A new perspective on the Frenkel–Zhu fusion rule theorem

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Abstract

In this paper we prove a formula for fusion coefficients of affine Kac–Moody algebras first conjectured by Walton [M.A. Walton, Tensor products and fusion rules, Canad. J. Phys. 72 (1994) 527–536], and rediscovered by Feingold [A. Feingold, Fusion rules for affine Kac–Moody algebras, in: N. Sthanumoorthy, Kailash Misra (Eds.), Kac–Moody Lie Algebras and Related Topics, Ramanujan International Symposium on Kac–Moody Algebras and Applications, Jan. 28–31, 2002, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai, India, in: Contemp. Math., vol. 343, American Mathematical Society, Providence, RI, 2004, pp. 53–96]. It is a reformulation of the Frenkel–Zhu affine fusion rule theorem [I.B. Frenkel, Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1992) 123–168], written so that it can be seen as a beautiful generalization of the classical Parthasarathy–Ranga Rao–Varadarajan tensor product theorem [K.R. Parthasarathy, R. Ranga Rao, V.S. Varadarajan, Representations of complex semi-simple Lie groups and Lie algebras, Ann. of Math. (2) 85 (1967) 383–429].
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1. Introduction

Fusion rules play a very important role in conformal field theory [11], in the representation theory of vertex operator algebras [8–10], and in quite a few other areas. For example, fusion rules were used in [6] to obtain information on D-brane charge groups in string theory which on the other hand correspond to certain twisted K-groups. This line of research found a mathematical culmination in the theorem by Freed, Hopkins and Teleman [7], showing that twisted equivariant K-theory can be identified with a fusion ring. In [1] a connection was found between the fusion rules for the Virasoro minimal models and elementary abelian 2-groups. Further work in [5] extended this idea to find a connection between the fusion rules for type A_1 and A_2 on all levels, and elementary abelian 2-groups and 3-groups. This was extended as far as was possible in [17,18] to the case of A_ℓ for any rank ℓ and any level.

In [4] an introduction was given to the subject with major focus on the algorithmic aspects of computing fusion rules for affine Kac–Moody algebras. In particular, it was emphasized that the Kac–Walton algorithm [12,14,19] for fusion coefficients is closely related to the Racah–Speiser algorithm for tensor product decompositions, which was the subject of earlier work [2,3]. [4] included a conjecture on fusion coefficients which restates the Frenkel–Zhu theorem [10] in a form which shows it to be a beautiful generalization of the classical Parthasarathy–Ranga Rao–Varadarajan tensor product theorem [16]. That conjecture had already been made by Walton [20] in 1994, but we believe that it has not been proven up until now.

An outline of the organization of the paper is as follows. We give the definition of a fusion algebra in section two, then we give notation and background about finite-dimensional simple Lie algebras in section three. This includes facts about irreducible representations, contravariant Hermitian forms on them, special results for sl_2 and its representations, and projection operators. In section four we briefly give notations about affine algebras leading to the level k fusion algebra associated with simple Lie algebra **g**. In section five we discuss tensor products of irreducible finite-dimensional modules for **g** and the PRV theorem. In section six we state the Frenkel–Zhu fusion rule theorem, the Walton conjecture, what it says in the special case when $\mathbf{g} = sl_2$, and a corollary relating fusion coefficients to tensor product multiplicities. We begin the proof of the Walton conjecture by rewriting the Frenkel–Zhu theorem in several ways. In section seven we review the proof of the PRV theorem and refine it to help find a relationship between the spaces which occur in the Frenkel–Zhu theorem and the Walton conjecture. In section eight we put all these pieces together to finish the proof of the Walton conjecture.

2. Definition of fusion algebra

Let us begin with the definition of fusion algebra given by J. Fuchs [11]. A fusion algebra F is a finite-dimensional commutative associative algebra over $\mathbf{Q}$ with some basis

$$B = \{x_a \mid a \in A\}$$

so that the structure constants $N_{a,b}^c$ defined by

$$x_a \cdot x_b = \sum_{c \in A} N_{a,b}^c x_c$$

are nonnegative integers. There must be a distinguished index $\Omega \in A$ with the following properties. It is required that the matrix

$$C = [C_{a,b}] = \left[N_{a,b}^\Omega \right]$$

satisfies $C^2 = I$. Because $0 \leqslant N_{a,b}^c \in \mathbf{Z}$, either $C = I$ or C must be an order 2 permutation matrix, that is, there is a permutation $\sigma : A \to A$ with $\sigma^2 = 1$ and

$$C_{a,b} = \delta_{a,\sigma(b)}.$$

Write $\sigma(a) = a^*$ and call x_{a^*} the conjugate of x_a. Use it to define the nonnegative integers

$$N_{a,b,c} = N_{a,b}^{c^*}$$

which, by commutativity and associativity of the algebra product, are completely symmetric in a, b and c. Using this we also find that x_Ω is a multiplicative identity element in F and $\Omega^* = \Omega$.

In this paper we are interested in the structure constants of fusion algebras that are associated to affine Lie algebras.

3. Background and notation for finite-dimensional Lie algebras

Now we will introduce notations and review some basic results needed later. Let $\mathbf{g}$ be a finite-dimensional simple Lie algebra of rank ℓ with Cartan matrix $A = [a_{ij}]$ and Cartan subalgebra H. The simple roots and the fundamental weights of $\mathbf{g}$ are linear functionals

$$\alpha_1, \ldots, \alpha_\ell \quad \text{and} \quad \lambda_1, \ldots, \lambda_\ell,$$

respectively, in the dual space H^*. Let the integral weight lattice P be the $\mathbf{Z}$-span of the fundamental weights, and let

$$P^+ = \{n_1 \lambda_1 + \cdots + n_\ell \lambda_\ell \mid 0 \leqslant n_1, \ldots, n_\ell \in \mathbf{Z}\}$$

be the set of dominant integral weights of $\mathbf{g}$, and let

$$\theta = \sum_{i=1}^\ell \theta_i \alpha_i$$

be the highest root of **g**. The symmetric bilinear form $(\cdot, \cdot)$ on H^* is determined by

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad 1 \leqslant i, j \leqslant \ell,$$

and the normalization $(\theta, \theta) = 2$. The fundamental weights are determined by the conditions $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant \ell$, and the special "Weyl vector"

$$\rho = \sum_{i=1}^{\ell} \lambda_i$$

will play an important role in several formulas. It is useful to define

$$\check{\lambda} = \frac{2\lambda}{(\lambda, \lambda)} \quad \text{for any } 0 \neq \lambda \in H^*,$$

so we can write $(\lambda_i, \check{\alpha}_j) = \delta_{ij}$ and $a_{ij} = (\alpha_i, \check{\alpha}_j)$. We may also express

$$\theta = \sum_{i=1}^{\ell} \check{\theta}_i \check{\alpha}_i \quad \text{so} \quad \check{\theta}_i = \frac{\theta_i(\alpha_i, \alpha_i)}{2}.$$

The Weyl group W of **g** is defined to be the group of endomorphisms of H^* generated by the simple reflections corresponding to the simple roots,

$$r_i(\lambda) = \lambda - (\lambda, \check{\alpha}_i)\alpha_i, \quad 1 \leqslant i \leqslant \ell.$$

This is a finite group of isometries which preserve P. There is a partial order defined on H^* defined by

$$\lambda \leqslant \mu \quad \text{if and only if} \quad \mu - \lambda = \sum_{i=1}^{\ell} k_i \alpha_i \quad \text{for some } 0 \leqslant k_i \in \mathbf{Z}.$$

For $\lambda \in P^+$ let V^λ denote the finite-dimensional irreducible **g**-module with highest weight λ. It has the weight space decomposition $V^\lambda = \bigoplus_{\beta \in H^*} V_\beta^\lambda$, where

$$V_\beta^\lambda = \{ v \in V^\lambda \mid h \cdot v = \beta(h)v, \ \forall h \in H \}$$

is the β weight space of V^λ. Of course, there are only finitely many $\beta \in H^*$ such that V_β^λ is nonzero, and we denote by Π^λ that finite set of such β. Since $\dim(V_\lambda^\lambda) = 1$, a nonzero highest weight vector $v_\lambda^\lambda \in V_\lambda^\lambda$ is determined up to a scalar. The dual space $(V^\lambda)^* = \text{Hom}(V^\lambda, \mathbf{C})$ is also an irreducible highest weight **g**-module, called the contragredient module of V^λ. The action of **g** on $(V^\lambda)^*$ is given by

$$(x \cdot f)(v) = -f(x \cdot v) \quad \text{for } x \in \mathbf{g}, \ f \in (V^\lambda)^*, \ v \in V^\lambda.$$

The highest weight of $(V^\lambda)^*$ is denoted by λ^*, and equals the negative of the lowest weight of V^λ. For example, in the case when $\mathbf{g}$ is of type A_ℓ, if $\lambda = \sum_{i=1}^{\ell} n_i \lambda_i$ then $\lambda^* = \sum_{i=1}^{\ell} n_{\ell+1-i} \lambda_i$.

