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Observational constraints on loop quantum cosmology

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In the inflationary scenario of loop quantum cosmology (LQC) in the presence of inverse-volume corrections, we give analytic formulas for the power spectra of scalar and tensor perturbations convenient to confront with observations. Since inverse-volume corrections can provide strong contributions to the running spectral indices, inclusion of terms higher than the second-order runnings in the power spectra is crucially important. Using the recent data of cosmic microwave background (CMB) and other cosmological experiments, we place bounds on the quantum corrections for a quadratic inflaton potential.

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One of the motivations to search for a quantum theory of gravity is the desire to unify general relativity with quantum mechanics and, in doing so, resolve classical singularities such as the big bang or those associated with black holes. Although there exist some candidates which apparently cure these pathologies, their significance as models of nature is hard to assess due to the lack of a connection between theory and observations.

Observational implications of quantum gravity present a delicate issue. Corrections to the general relativistic dynamics are expected to arise in different ways. For instance, loop corrections are always present in perturbative graviton field theory, which can be captured in effective actions with higher-curvature corrections to the Einstein-Hilbert action. The additional terms change the Newton potential as well as the cosmological dynamics. However, in currently observable regimes the curvature scale is very small, and so one expects only tiny corrections of (adimensional) size at most $\ell_{\rm Pl}H$, where $\ell_{\rm Pl}$ is the Planck length and $H^{-1}=a/\dot{a}$ is the radius of the Hubble region (a is the scale factor in the flat Friedmann-Robertson-Walker background and dots denote derivatives with respect to cosmic time t). In such a case, tests of quantum gravity are possible at best indirectly, for instance if it provides concrete and sufficiently constrained models for inflation. So far, however, models do not appear tight enough.

There is a different route of testing quantum gravity, or at least those approaches that change not only the dynamics but even the underlying spacetime structure. This happens especially in background independent frameworks such as loop quantum gravity (LQG) [1]. Stronger modifications of the theory are possible since the usual covariant continuum dynamics is generalized, and entirely new effects may be contemplated. Still, the modifications are controlled by the requirement that covariance not be broken but at most deformed, leading to anomaly-free and consistent sets of equations.

A concrete realization requires detailed calculations and new techniques, since action principles of the usual covariant form are no longer available in the presence of a modified spacetime structure. In canonical quantum gravity, in particular in the loop quantization some of whose corrections are used here, the covariance principle and the dynamics are implemented by constraint equations: Constraint functionals generate gauge transformations as well as the dynamics by Hamiltonian equations of motion. If constraints are modified, not only the dynamics but also gauge transformations change, sometimes even producing a deformed algebra that no longer generates spacetime diffeomorphisms or hypersurface deformations. In this way, new spacetime structures become apparent. It may be difficult to find concrete, non-manifold realizations of the deformed structures, but the dynamics and gauge-invariant observables are accessible. This provides the basis for a cosmological analysis.

Equations that consistently implement such a radical form of quantum corrections are now available for one characteristic effect of LQG: In this theory, the spatial geometry is discrete as a consequence of metric operators acquiring discrete spectra with zero as an eigenvalue [2]. An inverse of those metric operators is needed to construct Hamiltonian operators or the quantized constraints, but a direct inverse of an operator with zero in the discrete part of its spectrum does not exist. By more indirect quantization procedures [3] one can nevertheless construct suitable densely defined operators, but they contain quantum corrections sensitive to the discreteness scale [4]. These inverse-volume corrections constitute the first example that has been consistently implemented, when they are small, in the dynamics [5]; then, covariance is not destroyed but deformed. Equations are thus consistent and can be analyzed for further properties such as cosmological ones, looking for possible new effects. (Consistency has so far been obtained only for small inverse-triad corrections; thus no gauge-independent equations exist for an analysis of inflationary scenarios driven entirely by quantum-geometry effects [6].)

No complete anomaly-free system with all corrections from loop quantum gravity is known at present, but consequences of inverse-volume corrections are already so characteristic that they are not likely to be eliminated when others are included. Other corrections might lead to rather different modifications of the constraints, but they could not manage to restore the classical notion of covariance, undoing the deformation.

