coupling constant $2\pi/\sqrt{\lambda}$ defines the T-matrix

$$\mathbb{S} = \mathbb{I} + \frac{2\pi i}{\sqrt{\lambda}}\,\mathbb{T} + \mathcal{O}\left(\frac{1}{\lambda}\right) \ . \tag{3.11}$$

which inherits the factorised form $\mathbb{T} = \mathbb{I} \otimes \mathrm{T} + \mathrm{T} \otimes \mathbb{I}$ from the S-matrix. Furthermore, since $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathrm{PSU}(2|2)$ is a manifest symmetry of the gauge-fixed worldsheet theory, T may be parametrised in terms of ten unknown functions of the momenta p and p'. These functions, to leading order in $1/\sqrt{\lambda}$, can be easily extracted from the matrix elements of quartic terms of the light-cone Hamiltonian (3.5) (see [70] where explicit expressions for T can be found). Equivalently one can Legendre transform with respect to the transverse fields to find the light-cone Lagrangian and then use the usual LSZ reduction to calculate the worldsheet scattering amplitudes perturbatively.

Properties of the S-matrix

- The explicit perturbative calculation does indeed show that the two-body S-matrix has the factorised form (3.10). Furthermore, it can be explicitly checked to leading order that the ten functions in the T-matrix agree with the corresponding functions in the strong coupling BA S-matrix. It can be shown explicity that there is no two-to-four particle scattering [70].

- In calculating the S-matrix we relax the level-matching constraint. In this "off-shell" formulation of the theory the symmetries become extended by two additional central charges related to the worldsheet momentum [71] (the same as found in the spin chain [75]). Furthermore, as the supersymmetry generators, $Q \sim \int e^{ix^-}\Omega(Z, Y, \Upsilon, \Psi)$, depend on the zero mode of the longitudinal coordinate, $x^- \sim \int d\sigma \partial_\sigma x^-$, there is a mild non-locality in the action of the symmetries which thus satisfy a Hopf algebra [70, 72].

- The integrable structures of the perturbative string S-matrix have been further studied including the construction of the classical r-matrix e.g. [76]. Furthermore, assuming the quantum integrability of the full worldsheet theory, and using the global symmetries, the worldsheet S-matrix was uniquely determined up to an overall phase. We refer the reader to [12, 77] for a more complete discussion of these and other exact properties of the worldsheet S-matrix.

3.2 Simplifying Limits

Due to the complexity of the world sheet theory, going beyond the leading perturbative term is challenging. One simplifying limit which has proved useful is the "near-flat limit" [78]. This limit corresponds to studying the worldsheet near a constant density solution boosted with rapidity $\lambda^{1/4}$ in the worldsheet light-cone direction, σ^-. The left- and right-moving excitations on the worldsheet scale differently and the right movers essentially decouple. The resulting theory has only quartic interactions and is much

more tractable. The one-loop and two-loop [79] corrections to the S-matrix have been calculated and shown to match the all-order conjecture [22]; furthermore factorization at one-loop was explicitly shown. In the two-loop calculation radiative corrections induce a correction to the relativistic dispersion relation which corresponds to the expansion of the sine function, natural from a spin chain perspective, which appears in the exact dispersion relation [75].

Another interesting formulation of the theory is found via a generalisation of the Pohlmeyer reduction [80] which is used to relate, at a classical level, the string theory on $AdS_5 \times S^5$ to a massive, Lorentz invariant theory which only involves the physical fields. Applied to strings on $\mathbb{R} \times S^3$ this method consists of gauge fixing and solving the Virasoro constraints so that the remaining degree of freedom satisfies the sine-Gordon equation of motion [81]. Generalised to the full superstring [82] the reduced theory is a massive deformation of a gauged WZW model with an integrable potential. The resulting model has been explicitly shown to be UV finite to two-loops and there is evidence that the equivalence to the standard formulation persists at the quantum level [83]. The two-particle S-matrix was calculated in this formalism in [84] where it was shown that it has the appropriate group factorisation properties. Being manifestly Lorentz invariant this formalism may provide a better basis for understanding the quantum theory.

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