On V^λ with a chosen highest weight vector, $v_\lambda^\lambda \in V_\lambda^\lambda$, we have a positive definite contravariant Hermitian form [14] $(\cdot, \cdot): V^\lambda \times V^\lambda \to \mathbf{C}$ determined by the following conditions: (1) $(v_\lambda^\lambda, v_\lambda^\lambda) = 1$, (2) For any $v, v' \in V^\lambda$, and any $x \in \mathbf{g}$, we have $(x \cdot v, v') = -(v, x^\dagger \cdot v')$, where the map $x \to x^\dagger$ is the Chevalley involutive automorphism of $\mathbf{g}$ determined by its action on the generators

$$ e_i^\dagger = -f_i, \qquad f_i^\dagger = -e_i, \qquad h_i^\dagger = -h_i, \quad 1 \leqslant i \leqslant \ell. $$

Note that for any $v \in V_\beta^\lambda$, $v' \in V_{\beta'}^\lambda$, we have

$$ \beta(h_i)(v, v') = (h_i \cdot v, v') = -(v, -h_i \cdot v') = \beta'(h_i)(v, v') $$

so $0 = (\beta - \beta')(h_i)(v, v')$ for any Cartan generator h_i. This means that if $\beta \neq \beta'$ then $(v, v') = 0$ so different weight spaces are orthogonal. Let $\operatorname{Proj}_\beta^\lambda : V^\lambda \to V_\beta^\lambda$ denote the orthogonal projection operator.

If V^λ and V^μ are two irreducible highest weight modules with chosen highest weight vectors and positive definite contravariant Hermitian forms as above, then we have a positive definite contravariant Hermitian form on the tensor product $V^\lambda \otimes V^\mu$ given by

$$ \left(v_1^\lambda \otimes v_1^\mu, v_2^\lambda \otimes v_2^\mu\right) = \left(v_1^\lambda, v_2^\lambda\right)\left(v_1^\mu, v_2^\mu\right). $$

If V^ν is an irreducible submodule of $V^\lambda \otimes V^\mu$ then its orthogonal complement $(V^\nu)^\perp = \{v \in V^\lambda \otimes V^\mu \mid (v, V^\nu) = 0\}$ is clearly a $\mathbf{g}$-submodule since

$$ \left(x \cdot v, V^\nu\right) = -\left(v, x^\dagger \cdot V^\nu\right) = 0, \quad \text{for all } x \in \mathbf{g}, \ v \in \left(V^\nu\right)^\perp. $$

This shows that when the tensor product $V^\lambda \otimes V^\mu$ is decomposed into a direct sum of irreducible $\mathbf{g}$-modules, the distinct modules obtained are mutually orthogonal with respect to the contravariant Hermitian form. Let $\operatorname{Proj}_{V^\nu}^{\lambda, \mu} : V^\lambda \otimes V^\mu \to V^\nu$ denote the orthogonal projection operator from the tensor product to a particular irreducible submodule V^ν.

We will use certain facts about the representation theory of the simple Lie algebra $\mathbf{g} = \mathrm{sl}_2$ of type A_1 whose standard basis $\{e, f, h\}$ has the brackets $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. An irreducible finite-dimensional sl_2-module V^λ is determined by its highest weight, the non-negative integer $\lambda(h) = m$, so we write $V^\lambda = V(m)$. If v_0 is a highest weight vector then a basis of $V(m)$ can be written as $\{v_i \mid 0 \leqslant i \leqslant m\}$ where $v_i = \frac{1}{i!} f^i v_0$ and the action of $\mathbf{g}$ is given by the formulas:

$$ h \cdot v_i = (m - 2i)v_i, \qquad f \cdot v_i = (i + 1)v_{i+1}, \qquad e \cdot v_i = (m - i + 1)v_{i-1} $$

for $0 \leqslant i \leqslant m$ with the understanding that $v_j = 0$ for j outside that range. For any integer $p \geqslant 0$, we understand the operators e^p and f^p on $V(m)$ to mean p repetitions of the operators e and f, respectively. It is easy to see that the contravariant form has values $(v_i, v_j) = \delta_{i,j}\binom{m}{i}$, for $0 \leqslant i \neq j \leqslant m$, so the form is positive definite.

Lemma 3.1. *Let* $\mathbf{g} = \mathrm{sl}_2$ *and* $V(m)$ *be the irreducible finite-dimensional* sl_2*-module with highest integral weight* $m \geqslant 0$. *Then for any integer* $p \geqslant 1$, *with respect to the positive definite contravariant Hermitian form on* $V(m)$, *we have an orthogonal direct sum decomposition*

$$V(m) = \ker\big(f^p\big) \oplus \mathrm{Im}\big(e^p\big).$$

Proof. From the explicit formulas for the action, it is clear that $\ker(f^p)$ is the subspace of the p lowest weight spaces with basis $\{v_{m-p+1}, \ldots, v_m\}$ and that $\mathrm{Im}(e^p) = (\ker(f^p))^\perp$ is the subspace of all other weight spaces with basis $\{v_0, \ldots, v_{m-p}\}$. $\square$

We now go back to the general case of any finite-dimensional simple $\mathbf{g}$. Let V^λ be an irreducible $\mathbf{g}$-module, α any root of $\mathbf{g}$, and let $\mathbf{g}_\alpha$ be the corresponding subalgebra of $\mathbf{g}$ isomorphic to sl_2 with standard basis $\{e_\alpha, f_\alpha, h_\alpha\}$. The Chevalley involution acts on $\mathbf{g}_\alpha$ by $e_\alpha^\dagger = -f_\alpha$, $f_\alpha^\dagger = -e_\alpha$ and $h_\alpha^\dagger = -h_\alpha$. The complete reducibility of finite-dimensional sl_2-modules gives a direct sum decomposition

$$V^\lambda = \bigoplus_i V^\lambda_{\gamma_i}(m_i)$$

into irreducible $\mathbf{g}_\alpha$-modules, where $V^\lambda_{\gamma_i}(m_i)$ has $\mathbf{g}$-highest weight γ_i, and $\mathbf{g}_\alpha$-highest weight $\gamma_i(h_\alpha) = m_i$. If $V^\lambda_{\gamma_1}(m_1)$ is one of these, then its orthogonal complement is clearly a $\mathbf{g}_\alpha$-submodule by the same argument as given above for the decomposition of a tensor product. It means that this decomposition is an orthogonal direct sum decomposition with respect to the contravariant Hermitian form on V^λ.

Lemma 3.2. *Let* V^λ *be an irreducible* $\mathbf{g}$*-module,* α *any root of* $\mathbf{g}$, *and* $\mathbf{g}_\alpha$ *be the corresponding subalgebra of* $\mathbf{g}$ *isomorphic to* sl_2. *Let* $\beta \in \Pi^\lambda$ *be any weight of* V^λ. *Then, for any integer* $p \geqslant 0$ *such that* $p + \langle \beta, \alpha \rangle \geqslant 0$, *we have*

$$\big\{v \in V^\lambda_\beta \mid e_\alpha^p(v) = 0\big\} = \big\{v \in V^\lambda_\beta \mid f_\alpha^{p+\langle\beta,\alpha\rangle}(v) = 0\big\}.$$

Proof. The Weyl group reflection r_α acts on the weights Π^λ and $r_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$. It is also well known that the operator

$$R_\alpha = \big(\exp(f_\alpha)\big)\big(\exp(-e_\alpha)\big)\big(\exp(f_\alpha)\big) \in \mathrm{GL}\big(V^\lambda\big)$$

satisfies $R_\alpha(V^\lambda_\beta) = V^\lambda_{r_\alpha(\beta)}$ for any weight $\beta \in \Pi^\lambda$. It is clear from the definition of R_α that it acts on each of the $\mathbf{g}_\alpha$ submodules in the decomposition of V^λ given in the paragraph above the lemma. For any $0 \neq v \in V^\lambda_\beta$ we have $0 \neq R_\alpha(v) \in V^\lambda_{r_\alpha(\beta)}$. We can write $v = \sum_i v_i$ where $v_i \in V^\lambda_{\gamma_i}(m_i)$, and $e_\alpha^p(v) = 0$ iff $e_\alpha^p(v_i) = 0$ for each i, so we may assume v is in one such irreducible $\mathbf{g}_\alpha$-module. The condition $e_\alpha^p(v) = 0$ means v is in one of the top p weight spaces of its irreducible $\mathbf{g}_\alpha$-module. This is equivalent to saying that $R_\alpha(v)$ is in one of the bottom p weight spaces, that is, $f_\alpha^p(R_\alpha(v)) = 0$.

If $\langle\beta,\alpha\rangle \geqslant 0$ then $R_\alpha(v) = c f_\alpha^{\langle\beta,\alpha\rangle}(v)$ for some nonzero scalar c, which means $0 = f_\alpha^p(c f_\alpha^{\langle\beta,\alpha\rangle}(v)) = c f_\alpha^{p+\langle\beta,\alpha\rangle}(v)$.