A simplified implementation of corrections expected from LQG in cosmological scenarios via perturbations around homogeneous or other reduced models can be achieved in LQC [7]. There, with a phenomenological approach to effective dynamics, all the coupled equations of motion generated by the constraints [5] can be summarized in a single Mukhanov equation for the gauge-invariant scalar perturbation u_k [8], $u_k''+(s^2k^2-z''/z)u_k=0$ in momentum space, where primes denote derivatives with respect to conformal time $\tau=\int dt\, a^{-1}$, k is the comoving wavenumber, z is a background function, and $s^2=1+\chi(\sigma,\alpha_0,\nu_0)\delta_{\rm Pl}$ is the propagation speed squared. The quantum correction is characterized (i) by a specific function χ of the free constant parameters σ, α_0 and ν_0, that are not yet fixed by the quantization, and (ii) by the function $\delta_{\rm Pl}\propto a^{-\sigma}$ determining the size of inverse-volume corrections with respect to a reference scale set by the underlying discrete quantum state. A conservative theoretical estimate is $\delta_{\rm Pl}\sim O(10^{-8})$ [8], but the value may be higher by a few orders of magnitude especially for small σ. The aim of this paper is to restrict $\delta_{\rm Pl}$ by observations. While σ takes values in the range $0<\sigma\le 6$, the size of $\delta_{\rm Pl}$ does not depend on the values of α_0 and ν_0. However, the parameters α_0,ν_0 are related to σ by the consistency condition [8]

$$\nu_0(\sigma-3)=3\alpha_0(\sigma-6)/(\sigma+6)\,. \tag{1}$$

We will mainly place bounds on the combination $\alpha_0\delta_{\rm Pl}$ during slow-roll inflation, for which the precise origin of α_0 and of the scale hidden in $\delta_{\rm Pl}$ is not essential.

Corrections in the Mukhanov equation arise only in the form of z and in the k^2-term, yet the equation is covariant according to the corrected gauge transformations, as a direct consequence of the first-class constraint algebra found in [5]. Thus, one typical assumption of higher-curvature theories is violated. A second implication arises by comparison with tensor modes [9]. They can be summarized by the equation $w_k''+(\alpha^2k^2-\tilde a''/\tilde a)w_k=0$, which is also covariant, yet differs from the scalar equation by the functions α and $\tilde a$ instead of s and z. The speed here is corrected by $\alpha\approx 1+\alpha_0\delta_{\rm Pl}$ with $2\alpha_0\neq\chi$. Again, this is only possible with the change in the underlying manifold and gauge structure, and gives rise to additional characteristic effects. With different types of equations for scalar and tensor modes, there are changes to the standard inflationary spectra and the tensor-to-scalar ratio.

In Ref. [8], two of us evaluated the inflationary observables in terms of the three slow-roll (SR) parameters $\epsilon\equiv-\dot H/H^2$, $\eta\equiv-\ddot\varphi/(H\dot\varphi)$, and $\xi^2\equiv(\ddot\varphi/\dot\varphi)\dot{}/H^2$, where φ is a scalar field with potential $V(\varphi)$. In order to place observational bounds on concrete inflaton potentials, it is more convenient to use SR parameters expressed by V and its derivatives:

$$\epsilon_V\equiv\frac{1}{2\kappa^2}\left(\frac{V_{,\varphi}}{V}\right)^2,\quad \eta_V\equiv\frac{1}{\kappa^2}\frac{V_{,\varphi\varphi}}{V},\quad \xi_V^2\equiv\frac{V_{,\varphi}V_{,\varphi\varphi\varphi}}{\kappa^4V^2}, \tag{2}$$

where $\kappa^2=8\pi G$ (G is the gravitational constant). Conversion formulas from ϵ,η,ξ^2 to $\epsilon_V,\eta_V,\xi_V^2$ will be presented in a separate publication [10], to which we refer for all the technical details, together with a discussion of cosmic variance. Here we report only the minimum set of expressions required for the numerical analysis.