If $\langle \beta, \alpha \rangle < 0$ but $p + \langle \beta, \alpha \rangle \geqslant 0$ then $R_\alpha(v) = c e_\alpha^{-\langle \beta, \alpha \rangle}(v)$ for some nonzero scalar c which means $0 = f_\alpha^p(c e_\alpha^{-\langle \beta, \alpha \rangle}(v)) = d f_\alpha^{p+\langle \beta, \alpha \rangle}(v)$ for a nonzero scalar d. $\square$

If V is any finite-dimensional vector space with a positive definite Hermitian form and W is any subspace of V then W has an orthogonal complement $W^\perp = \{v \in V \mid (v, w) = 0, \ \forall w \in W\}$ such that $V = W \oplus W^\perp$. Let $P_W : V \to W$ be the orthogonal projection of V onto W defined by $P_W(v) = w$ where $v = w + w'$ is the unique expression for $v \in V$ with $w \in W$ and $w' \in W^\perp$. If $L : V \to V$ is any linear transformation, there is a unique adjoint linear transformation $L^\dagger : V \to V$ determined by the conditions

$$(L(v), v') = (v, L^\dagger(v')), \quad \text{for all } v, v' \in V.$$

We call L self-adjoint when $L = L^\dagger$. Note that any orthogonal projection map is self-adjoint because if $v_1 = w_1 + w_1'$ and $v_2 = w_2 + w_2'$ for $w_1, w_2 \in W$ and $w_1', w_2' \in W^\perp$, then

$$(P_W(v_1), v_2) = (w_1, w_2 + w_2') = (w_1, w_2) = (w_1 + w_1', w_2) = (v_1, P_W(v_2))$$

so $P_W^\dagger = P_W$. Also, it is clear that $P_W^2 = P_W$.

Finally, later we will need the following lemma.

Lemma 3.3. *Let $V = U_1 \oplus U_2$ be an orthogonal direct sum decomposition of a finite-dimensional vector space V with a positive definite Hermitian form, and let W be any subspace of V. Then we have the orthogonal direct sum decomposition of W:*

$$W = P_W(U_1) \oplus (W \cap U_2).$$

Proof. Let $v \in W$ be in the orthogonal complement of $P_W(U_1)$. This means that for any $u_1 \in U_1$, we have

$$0 = (P_W(u_1), v) = (u_1, P_W^\dagger(v)) = (u_1, P_W(v)) = (u_1, v)$$

which means $v \in W \cap U_1^\perp = W \cap U_2$. $\square$

4. Notation for affine Lie algebras

Let

$$\hat{\mathbf{g}} = \mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

be the affine algebra constructed from $\mathbf{g}$ with derivation $d = -t\frac{d}{dt}$ adjoined as usual, and with Cartan subalgebra

$$\mathcal{H} = H \oplus \mathbf{C}c \oplus \mathbf{C}d.$$

The simple roots and the fundamental weights of $\hat{\mathbf{g}}$ are linear functionals

$$\alpha_0, \alpha_1, \ldots, \alpha_\ell \quad \text{and} \quad \Lambda_0, \Lambda_1, \ldots, \Lambda_\ell,$$

respectively, in the dual space $\mathcal{H}^*$. The simple roots of $\mathbf{g}$ form a basis of H^* (as do the fundamental weights), and we identify them with linear functionals in $\mathcal{H}^*$ having the same values on $H \subseteq \mathcal{H}$ and being zero on c and d. Let c^* and d^* in $\mathcal{H}^*$ be the functionals which are zero on H and which satisfy

$$c^*(c) = 1, \qquad c^*(d) = 0, \qquad d^*(c) = 0, \qquad d^*(d) = 1.$$

Extend the bilinear form $(\cdot, \cdot)$ to $\mathcal{H}^*$ by letting

$$\left(c^*, H^*\right) = 0 = \left(d^*, H^*\right), \qquad \left(c^*, c^*\right) = 0 = \left(d^*, d^*\right), \quad \text{and} \quad \left(c^*, d^*\right) = 1.$$

Then $\alpha_0 = d^* - \theta$ and

$$\Lambda_0 = c^*, \qquad \Lambda_i = \theta_i \frac{(\alpha_i, \alpha_i)}{2} c^* + \lambda_i = \check{\theta}_i c^* + \lambda_i, \quad 1 \leqslant i \leqslant \ell,$$

are determined by the conditions $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ for $0 \leqslant i, j \leqslant \ell$. Let the integral weight lattice $\hat{P}$ be the $\mathbf{Z}$-span of the fundamental weights, and let

$$\hat{P}^+ = \left\{ \sum_{i=0}^{\ell} n_i \Lambda_i \;\middle|\; 0 \leqslant n_i \in \mathbf{Z} \right\}$$

be the set of dominant integral weights of $\hat{\mathbf{g}}$.

The affine Weyl group $\widehat{W}$ of $\hat{\mathbf{g}}$ is the group of endomorphisms of $\mathcal{H}^*$ generated by the simple reflections corresponding to the simple roots,

$$r_i(\Lambda) = \Lambda - (\Lambda, \check{\alpha}_i)\alpha_i, \quad 0 \leqslant i \leqslant \ell.$$

This is an infinite group of isometries which preserve $\hat{P}$. The canonical central element, $c \in \hat{\mathbf{g}}$ acts on an irreducible $\hat{\mathbf{g}}$-module as a scalar k, called the level of the module. We will only discuss modules with highest weight $\Lambda \in \hat{P}^+$, which are the "nicest" in that they have affine Weyl group symmetry and satisfy the Weyl–Kac character formula. An irreducible highest weight $\hat{\mathbf{g}}$-module is uniquely determined by its highest weight

$$\Lambda = \sum_{i=0}^{\ell} n_i \Lambda_i \in \hat{P}^+$$

and, if we define $\theta_0 = 1 = \check{\theta}_0$, then

$$k = \Lambda(c) = \sum_{i=0}^{\ell} n_i \Lambda_i(c) = \sum_{i=0}^{\ell} n_i \theta_i \frac{(\alpha_i, \alpha_i)}{2} = \sum_{i=0}^{\ell} n_i \check{\theta}_i.$$

For fixed k there are only finitely many $\Lambda \in \hat{P}^+$ with $\Lambda(c) = k$, and we denote that finite set by $\hat{P}_k^+$. It is easy to see that $\widehat{W}$ preserves the level k weights $\{\Lambda \in \hat{P} \mid \Lambda(c) = k\}$. The affine hyperplane determined by the condition $\Lambda(c) = k$ can be projected onto H^* and the corresponding action of $\widehat{W}$ is such that the simple reflections r_i for $1 \leqslant i \leqslant \ell$ act as they were defined originally

on H^*, as isometries generating the finite Weyl group W of **g**. But the new affine reflection r_0 acts as $r_0(\lambda) = \lambda - (\lambda, \theta)\theta + k\theta = r_\theta(\lambda) + k\theta$, the composition of reflection r_θ and the translation by $k\theta$, which is not an isometry on H^*.

Irreducible $\hat{\mathbf{g}}$-modules $\hat{V}^\Lambda$ of level $k \geqslant 1$ are indexed by $\hat{P}_k^+$, but we can also index them by certain weights of **g** as follows. From the formulas above we can write

$$\Lambda = \sum_{i=0}^{\ell} n_i \Lambda_i = kc^* + \sum_{i=1}^{\ell} n_i \lambda_i.$$

So there is a bijection between $\hat{P}_k^+$ and the set of weights $\lambda = \sum_{i=1}^{\ell} n_i \lambda_i$ such that

$$k = n_0 + \sum_{i=1}^{\ell} n_i \theta_i \frac{(\alpha_i, \alpha_i)}{2} = n_0 + \sum_{i=1}^{\ell} n_i \check{\theta}_i = n_0 + \langle \lambda, \theta \rangle.$$

Since $n_0 \geqslant 0$, this is equivalent to the "level k condition"

$$\langle \lambda, \theta \rangle = \sum_{i=1}^{\ell} n_i \check{\theta}_i \leqslant k.$$

Define the set

$$P_k^+ = \left\{ \lambda = \sum_{i=1}^{\ell} n_i \lambda_i \in P^+ \,\middle|\, \langle \lambda, \theta \rangle \leqslant k \right\}$$

and let the index set A (as in the fusion algebra definition) be P_k^+. Then we see that irreducible modules on level k correspond to $\lambda \in P_k^+$. Fix level $k \geqslant 1$ and write the fusion algebra product (which has not been defined yet!)

$$[\lambda] \cdot [\mu] = \sum_{\nu \in P_k^+} N_{\lambda,\mu}^{(k)\nu} [\nu].$$

The distinguished identity element, $[0]$, corresponds to $\Lambda = kc^*$, and for each $[\lambda]$ there is a distinguished conjugate $[\lambda^*]$ such that $N_{\lambda,\mu}^{(k)0} = \delta_{\mu,\lambda^*}$. Knowing $N_{\lambda,\mu}^{(k)\nu}$ is equivalent to knowing the completely symmetric coefficients

$$N_{\lambda,\mu,\nu}^{(k)} = N_{\lambda,\mu}^{(k)\nu^*}.$$

Let $\mathcal{F}(\mathbf{g}, k)$ denote this fusion algebra.

5. Tensor product decompositions

There is a close relationship between the product in fusion algebras associated with an affine Kac–Moody algebra $\hat{\mathbf{g}}$ and tensor product decompositions of irreducible **g**-modules. Let V^λ

be the irreducible finite-dimensional **g**-submodule of $\hat{V}^\Lambda$ generated by a highest weight vector. In the special case when $\Lambda = k\Lambda_0 = kc^*$, that finite-dimensional **g**-module is V^0, the one-dimensional trivial **g**-module. Since **g** is semisimple, any finite-dimensional **g**-module is completely reducible. Therefore, we can write the tensor product of irreducible **g**-modules

$$V^\lambda \otimes V^\mu = \sum_{\nu \in P^+} \mathrm{Mult}^\nu_{\lambda,\mu} \, V^\nu$$

as the direct sum of irreducible **g**-modules, including multiplicities. This decomposition is independent of the level k and is part of the basic representation theory of **g**. The fusion products $[\lambda] \cdot [\mu]$ are obtained by a subtle truncation of the above summation.

The Racah–Speiser algorithm gives the formula

$$\mathrm{Mult}^\nu_{\lambda,\mu} = \sum_{w \in W} \epsilon(w) \, \mathrm{Mult}_\lambda \big(w(\nu + \rho) - \mu - \rho \big)$$

where W is the Weyl group of **g**, $\epsilon(w) = (-1)^{\mathrm{length}(w)}$ is the sign of w, the Weyl vector $\rho = \sum \lambda_i$ is the sum of the fundamental weights of **g**, and $\mathrm{Mult}_\lambda(\beta) = \dim(V^\lambda_\beta)$ is the inner multiplicity of the weight β in V^λ. Recall that $\Pi^\lambda = \{\beta \in H^* \mid \dim(V^\lambda_\beta) > 0\}$ denotes the set of all weights of V^λ. In fact, the only weights ν for which $\mathrm{Mult}^\nu_{\lambda,\mu}$ may be nonzero are those of the form $\nu = \beta + \mu$ where $\beta \in \Pi^\lambda$.

This algorithm assumes that you can already produce the weight diagram of any irreducible module, V^λ, so we should have discussed that first, but in fact the special case of the Racah–Speiser algorithm when $\mu = 0$ gives a recursion for the inner multiplicities of V^λ. Since V^0 is the trivial one-dimensional module, $V^\lambda \otimes V^0 = V^\lambda$, so $\mathrm{Mult}^\nu_{\lambda,0} = \delta_{\lambda,\nu}$ and therefore

$$0 = \sum_{w \in W} \epsilon(w) \, \mathrm{Mult}_\lambda \big(w(\nu + \rho) - \rho \big)$$

for $\nu \neq \lambda$. One knows that $\mathrm{Mult}_\lambda(w\lambda) = 1$ and $\mathrm{Mult}_\lambda(w\nu) = \mathrm{Mult}_\lambda(\nu)$ for all $w \in W$, so the above formula implies that

$$\mathrm{Mult}_\lambda(\nu) = - \sum_{1 \neq w \in W} \epsilon(w) \, \mathrm{Mult}_\lambda(\nu + \rho - w\rho)$$

for $\nu \neq \lambda$. Since $\rho > w\rho$ in the partial ordering on weights, this gives an effective recursion for $\mathrm{Mult}_\lambda(\nu)$.

In [2,3] Feingold studied certain patterns which occur in the tensor product decomposition of a fixed irreducible **g**-module, V^λ, with all other modules V^μ. For fixed λ, as μ varies there are only a finite number of different patterns of outer multiplicities which can occur, and there are sets of values for μ for which the pattern is constant, called zones of stability for tensor product decompositions. We have the following precise result from [3] about when a particular weight β of V^λ, reaches the zone of stability.

Theorem 5.1. *Let* $\lambda, \mu \in P^+$ *and* $\beta \in \Pi^\lambda$ *be such that* $\beta + \mu \in P^+$. *Let*

$$\beta - r_{\beta,j}\alpha_j, \ldots, \beta, \ldots, \beta + q_{\beta,j}\alpha_j$$

be the α_j weight string through β. If $\langle \mu, \alpha_j \rangle \geqslant q_{\beta, j}$ then

$$\text{Mult}_{\lambda, \mu}^{\beta + \mu} = \text{Mult}_{\lambda, \mu + \lambda_j}^{\beta + \mu + \lambda_j} \ .$$

Since $\langle \mu + \lambda_j, \alpha_j \rangle = \langle \mu, \alpha_j \rangle + 1$, it is clear that $\langle \mu, \alpha_j \rangle \geqslant q_{\beta, j}$ implies

$$\text{Mult}_{\lambda, \mu}^{\beta + \mu} = \text{Mult}_{\lambda, \mu + m\lambda_j}^{\beta + \mu + m\lambda_j} \quad \text{for all } m \geqslant 1.$$

This result shows that for fixed $\lambda \in P^+$ and fixed $\beta \in \Pi^\lambda$, the tensor product multiplicities $\text{Mult}_{\lambda, \mu}^{\beta + \mu}$ have zones of stability as μ varies, and it is sufficient to study the finite number of μ such that $\langle \mu, \alpha_j \rangle \leqslant q_{\beta, j}$ for $1 \leqslant j \leqslant \ell$.

There is another important result about tensor product coefficients which played a role in [2,3]. In 1977 Prof. Bertram Kostant drew the attention of Feingold to the following beautiful result of Parthasarathy, Ranga Rao and Varadarajan [16], which is here rewritten slightly.

Theorem 5.2. *(See [16].) Let $\lambda, \mu \in P^+$ and $\beta \in \Pi^\lambda$ be such that $\beta + \mu \in P^+$. Let $\ell = \text{rank}(\mathbf{g})$ and let $0 \neq e_j \in \mathbf{g}_{\alpha_j}$ be a root vector corresponding to the simple root α_j for $1 \leqslant j \leqslant \ell$. Then*

$$\text{Mult}_{\lambda, \mu}^{\beta + \mu} = \dim \{ v \in V_\beta^\lambda \mid e_j^{\langle \mu, \alpha_j \rangle + 1} v = 0, \ 1 \leqslant j \leqslant \ell \}.$$

6. The Frenkel–Zhu theorem and its reformulation

Now let us turn to the Frenkel–Zhu fusion rule theorem for affine Kac–Moody algebras. (Note that this is closely related to results of Gepner–Witten [13], which appeared much earlier in the physics literature. Also, see Haisheng Li [15].)

Theorem 6.1. *(See [10].) Let $\lambda, \mu, \nu \in P_k^+$, and let $0 \neq e_\theta \in \mathbf{g}_\theta$ be a root vector of $\mathbf{g}$ in the θ root space of $\mathbf{g}$. Let $v_\nu^\nu \in V^\nu$ be a highest weight vector and write*

$$\mathcal{H}' = \text{Hom}_{\mathbf{g}} (V^\lambda \otimes V^\mu \otimes V^\nu, \mathbf{C}).$$

Then the level k fusion coefficient $N_{\lambda, \mu, \nu}^{(k)}$, which is completely symmetric in λ, μ and ν, equals the dimension of the vector space

$$\text{FZ}_k(\lambda, \mu, \nu) = \{ f \in \mathcal{H}' \mid f (e_\theta^{k - \langle \nu, \theta \rangle + 1} V^\lambda \otimes V^\mu \otimes v_\nu^\nu) = 0 \}.$$

We now state the main result of this paper, the theorem, conjectured by Walton, which is a blending of the PRV and FZ theorems, showing that the FZ theorem is actually a beautiful generalization of the PRV theorem.

Theorem 6.2. *For $\lambda, \mu \in P_k^+$, $\beta \in \Pi^\lambda$ such that $\beta + \mu \in P_k^+$, we have $N_{\lambda, \mu}^{(k)(\beta + \mu)}$ equals the dimension of the space*

$$W_k^+(\lambda, \beta, \mu) = \{ v \in V_\beta^\lambda \mid e_j^{\langle \mu, \alpha_j \rangle + 1} v = 0, \ 1 \leqslant j \leqslant \ell, \text{ and } e_\theta^{k - \langle \beta + \mu, \theta \rangle + 1} v = 0 \}.$$

In [20] the statement of the conjecture is slightly different from above, with the condition $e_\theta^{k-\langle\beta+\mu,\theta\rangle+1}v = 0$ replaced by the condition $f_\theta^{k-\langle\mu,\theta\rangle+1}v = 0$. The equivalence of these two conditions is precisely the content of Lemma 3.2.

Theorem 6.2 implies the following result, which tells the level k at which the fusion coefficient associated with a single weight $\beta \in \Pi^\lambda$ equals the tensor product multiplicity associated with that weight.

Corollary 6.3. *Suppose* $\lambda, \mu \in P_k^+$, *and* $\beta \in \Pi^\lambda$ *is such that* $\beta + \mu \in P_k^+$. *Let the* θ *weight string through* β *in* Π^λ *be* $\beta - r\theta, \ldots, \beta, \ldots, \beta + q\theta$. *Then* $k \geqslant \langle\mu, \theta\rangle + r$ *implies* $N_{\lambda,\mu}^{(k)\,(\beta+\mu)} = \text{Mult}_{\lambda,\mu}^{\beta+\mu}$.

Before starting the proof of the theorem, we will show how it reproduces the well-known fusion coefficients in the special case when $\mathbf{g} = \text{sl}_2$, where $\ell = 1$, $\theta = \alpha_1$, and $P_k^+ = \{n_1\lambda_1 \mid n_1 \in \mathbf{Z}, 0 \leqslant n_1 \leqslant k\}$. In this case we use the notation $[n_1]$ instead of $n_1\lambda_1$, so $V^{[n_1]} = V(n_1)$ is the irreducible $\mathbf{g}$-module with highest weight $[n_1]$. The weights of $V^{[n_1]}$ are $\{\beta = [n_1 - 2i] \mid 0 \leqslant i \leqslant n_1\}$ and each weight space $V_{[n_1-2i]}^{[n_1]}$ is one-dimensional. For $0 \leqslant n_1 \leqslant n_2 \in \mathbf{Z}$, the tensor product decomposition

$$V^{[n_1]} \otimes V^{[n_2]} = \bigoplus_{i=0}^{n_1} V^{[n_1+n_2-2i]}$$

is well known. If $[n_1], [n_2] \in P_k^+$ then the fusion product corresponds to a truncation of this tensor product, so that only terms $[n_1 + n_2 - 2i] \in P_k^+$ could appear, with coefficients no larger than 1. Note that the following Corollary 6.4 says the truncation is somewhat stronger than that, requiring $n_1 + n_2 - 2i \leqslant k - i$. Since there is a symmetry between n_1 and n_2, it is not surprising to also find the condition $i \leqslant n_2$ symmetric to the assumption $i \leqslant n_1$.