The power spectra of scalar and tensor perturbations, evaluated at the Hubble horizon crossing during inflation ($k\approx aH$), are given, respectively, by [8]

$$\mathcal{P}_{\rm s}=\frac{GH^2}{\pi\epsilon}(1+\gamma_{\rm s}\delta_{\rm Pl})\,,\quad \mathcal{P}_{\rm t}=\frac{16GH^2}{\pi}(1+\gamma_{\rm t}\delta_{\rm Pl})\,, \tag{3}$$

where $\gamma_{\rm s}=\nu_0(\sigma/6+1)+\sigma\alpha_0/(2\epsilon)-[\sigma\nu_0(\sigma+6)+3\alpha_0(15-\sigma)]/[18(\sigma+1)]$ and $\gamma_{\rm t}=(\sigma-1)\alpha_0/(\sigma+1)$. We expand the scalar spectrum about a pivot wavenumber k_0, as

$$\begin{aligned}\ln\mathcal{P}_{\rm s}(k) &= \ln\mathcal{P}_{\rm s}(k_0)+[n_{\rm s}(k_0)-1]\,x\\ &\quad+\frac{\alpha_{\rm s}(k_0)}{2}x^2+\sum_{m=3}^{\infty}\frac{\alpha_{\rm s}^{(m)}(k_0)}{m!}x^m\,,\end{aligned} \tag{4}$$

where $x\equiv\ln(k/k_0)$, $n_{\rm s}(k)-1\equiv d\ln\mathcal{P}_{\rm s}(k)/d\ln k$, $\alpha_{\rm s}(k)\equiv dn_{\rm s}/d\ln k$, and $\alpha_{\rm s}^{(m)}(k)\equiv d^{m-2}\alpha_{\rm s}/(d\ln k)^{m-2}$. The tensor spectrum can be expanded in a similar way with a different index $n_{\rm t}(k)\equiv d\ln\mathcal{P}_{\rm t}(k)/d\ln k$. The spectral indices are

$$n_{\rm s}-1=-6\epsilon_V+2\eta_V-c_{n_{\rm s}}\delta_{\rm Pl}\,,\quad n_{\rm t}=-2\epsilon_V-c_{n_{\rm t}}\delta_{\rm Pl}\,, \tag{5}$$

where

$$\begin{aligned}c_{n_{\rm s}} &= \sigma[3\alpha_0(13\sigma-3)+\nu_0\sigma(6+11\sigma)]/[18(\sigma+1)]\\ &\quad-\left[6\alpha_0\,(1-\sigma)-\nu_0\left(6-13\sigma/3+2\sigma^2/9\right)\right]\epsilon_V\\ &\quad-\left[\alpha_0\,(7\sigma/3-2)+2\nu_0\,(1-2\sigma/3)\right]\eta_V\,,\end{aligned} \tag{6}$$

$$\begin{aligned}c_{n_{\rm t}} &= 2\sigma^2\alpha_0/(\sigma+1)\\ &\quad-[2\alpha_0(1-\sigma)+\nu_0(\sigma-2)]\,\epsilon_V-2\sigma\alpha_0\eta_V/3\,.\end{aligned} \tag{7}$$

The dominant LQC corrections to $n_{\rm s,t}$ correspond to the first terms in Eqs. (6) and (7), i.e., $f_{\rm s}\equiv\sigma[3\alpha_0(13\sigma-3)+\nu_0\sigma(6+11\sigma)]/[18(\sigma+1)]$ and $f_{\rm t}\equiv 2\sigma^2\alpha_0/(\sigma+1)$. For $\sigma\gtrsim O(1)$ the variation of $\delta_{\rm Pl}$ is fast ($\delta_{\rm Pl}\propto a^{-\sigma}\propto k^{-\sigma}$ at Hubble crossing), so that $f_{\rm s,t}$ provide dominant

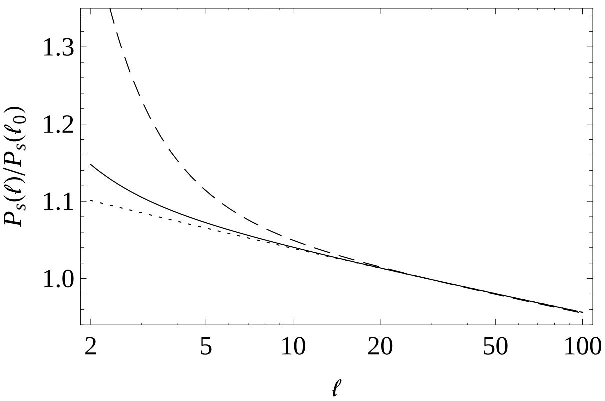

FIG. 1: Primordial scalar power spectrum $\mathcal{P}_{\rm s}(\ell)$ for the case $n=2$, $\sigma=2$, and $\epsilon_V(k_0)=0.009$ with three different values of $\delta(k_0)$: 0 (classical case, dotted line), 7×10^{-5} (observational upper bound, solid line), 4.8×10^{-4} [1/10 of the a-priori upper bound (10), dashed line]. Here the pivot wavenumber is $k_0=0.002\,{\rm Mpc}^{-1}$, which corresponds to $\ell_0=29$.