Corollary 6.4. *For* $0 \leqslant n_1, n_2 \leqslant k$, $0 \leqslant i \leqslant n_1$ *with* $0 \leqslant n_1 - 2i + n_2 \leqslant k$, *the* sl_2 *fusion coefficient* $N_{[n_1],[n_2]}^{(k)[n_1+n_2-2i]}$ *equals* 1 *if* $i \leqslant n_2$ *and* $n_1 + n_2 - 2i \leqslant k - i$, *zero otherwise.*

Proof. For $1 \leqslant i \leqslant n_1$, the raising operator $e_1 = e_\theta$ sends $V_{[n_1-2i]}^{[n_1]}$ isomorphically onto $V_{[n_1-2i+2]}^{[n_1]}$, and kills the highest weight space $V_{[n_1]}^{[n_1]}$. This means that for $v \in V_{[n_1-2i]}^{[n_1]}$ and $p \geqslant 0$,

$$e_1^{p+1}v = 0 \quad \text{iff } n_1 < n_1 - 2i + 2(p+1) \quad \text{iff } i \leqslant p.$$

The conditions on v in the Walton space

$$W_k^+\big([n_1], [n_1 - 2i], [n_2]\big) = \big\{v \in V_{[n_1-2i]}^{[n_1]} \mid e_1^{n_2+1}v = 0 \text{ and } e_1^{k-(n_1+n_2-2i)+1}v = 0\big\}$$

are then $i \leqslant n_2$ and $n_1 + n_2 - 2i \leqslant k - i$. When these are satisfied, we have

$$W_k^+\big([n_1], [n_1 - 2i], [n_2]\big) = V_{[n_1-2i]}^{[n_1]}$$

so the sl_2 fusion coefficient $N_{[n_1],[n_2]}^{(k)\,[n_1+n_2-2i]} = 1$, and otherwise, it is zero. $\quad\square$

In order to prove Theorem 6.2 we must understand the connection between the PRV theorem, the statement of the theorem and the FZ theorem. We begin by rewriting the FZ theorem in a slightly different form. We can define a **g**-module map

$$\Phi : \mathrm{Hom}\big(V^\lambda \otimes V^\mu, V^{\nu^*}\big) \to \mathrm{Hom}\big(V^\lambda \otimes V^\mu \otimes V^\nu, \mathbf{C}\big)$$

by

$$(\Phi f)\big(v^\lambda \otimes v^\mu \otimes v^\nu\big) = \big(f\big(v^\lambda \otimes v^\mu\big)\big)\big(v^\nu\big).$$

It is easy to check that this is a **g**-module map and an isomorphism. In general, for V and W any two **g**-modules, $\mathrm{Hom}(V, W)$ is a **g**-module under the action, $(x \cdot L)(v) = x \cdot (L(v)) - L(x \cdot v)$ for any $v \in V$ and any $L \in \mathrm{Hom}(V, W)$. It may be helpful to use the notations $\pi_V : \mathbf{g} \to \mathrm{End}(V)$, $\pi_W : \mathbf{g} \to \mathrm{End}(W)$, and $\pi : \mathbf{g} \to \mathrm{End}(\mathrm{Hom}(V, W))$ to distinguish the representations of **g** on these three spaces. Then the above equation is saying that $\pi(x)(L) = \pi_W(x) \circ L - L \circ \pi_V(x)$.

We also have the definition of the space of **g**-module maps from V to W,

$$\mathrm{Hom}_{\mathbf{g}}(V, W) = \big\{ L \in \mathrm{Hom}(V, W) \,\big|\, \pi(x)(L) = 0, \ \forall x \in \mathbf{g} \big\}$$

$$= \big\{ L \in \mathrm{Hom}(V, W) \,\big|\, \pi_W(x) \circ L = L \circ \pi_V(x), \ \forall x \in \mathbf{g} \big\}.$$

If $v \in V_\beta$ is a weight vector of weight β, that is, for any $h \in H$, $\pi_V(h)v = \beta(h)v$, and L is any **g**-module map, then $\pi_W(h)L(v) = L(\pi_V(h)v) = L(\beta(h)v) = \beta(h)L(v)$ shows that $L(V_\beta) \subseteq W_\beta$. If $\mathrm{Proj}_\beta^V : V \to V_\beta$ and $\mathrm{Proj}_\beta^W : W \to W_\beta$ are the orthogonal projection operators, then it is easy to see that $L(\mathrm{Proj}_\beta^V(v)) = \mathrm{Proj}_\beta^W(L(v))$ for any $v \in V$.

Since Φ is a **g**-module isomorphism, it is clear that it restricts to an isomorphism

$$\Phi : \mathrm{Hom}_{\mathbf{g}}\big(V^\lambda \otimes V^\mu, V^{\nu^*}\big) \to \mathrm{Hom}_{\mathbf{g}}\big(V^\lambda \otimes V^\mu \otimes V^\nu, \mathbf{C}\big).$$

We wish to describe the preimage of the space $\mathrm{FZ}_k(\lambda, \mu, \nu)$ under Φ. Since Φ is an isomorphism, $f \in \mathrm{FZ}_k(\lambda, \mu, \nu)$ is of the form Φg for a unique element $g \in \mathrm{Hom}_{\mathbf{g}}(V^\lambda \otimes V^\mu, V^{\nu^*})$. The conditions on f mean that

$$\big(g\big(e_\theta^{k-\langle \nu, \theta \rangle + 1} V^\lambda \otimes V^\mu\big)\big)\big(v_\nu^\nu\big) = 0.$$

This allows us to rewrite the FZ theorem as follows.

Theorem 6.5. *(See [10].) Let* $\lambda, \mu, \nu \in P_k^+$, *and let* $0 \neq e_\theta \in \mathbf{g}_\theta$ *be a root vector of* **g** *in the* θ *root space of* **g**. *Let* $v_\nu^\nu \in V^\nu$ *be a highest weight vector and write*

$$\mathcal{H} = \mathrm{Hom}_{\mathbf{g}}\big(V^\lambda \otimes V^\mu, V^{\nu^*}\big).$$

Then the level k fusion coefficient $N_{\lambda,\mu,\nu}^{(k)}$ *equals the dimension of the space*

$$\mathrm{FZ}_k'(\lambda, \mu, \nu) = \big\{ g \in \mathcal{H} \,\big|\, g\big(e_\theta^{k-\langle \nu, \theta \rangle + 1} V^\lambda \otimes V^\mu\big)\big(v_\nu^\nu\big) = 0 \big\}. \tag{6.1}$$

There is a natural isomorphism of **g**-modules

$$\Psi : \mathrm{Hom}\big(V^*, W\big) \to W \otimes V \tag{6.2}$$

which is defined as follows. For any $L \in \mathrm{Hom}(V^*, W)$,

$$\Psi(L) = \sum_{j=1}^{d} L\big(v_j^*\big) \otimes v_j$$

where $d = \dim(V) = \dim(V^*)$, $\{v_1, \ldots, v_d\}$ is any basis of V and $\{v_1^*, \ldots, v_d^*\}$ is the dual basis of V^*, that is, the basis such that $v_i^*(v_j) = \delta_{ij}$. The inverse map sends a basic tensor $w \otimes v \in W \otimes V$ to the element in $\mathrm{Hom}(V^*, W)$ which sends any $f \in V^*$ to $f(v)w \in W$. We will always choose the basis of V to consist of weight vectors, and if v_j has weight μ_j, so that for any $h \in H$, $\pi_V(h)v_j = \mu_j(h)v_j$, then it is easy to see that the weight of the dual vector v_j^* is $-\mu_j$. Namely, by the definition of the representation of **g** on the dual space V^*, for $1 \leqslant i \leqslant d$ we have

$$\big(\pi_{V^*}(h)v_j^*\big)(v_i) = -v_j^*\big(\pi_V(h)v_i\big) = -v_j^*\big(\mu_i(h)v_i\big) = -\mu_i(h)v_j^*(v_i)$$
$$= -\mu_i(h)\delta_{ij} = -\mu_j(h)\delta_{ij} = -\mu_j(h)v_j^*(v_i)$$

which says that $\pi_{V^*}(h)v_j^* = -\mu_j(h)v_j^*$. So $\Pi^{\lambda^*} = -\Pi^\lambda$. This means that a highest weight vector $v_\nu^\nu \in V_\nu^\nu$ has a dual lowest weight vector $v_{-\nu}^{\nu^*} \in V_{-\nu}^{\nu^*}$, and all other weight vectors of V^{ν^*} with weights above $-\nu$ are zero on v_ν^ν. In other words, with respect to the positive definite Hermitian form on the irreducible module V^{ν^*}, the orthogonal complement of the lowest weight space $V_{-\nu}^{\nu^*}$ is the subspace of linear functionals in V^{ν^*} that send v_ν^ν to 0. We now see that

$$\mathrm{FZ}_k'(\lambda, \mu, \nu) = \big\{g \in \mathcal{H} \mid g\big(e_\theta^{k-\langle \nu, \theta \rangle + 1} V^\lambda \otimes V^\mu\big) \in \big(V_{-\nu}^{\nu^*}\big)^\perp\big\}. \tag{6.3}$$

For any $g \in \mathcal{H}$ we know that $\mathrm{Im}(g)$ is a submodule of V^{ν^*}, so if $g \neq 0$ then g is surjective. Also, g sends weight vectors to weight vectors of the same weight, and g sends highest (respectively, lowest) weight vectors to highest (respectively, lowest) weight vectors. V^{ν^*} has a one-dimensional highest weight space in which we have chosen a basis vector $v_{\nu^*}^{\nu^*} \in V_{\nu^*}^{\nu^*}$. V^{ν^*} also has a one-dimensional lowest weight space in which we have chosen a basis vector $v_{-\nu}^{\nu^*} \in V_{-\nu}^{\nu^*}$. The tensor product $V^\lambda \otimes V^\mu$ decomposes into the direct sum of irreducible modules, but g must send any highest (respectively, lowest) weight vector whose weight is not ν^* (respectively, not $-\nu$) to zero, so it sends all irreducible components whose highest weight is not ν^* to zero. The dimension of the space of highest (respectively, lowest) weight vectors in $V^\lambda \otimes V^\mu$ of weight ν^* (respectively, $-\nu$) is the tensor product multiplicity $M = \mathrm{Mult}_{\lambda,\mu}^{\nu^*}$, so we may choose a basis $\{u_1, \ldots, u_M\}$ of that HWV space U^+ (respectively, LWV space U^-) and determine $g_i \in \mathcal{H}$ uniquely by the conditions $g_i(u_j) = \delta_{i,j} v_{\nu^*}^{\nu^*}$ (respectively, $g_i(u_j) = \delta_{i,j} v_{-\nu}^{\nu^*}$) for $1 \leqslant i, j \leqslant M$. Then $\{g_1, \ldots, g_M\}$ is a basis of $\mathcal{H}$. Let us denote by $\mathcal{U}(\mathbf{g})$ the universal enveloping algebra of **g**. It is clear that g_i takes the submodule $\mathcal{U}(\mathbf{g})u_i$ isomorphically to V^{ν^*} and sends all other irreducible submodules $\mathcal{U}(\mathbf{g})u_j$, $j \neq i$, of the tensor product to zero, so it is essentially an orthogonal projection from the tensor product to one of its components followed by an isomorphism. Let $\mathrm{Proj}_{U^+}^{\lambda,\mu}$ be the orthogonal projection from $V^\lambda \otimes V^\mu$ to the subspace of highest weight vectors of weight ν^*,

and let $\mathrm{Proj}_{U^-}^{\lambda,\mu}$ be the orthogonal projection from $V^\lambda \otimes V^\mu$ to the subspace of lowest weight vectors of weight $-\nu$. Then for any $v \in V^\lambda \otimes V^\mu$, write $v = u + v' + v''$ where $u = \mathrm{Proj}_{U^-}^{\lambda,\mu}(v) \in U^-$, v' is of weight $-\nu$ but is orthogonal to U^- so is not a lowest weight vector and must be a sum of vectors from irreducible components whose highest weights are not ν^*, and v'' is a sum of vectors of weights not $-\nu$. Then $g(v) = g(u) + g(v') + g(v'')$ with $g(u) \in V_{-\nu}^{\nu^*}$, and $g(v') = 0$ and $g(v'')$ is a sum of vectors of weights not $-\nu$, so $\mathrm{Proj}_{-\nu}^{\nu^*}(g(v)) = g(u) = g(\mathrm{Proj}_{U^-}^{\lambda,\mu}(v))$. A similar argument applies to U^+, so we have shown that for any $g \in \mathcal{H}$ we have

$$g \circ \mathrm{Proj}_{U^+}^{\lambda,\mu} = \mathrm{Proj}_{\nu^*}^{\nu^*} \circ g, \tag{6.4}$$

$$g \circ \mathrm{Proj}_{U^-}^{\lambda,\mu} = \mathrm{Proj}_{-\nu}^{\nu^*} \circ g. \tag{6.5}$$

But this means that we can rewrite the Frenkel–Zhu space in (6.3) as

$$\mathrm{FZ}_k'(\lambda,\mu,\nu) = \left\{ g \in \mathcal{H} \mid \mathrm{Proj}_{-\nu}^{\nu^*} g\left(e_\theta^{k-\langle \nu,\theta \rangle + 1} V^\lambda \otimes V^\mu \right) = 0 \right\} \tag{6.6}$$

$$= \left\{ g \in \mathcal{H} \mid g\left(\mathrm{Proj}_{U^-}^{\lambda,\mu}\left(e_\theta^{k-\langle \nu,\theta \rangle + 1} V^\lambda \otimes V^\mu \right) \right) = 0 \right\}. \tag{6.7}$$

7. Review of the proof of the PRV theorem

Now we will review the proof of the PRV theorem and see if it allows us to find an isomorphism between the Frenkel–Zhu space $\mathrm{FZ}_k'(\lambda,\mu,\nu)$ and the Walton space $W_k^+(\lambda,\beta,\mu)$ when $\nu^* = \beta + \mu$.

In the proof of the PRV theorem one looks at the **g**-module $V = \mathrm{Hom}(V^{\mu^*}, V^\lambda)$, where $\pi : \mathbf{g} \to \mathrm{End}(V)$ denotes the representation. As noted above (see Eq. (6.2)), $V \cong V^\lambda \otimes V^\mu$, and this isomorphism is given by the map Ψ which sends irreducible components in V to isomorphic irreducible components in $V^\lambda \otimes V^\mu$. The proof begins by considering the subspace of all lowest weight vectors (LWVs) in V,

$$U = \left\{ L \in V \mid \pi(f_i)L = 0, \ 1 \leqslant i \leqslant \ell \right\}$$

where $\ell = \mathrm{rank}(\mathbf{g})$ and e_i, f_i, h_i are the generators of **g** with the usual Serre relations. Then

$$L \in U \quad \mathrm{iff} \quad \pi_\lambda(f_i) \circ L = L \circ \pi_{\mu^*}(f_i), \quad \mathrm{for}\ 1 \leqslant i \leqslant \ell.$$

It is clear that U is invariant under the operators $\pi(h_j)$, so it has a weight space decomposition

$$U = \bigoplus_{m=1}^r U_m$$

where $U_m = \{ L \in U \mid \pi(h)L = -\nu_m(h)L, \ \forall h \in H \}$ is the $-\nu_m$-weight space, $-\nu_1, \ldots, -\nu_r$ are the distinct lowest weights of irreducible components in V whose corresponding highest weights are $\nu_1^*, \ldots, \nu_r^*$. Furthermore,

$$\dim(U_m) = \mathrm{Mult}_{\lambda,\mu}^{\nu_m^*}$$

is the multiplicity of $V^{v_m^*}$ in the tensor product $V^\lambda \otimes V^\mu$ because the independent vectors in U_m each generate a distinct irreducible component in V. Let $v_1^* = v_{\mu^*}^{\mu^*}$ be a highest weight vector (HWV) in V^{μ^*} of weight μ^* dual to $v_1 = v_{-\mu^*}^\mu$ a LWV in V^μ of weight $-\mu^*$. The key step in the proof of the PRV theorem is the following lemma.

Lemma 7.1. *Define the linear map $\xi : U \to V^\lambda$ by*

$$\xi(L) = L(v_1^*), \quad \forall L \in U.$$

Then ξ is injective and the range of ξ equals

$$V' = \{ v \in V^\lambda \mid \pi_\lambda(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} v = 0, \ 1 \leqslant i \leqslant \ell \}.$$

Proof. Because the highest weight vector $v_1^* \in V^{\mu^*}$ satisfies

$$\pi_{\mu^*}(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} v_1^* = 0$$

for $1 \leqslant i \leqslant \ell$, we have

$$\pi_\lambda(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} L(v_1^*) = L(\pi_{\mu^*}(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} v_1^*) = 0$$

so $\xi(U) \subseteq V'$. Let $\mathbf{g} = \mathbf{g}^- \oplus H \oplus \mathbf{g}^+$ be the triangular decomposition of $\mathbf{g}$, where $\mathbf{g}^-$ is the Lie subalgebra of $\mathbf{g}$ generated by the negative root vectors, that is, the span of $f_1, \ldots, f_\ell$ and all their multibrackets, and similarly $\mathbf{g}^+$ is generated by the positive root vectors. Let $\mathcal{U}(\mathbf{g})$ be the universal enveloping algebra of $\mathbf{g}$ and extend the meaning of any representation of $\mathbf{g}$ to include the representation of the associative algebra $\mathcal{U}(\mathbf{g})$. We may also have use for the universal enveloping algebras $\mathcal{U}(\mathbf{g}^-)$ and $\mathcal{U}(\mathbf{g}^+)$. It is well known that $\mathcal{U}(\mathbf{g}^-)$ is spanned by all products of the form $y = f_{i_1} \cdots f_{i_s}$ for any $s \geqslant 0$ and any $1 \leqslant i_j \leqslant \ell$ for $1 \leqslant j \leqslant s$, and that $V^{\mu^*} = \mathcal{U}(\mathbf{g}^-) v_1^*$ is spanned by all vectors of the form