contributions to the scalar and tensor runnings as well, $\alpha_{\rm s,t}(k_0)=dn_{\rm s,t}/d\ln k|_{k=k_0}\approx\sigma f_{\rm s,t}\delta_{\rm Pl}(k_0)$. Similarly, the m-th order terms are $\alpha^{(m)}_{\rm s,t}(k_0)\approx(-1)^m\sigma^{m-1}f_{\rm s,t}\delta_{\rm Pl}(k_0)$ and hence we can evaluate the sum in Eq. (4) as

$$\sum_{m=3}^{\infty}\frac{\alpha^{(m)}_{\rm s,t}}{m!}x^m=\left[x\left(1-\frac{1}{2}\sigma x\right)+\frac{e^{-\sigma x}-1}{\sigma}\right]f_{\rm s,t}\delta_{\rm Pl}. \quad (8)$$

This expression is valid for any value of σ and of the pivot scale k_0 within the observational range of CMB. Since the LQC corrections to the runnings $\alpha_{\rm s,t}$ can be large, inclusion of the higher-order terms (8) is important to estimate the power spectra properly. For the CMB likelihood analysis we also take into account the second-order terms of slow-roll parameters, i.e., $\alpha_{\rm s}=-24\epsilon_V^2+16\epsilon_V\eta_V-2\xi_V^2+c_{\alpha_{\rm s}}\delta_{\rm Pl}$ and $\alpha_{\rm t}=-4\epsilon_V\left(2\epsilon_V-\eta_V\right)+c_{\alpha_{\rm t}}\delta_{\rm Pl}$, where the dominant contributions to $c_{\alpha_{\rm s,t}}$ correspond to $c_{\alpha_{\rm s,t}}\approx\sigma f_{\rm s,t}$. In the numerical code, the full expressions of the coefficients $c_{n_{\rm s,t}}$ and $c_{\alpha_{\rm s,t}}$ [10] are used.

Defining the tensor-to-scalar ratio at the pivot scale k_0 as $r(k_0)\equiv\mathcal{P}_{\rm t}(k_0)/\mathcal{P}_{\rm s}(k_0)$, it follows that

$$r(k_0)=16\epsilon_V(k_0)+c_r\delta_{\rm Pl}(k_0)\,, \quad (9)$$

where $c_r=8[3\alpha_0(3+5\sigma+6\sigma^2)-\nu_0\sigma(6+11\sigma)]\epsilon_V(k_0)/[9(\sigma+1)]-16\sigma\alpha_0\eta_V(k_0)/3$. These results show that LQC corrections modify the standard formulas in slow-roll inflation.

In the quasi-de Sitter background, $\delta_{\rm Pl}\propto k^{-\sigma}$ gives the relation $\delta_{\rm Pl}(k)\approx\delta_{\rm Pl}(k_0)(k/k_0)^{-\sigma}=\delta_{\rm Pl}(\ell_0)(\ell/\ell_0)^{-\sigma}$, where ℓ are the CMB multipoles related to k via $k\approx(h/10^4)\ell$ Mpc^{-1} ($h\approx0.7$ is the reduced Hubble constant). With the large-volume expansion of quantum corrections, we require that $\delta_{\rm Pl}(k)\ll1$ at all scales. For $\sigma>0$ the LQC correction is most significant on the largest scales observed in the CMB ($\ell=2$). This property can be clearly seen in Fig. 1, where the pivot scale for the scalar power spectrum is taken to be $\ell_0=29$. Imposing the condition $\delta_{\rm Pl}(\ell=2)\ll1$, this gives the bound

$$\delta_{\rm Pl}(\ell_0)\ll(2/\ell_0)^{\sigma} \quad (10)$$

at the multipole ℓ_0. For larger σ and ℓ_0, $\delta_{\rm Pl}(\ell_0)$ is constrained to be smaller.