$$\pi_{\mu^*}(y) v_1^* = \pi_{\mu^*}(f_{i_1}) \cdots \pi_{\mu^*}(f_{i_s}) v_1^*$$

for y as above. If $L(v_1^*) = 0$ for some $L \in U$ then we get

$$0 = \pi_\lambda(y) L(v_1^*) = L(\pi_{\mu^*}(y) v_1^*)$$

showing that $L = 0$ and therefore ξ is injective. Let $v \in V'$ be arbitrary and try to define $L \in V$ by

$$L(\pi_{\mu^*}(y) v_1^*) = \pi_\lambda(y) v$$

for any $y \in \mathcal{U}(\mathbf{g}^-)$. If $\pi_{\mu^*}(y) v_1^* = 0$ then it is known that y can be written

$$y = \sum_{i=1}^\ell y_i \, f_i^{\langle \mu^*, \alpha_i \rangle + 1}$$

for some $y_i \in \mathcal{U}(\mathbf{g}^-)$, so $\pi_\lambda(y)v = 0$. This means that L is well defined on $\pi_{\mu^*}(\mathcal{U}(\mathbf{g}^-))v_1^* = V^{\mu^*}$. By the definition of the linear map L we have

$$\big(\pi_\lambda(f_i) \circ L\big)\big(\pi_{\mu^*}(y)v_1^*\big) = \pi_\lambda(f_i y)v = L\big(\pi_{\mu^*}(f_i y)v_1^*\big) = \big(L \circ \pi_{\mu^*}(f_i)\big)\big(\pi_{\mu^*}(y)v_1^*\big)$$

which shows that $\pi_\lambda(f_i) \circ L = L \circ \pi_{\mu^*}(f_i)$ so $L \in U$. This completes the argument that ξ is an isomorphism from U to V'.   □

Now suppose that $L \in U_m$ for some $1 \leqslant m \leqslant r$, so $\pi(h)L = -\nu_m(h)L$ for any $h \in H$. But $\pi(h)L = \pi_\lambda(h) \circ L - L \circ \pi_{\mu^*}(h)$ so $\xi(L) \in V^\lambda_{\mu^*-\nu_m}$ has weight $\mu^* - \nu_m$ because

$$\pi_\lambda(h)\big(Lv_1^*\big) = L\big(\pi_{\mu^*}(h)v_1^*\big) - \nu_m(h)Lv_1^* = L\big(\mu^*(h)v_1^*\big) - \nu_m(h)Lv_1^*$$
$$= \big(\mu^* - \nu_m\big)(h)Lv_1^*.$$

This shows that ξ provides an isomorphism between each subspace U_m and

$$V'_{\mu^*-\nu_m} = \big\{v \in V^\lambda_{\mu^*-\nu_m} \mid \pi_\lambda(f_i)^{\langle \mu^*, \alpha_i \rangle + 1}v = 0, \ 1 \leqslant i \leqslant \ell\big\}.$$

The PRV notation for this subspace is $V^-(\lambda; \mu^* - \nu_m, \mu^*)$ and their result is the formula for the tensor product multiplicity

$$\mathrm{Mult}^{\nu_m^*}_{\lambda,\mu} = \dim\big(V^-\big(\lambda; \mu^* - \nu_m, \mu^*\big)\big).$$

Replacing f_i by e_i in the definition of the space $V^-(\lambda; \gamma, \mu^*)$ one gets another space,

$$V^+\big(\lambda; \gamma, \mu^*\big) = \big\{v \in V^\lambda_\gamma \mid \pi_\lambda(e_i)^{\langle \mu^*, \alpha_i \rangle + 1}v = 0, \ 1 \leqslant i \leqslant \ell\big\}.$$

In the proof of the PRV theorem it is shown that

$$\dim\big(V^-\big(\lambda; \gamma, \mu^*\big)\big) = \dim\big(V^+\big(\lambda; -\gamma^*, \mu\big)\big)$$

by using an automorphism coming from the longest element of the Weyl group, W. Then the final result of the PRV theorem is that

$$\mathrm{Mult}^{\nu_m^*}_{\lambda,\mu} = \dim\big(V^+\big(\lambda; \nu_m^* - \mu, \mu\big)\big).$$

To understand this we must discuss the longest element and a little bit of the theory of Lie groups. First it is necessary to know that the elements of the Weyl group are in one-to-one correspondence with the Weyl chambers in H^*. The dominant chamber, P^+, corresponding to the identity element in W, is also associated with a choice of simple roots, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, or with a choice of positive roots, R^+, by the condition $\lambda \in P^+$ iff $\langle \lambda, \alpha_i \rangle \geqslant 0$, for $1 \leqslant i \leqslant \ell$. The opposite chamber $-P^+$ defined by the conditions $\langle \lambda, \alpha_i \rangle \leqslant 0$ is related to P^+ by a unique element $w_0 \in W$ such that $w_0(P^+) = -P^+$, which means $w_0(\Delta) = -\Delta$, and $w_0(R^+) = R^-$. This is the longest element whose length is the number of positive roots and whose order is 2. For example, in type A_2, $w_0 = r_1 r_2 r_1 = r_\theta$, but in type B_2, $w_0 = r_1 r_2 r_1 r_2 \neq r_\theta$. Since $w_0(\Delta) = -\Delta$,

there is an order 2 permutation $\sigma \in S_\ell$ such that $w_0(\alpha_i) = -\alpha_{\sigma(i)}$ for $1 \leqslant i \leqslant \ell$. If $\nu \in P^+$ then $w_0(\nu) = -\nu^*$ is the lowest weight in Π^ν, so we have

$$\langle \nu, \alpha_i \rangle = \langle w_0(\nu), w_0(\alpha_i) \rangle = \langle -\nu^*, -\alpha_{\sigma(i)} \rangle = \langle \nu^*, \alpha_{\sigma(i)} \rangle.$$

We use $\nu^* = -w_0(\nu)$ to extend the definition of dual weight to any $\nu \in H^*$. Note that θ is the highest weight of the adjoint representation and $-\theta = w_0(\theta) = -\theta^*$ is the lowest weight, so $\theta^* = \theta$. Therefore, for any $\nu \in H^*$ we have

$$\langle \nu, \theta \rangle = \langle w_0(\nu), w_0(\theta) \rangle = \langle -\nu^*, -\theta \rangle = \langle \nu^*, \theta \rangle.$$

We say $\pi_V : \mathbf{g} \to \mathrm{End}(V)$ is an integrable representation when $\pi_V(H)$ acts diagonalizably on V and all $\pi_V(e_i)$ and $\pi_V(f_i)$ are locally nilpotent on V. This is certainly true for V any finite-dimensional $\mathbf{g}$-module, including the adjoint representation, $\mathbf{g}$ itself, so that $\exp(\pi_V(x)) \in \mathrm{GL}(V)$ and $\exp(\mathrm{ad}(x)) \in \mathrm{Aut}(\mathbf{g})$ for all $x = e_i$, $x = f_i$ and $x = h \in H$. It is not hard to check that

$$\left(\exp(\pi_V(x)) \right) \pi_V(y) \left(\exp(\pi_V(x)) \right)^{-1} = \pi_V \left(\exp(\mathrm{ad}(x)) y \right)$$

for all $y \in \mathbf{g}$. Of particular interest are the elements

$$r_i^{\pi_V} = \left(\exp(\pi_V(f_i)) \right) \left(\exp(\pi_V(-e_i)) \right) \left(\exp(\pi_V(f_i)) \right) \in \mathrm{GL}(V)$$

for $1 \leqslant i \leqslant \ell$. It is known [14] that $r_i^{\pi_V}(V_\mu) = V_{r_i(\mu)}$ for any weight μ of V, and $r_i^{\mathrm{ad}}(\mathbf{g}_\alpha) = \mathbf{g}_{r_i(\alpha)}$ for any root α of $\mathbf{g}$. If the longest element is written as a product of simple reflections, $w_0 = r_{i_1} \cdots r_{i_s}$, then we have corresponding elements

$$w_0^{\pi_V} = r_{i_1}^{\pi_V} \cdots r_{i_s}^{\pi_V} \in \mathrm{GL}(V) \quad \text{and} \quad w_0^{\mathrm{ad}} = r_{i_1}^{\mathrm{ad}} \cdots r_{i_s}^{\mathrm{ad}} \in \mathrm{Aut}(\mathbf{g})$$

such that

$$w_0^{\pi_V} \circ \pi_V(y) \circ \left(w_0^{\pi_V} \right)^{-1} = \pi_V \left(w_0^{\mathrm{ad}}(y) \right)$$

so using $y = h \in H$ we can get

$$w_0^{\pi_V}(V_\mu) = V_{w_0(\mu)} \quad \text{and} \quad w_0^{\mathrm{ad}}(\mathbf{g}_\alpha) = \mathbf{g}_{w_0(\alpha)}.$$