For concreteness, let us consider the power-law potential $V(\varphi)=\lambda\varphi^n$, for which $\epsilon_V=n^2/(2\kappa^2\varphi^2)$ and

$$\eta_V=\frac{2(n-1)}{n}\epsilon_V\,,\qquad \xi_V^2=\frac{4(n-1)(n-2)}{n^2}\epsilon_V^2\,. \quad (11)$$

Among the variables σ, α_0, and ν_0 we have the relation (1), a condition under which, for given n and σ, the inflationary observables can be expressed via ϵ_V and $\delta\equiv\alpha_0\delta_{\rm Pl}$ for $\sigma\neq3$, or by ϵ_V and $\tilde\delta\equiv\nu_0\delta_{\rm Pl}$ for $\sigma=3$.

We carry out the CMB likelihood analysis by varying the parameters ϵ_V and δ in the Cosmological Monte Carlo (CosmoMC) code [11]. We use the 7-year WMAP data combined with Baryon Acoustic Oscillations (BAO) and the Hubble constant measurement from the Hubble Space Telescope (HST), which are currently considered to be the best dataset for the estimate of cosmological parameters [12]. We take the pivot wavenumber $k_0=0.002$ Mpc^{-1} ($\ell_0\approx29$) used by the WMAP team. $\delta(k_0)$ and $\epsilon_V(k_0)$ are constrained at this scale. While the bound on δ depends on the pivot scale (and it tends to be smaller for larger k_0), that on $k_0^\sigma\delta(k_0)$ does not.

The exponential term $e^{-\sigma x}=(k_0/k)^\sigma$ in Eq. (8) gives rise to the enhancement of the power spectra on large scales, as we see in Fig. 1. For $\sigma\gtrsim3$, the growth of this term is so significant that $\delta_{\rm Pl}(\ell)$ must be very much smaller than 1 for most of the scales observed in the CMB, in order to satisfy the bound $\delta_{\rm Pl}(\ell=2)\ll1$. More precisely, LQC corrections manifest themselves mainly at $\ell=2,3$ where cosmic variance dominates, so it seems implausible to isolate these effects. For $\sigma<3$, the LQC modification to the classical power spectra also affects larger multipoles ℓ, and hence it seems possible to constrain it from CMB anisotropies.

In Fig. 2 we plot the 2D posterior distributions on the parameters $\delta(k_0)$ and $\epsilon_V(k_0)$ with $k_0=0.002$ Mpc^{-1} for $n=2$ and $\sigma=2$. The two parameters are constrained to be $\delta(k_0)<7\times10^{-5}$ and $\epsilon_V(k_0)<0.012$ (95% CL). The modification of the large-scale power spectra ($\ell\lesssim20$) shown in Fig. 1 leads to the upper bound on $\delta(k_0)$. The condition (10) gives the prior $\delta_{\rm Pl}(\ell_0)\ll4.8\times10^{-3}$ at $\ell_0=29$, so that for $\alpha_0=O(1)$ the observational bound is smaller by two orders of magnitude.

For smaller σ the observational upper bound on $\delta(k_0)$ tends to be larger, with milder enhancement of the power spectra on large scales. In Fig. 3 we show the likelihood results for $\sigma=1$, in which case the LQC correction is constrained to be $\delta(k_0)<3.7\times10^{-2}$ (95% CL). Meanwhile,

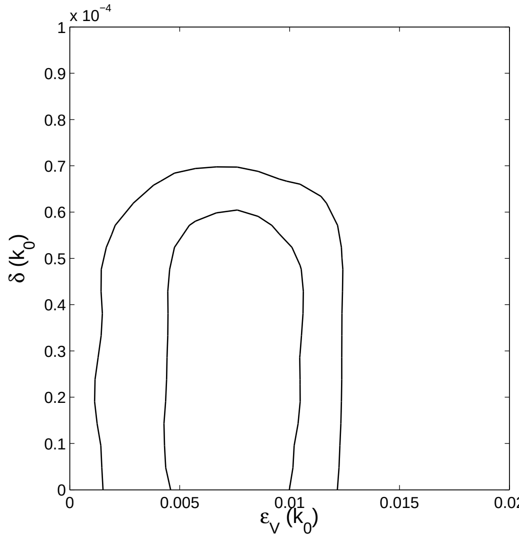

FIG. 2: 2D marginalized distribution for the quantum-gravity parameter $\delta(k_0) = \alpha_0 \delta_{\rm Pl}(k_0)$ and the slow-roll parameter $\epsilon_V(k_0)$ with the pivot $k_0 = 0.002$ Mpc^{-1} for $n = 2$ and $\sigma = 2$, using the data of WMAP7+BAO+HST. The internal and external solid lines correspond to the 68% and 95% confidence levels, respectively.

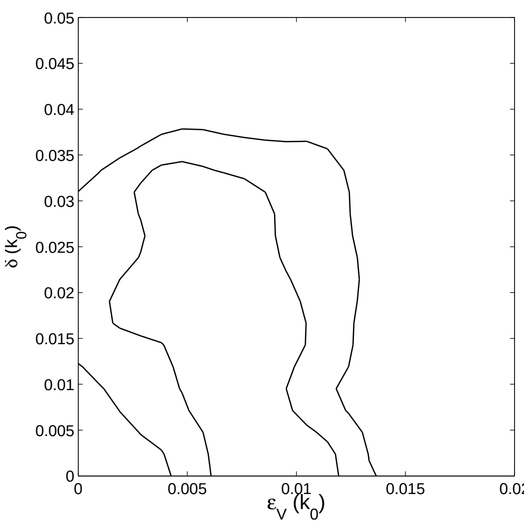

FIG. 3: 2D marginalized distributions as in Fig. 2, but for the case $n = 2$ and $\sigma = 1$.

the a-priori criterion (10) gives $\delta_{\rm Pl}(k_0) \ll 6.9 \times 10^{-2}$. For $\alpha_0 = O(1)$, the case $\sigma = 1$ is marginally consistent with the combined SR/$\delta_{\rm Pl}$ truncation.

For $\sigma \lesssim 1$, the exponential factor $e^{-\sigma x}$ does not change rapidly with smaller values of $f_{\rm s,t}$, so that the LQC effect on the power spectra would not be very significant even if $\delta(k_0)$ was as large as $\epsilon_V(k_0)$. Our likelihood analysis shows that the observational upper bound on $\delta(k_0)$ exceeds the a-priori upper limit of $\delta_{\rm Pl}(k_0)$ given by Eq. (10). Since $\delta(k_0)$ can be as large as 1, the validity of the approximation $\delta(k_0) < \epsilon_V(k_0)$ used in the main formulas may break down in such cases.

Under the conditions $\epsilon_V \ll 1$ and $\delta \ll 1$, it follows that $\epsilon_V \approx (\kappa^2/2)(\dot{\varphi}/H)^2$. Then the number of e-foldings during inflation is given by $N \equiv \int_t^{t_{\rm f}} d\tilde{t}\, H \approx \kappa \int_{\varphi_{\rm f}}^{\varphi} d\tilde{\varphi}/\sqrt{2\epsilon_V(\tilde{\varphi})}$, where $\varphi_{\rm f}$ is the field value at the end of inflation [which is determined by the condition $\epsilon_V \approx O(1)$]. For the power-law potentials one has $N \approx n/(4\epsilon_V) - n/4$, which gives $\epsilon_V \approx n/(4N+n)$. For $n = 2$, the theoretically constrained range $45 < N < 65$ corresponds to $0.008 < \epsilon_V < 0.011$. The probability distributions of ϵ_V in Figs. 2 and 3 are consistent with this range even in the presence of the LQC corrections, so the quadratic potential is compatible with observations as in standard cosmology.

In summary, in inflation combined with LQC inverse-volume corrections we provided general formulas for the scalar and tensor power spectra and placed observational bounds on the size of the correction $\delta_{\rm Pl}$ for a quadratic potential. We have shown that interesting and nontrivial effects can arise from the modified spacetime structure underlying the dynamics. Even though quantum-geometry corrections are small, they can significantly change the runnings of spectral indices. Thus, the observational bounds on $\delta_{\rm Pl}$ can be much closer to theoretical expectations than often thought in quantum gravity.

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