In particular, this means that for $1 \leqslant i \leqslant \ell$, we have

$$w_0^{\mathrm{ad}}(e_i) \in \mathbf{g}_{w_0(\alpha_i)} = \mathbf{g}_{-\alpha_{\sigma(i)}}$$

so $w_0^{\mathrm{ad}}(e_i) = c_i f_{\sigma(i)}$ for some $0 \neq c_i \in \mathbf{C}$ and $w_0^{\mathrm{ad}}(f_i) = c_i^{-1} e_{\sigma(i)}$. Then we have

$$w_0^{\pi_V} \circ \pi_V(f_i) = \pi_V \left(w_0^{\mathrm{ad}}(f_i) \right) \circ w_0^{\pi_V} = c_i^{-1} \pi_V(e_{\sigma(i)}) \circ w_0^{\pi_V}$$

and for any power, p_i,

$$w_0^{\pi_V} \circ \pi_V(f_i)^{p_i} = c_i^{-p_i} \pi_V(e_{\sigma(i)})^{p_i} \circ w_0^{\pi_V}.$$

Using $p_i = \langle \mu^*, \alpha_i \rangle + 1$ and $V = V^\lambda$, we see that $w_0^{\pi_V}$ provides an isomorphism between

$$V^-(\lambda; \gamma, \mu^*) = \{v \in V_\gamma^\lambda \mid \pi_\lambda(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} v = 0, \; 1 \leqslant i \leqslant \ell\}$$

and

$$V^+(\lambda; -\gamma^*, \mu) = \{v \in V_{-\gamma^*}^\lambda \mid \pi_\lambda(e_i)^{\langle \mu, \alpha_i \rangle + 1} v = 0, \; 1 \leqslant i \leqslant \ell\}.$$

Since $w_0^{\mathrm{ad}}(\mathbf{g}_\theta) = \mathbf{g}_{-\theta}$ we also have $w_0^{\mathrm{ad}}(e_\theta) = c f_\theta$ for some $0 \neq c \in \mathbf{C}$ and for any power, p,

$$w_0^{\pi_V} \circ \pi_V(f_\theta)^p = c^{-p} \, \pi_V(e_\theta)^p \circ w_0^{\pi_V}.$$

Applying $w_0^{\pi_\lambda}$ to the space $W_k^+(\lambda, \beta, \mu)$ in Theorem 6.2 gives the isomorphic space

$$W_k^-(\lambda, -\beta^*, \mu^*)$$
$$= \{v \in V_{-\beta^*}^\lambda \mid \pi_\lambda(f_j)^{\langle \mu^*, \alpha_j \rangle + 1} v = 0, \; 1 \leqslant j \leqslant \ell, \text{ and } \pi_\lambda(f_\theta)^{k - \langle \beta + \mu, \theta \rangle + 1} v = 0\}. \quad (7.1)$$

It is clear that $W_k^-(\lambda, -\beta^*, \mu^*)$ is a subspace of $V^-(\lambda; -\beta^*, \mu^*)$,

$$W_k^-(\lambda, -\beta^*, \mu^*) = \{v \in V^-(\lambda; -\beta^*, \mu^*) \mid \pi_\lambda(f_\theta)^{k - \langle \beta + \mu, \theta \rangle + 1} v = 0\}$$

which corresponds by ξ to a subspace of U. Our next step is to find the condition on $L \in U$ which corresponds to this subspace.

8. Conclusion of the proof

The root vector $f_\theta \in \mathbf{g}_{-\theta}$ can be expressed as some multibracket of the simple root vectors $f_1, \ldots, f_\ell$, so $L \in U$ implies that $\pi(f_\theta) L = 0$ so $\pi_\lambda(f_\theta) \circ L = L \circ \pi_{\mu^*}(f_\theta)$. Furthermore, since $-\theta$ is the lowest root of $\mathbf{g}$, $[f_\theta, f_i] = 0$ for $1 \leqslant i \leqslant \ell$, so in any representation of $\mathbf{g}$, the representatives of these root vectors commute. For any $p \geqslant 1$ define the subspace of V'

$$V'(p) = \{v \in V^\lambda \mid \pi_\lambda(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} v = 0, \; 1 \leqslant i \leqslant \ell, \; \pi_\lambda(f_\theta)^p v = 0\}.$$

Then for any $L \in U$, $\xi(L) \in V'(p)$ iff $\pi_\lambda(f_\theta)^p L(v_{\mu^*}^{\mu^*}) = 0$ iff $\pi_\lambda(y) \pi_\lambda(f_\theta)^p L(v_{\mu^*}^{\mu^*}) = 0$ for all $y \in \mathcal{U}(\mathbf{g}^-)$. But since $\pi_\lambda(y)$ commutes with $\pi_\lambda(f_\theta)$, and since $\pi_\lambda(y) L(v_{\mu^*}^{\mu^*}) = L(\pi_{\mu^*}(y) v_{\mu^*}^{\mu^*})$ and $\mathcal{U}(\mathbf{g}^-) v_{\mu^*}^{\mu^*} = V^{\mu^*}$, so

$$\xi(L) \in V'(p) \quad \text{iff} \quad \pi_\lambda(f_\theta)^p L(V^{\mu^*}) = 0 \quad \text{iff} \quad L(V^{\mu^*}) \subseteq \mathrm{Ker}(\pi_\lambda(f_\theta)^p).$$

Then ξ provides an isomorphism from the subspace

$$U(p) = \{L \in U \mid \pi_\lambda(f_\theta)^p L(V^{\mu^*}) = 0\} = \{L \in U \mid L(V^{\mu^*}) \subseteq \mathrm{Ker}(\pi_\lambda(f_\theta)^p)\}$$

to $V'(p)$. Let $-\nu$ be one of the weights $-\nu_m$ which occur in the weight space decomposition of U, corresponding to a highest weight module V^{ν^*} where $\nu^* = \beta + \mu$ so $\langle \beta + \mu, \theta \rangle = \langle \nu^*, \theta \rangle =$

$\langle \nu, \theta \rangle$. We have seen that ξ provides an isomorphism between $U_{-\nu}$ and $V'_{\mu^*-\nu} = V^-(\lambda; \mu^* - \nu, \mu^*) = V^-(\lambda; -\beta^*, \mu^*)$, so it also provides an isomorphism between corresponding weight spaces

$$
\begin{aligned}
U_{-\nu}(p) &= \left\{ L \in U_{-\nu} \mid \pi_\lambda(f_\theta)^p L(V^{\mu^*}) = 0 \right\} \\
&= \left\{ L \in U_{-\nu} \mid L(V^{\mu^*}) \subseteq \mathrm{Ker}(\pi_\lambda(f_\theta)^p) \right\}
\end{aligned}
\tag{8.1}
$$

and

$$
V'_{-\beta^*}(p) = \left\{ v \in V^\lambda_{-\beta^*} \mid \pi_\lambda(f_i)^{\langle \mu^*, \alpha_i \rangle + 1} v = 0, \ 1 \leqslant i \leqslant \ell, \ \pi_\lambda(f_\theta)^p v = 0 \right\},
$$

which will equal the Walton space $W_k^-(\lambda, -\beta^*, \mu^*)$ when $p = k - \langle \nu, \theta \rangle + 1$.

Lemma 8.1. *For any integer $p \geqslant 1$ we have*

$$
\Psi(U_{-\nu}(p)) = \left(\mathrm{Ker}(\pi_\lambda(f_\theta)^p) \otimes V^\mu \right) \cap \Psi(U_{-\nu})
$$

and we have the orthogonal direct sum decomposition

$$
\Psi(U_{-\nu}) = \Psi(U_{-\nu}(p)) \oplus \mathrm{Proj}^{\lambda, \mu}_{\Psi(U_{-\nu})} \left(\mathrm{Im}(\pi_\lambda(e_\theta)^p) \otimes V^\mu \right).
$$

Proof. Apply the isomorphism Ψ to $U_{-\nu}(p)$ to get the subspace

$$
\Psi(U_{-\nu}(p)) = \left\{ \Psi(L) \in V^\lambda \otimes V^\mu \mid L \in U_{-\nu}(p) \right\}
$$

of certain lowest weight vectors of weight $-\nu$ in $V^\lambda \otimes V^\mu$. Recall the definition

$$
\Psi(L) = \sum_{j=1}^d L(v_j^*) \otimes v_j
$$

where $d = \dim(V^\mu) = \dim(V^{\mu^*})$, $\{v_1, \ldots, v_d\}$ is a basis of V^μ and $\{v_1^*, \ldots, v_d^*\}$ is the dual basis of V^{μ^*}. Then we see that

$$
\Psi(L) \in \mathrm{Ker}(\pi_\lambda(f_\theta)^p) \otimes V^\mu, \quad \text{for all } L \in U_{-\nu}(p),
$$

since $L(v_j^*) \in \mathrm{Ker}(\pi_\lambda(f_\theta)^p)$ for $1 \leqslant j \leqslant d$. Of course, $\Psi(L) \in \Psi(U_{-\nu})$, so we get containment in one direction. Now suppose that $\Psi(L) \in \Psi(U_{-\nu})$ and $\Psi(L) \in \mathrm{Ker}(\pi_\lambda(f_\theta)^p) \otimes V^\mu$, so for $1 \leqslant j \leqslant d$ we have $L(v_j^*) \in \mathrm{Ker}(\pi_\lambda(f_\theta)^p)$, giving $L \in U_{-\nu}(p)$ so $\Psi(L) \in \Psi(U_{-\nu}(p))$.

Let $\mathbf{g}_\theta \cong \mathrm{sl}_2$ be the subalgebra with basis e_θ, f_θ and $h_\theta = [e_\theta, f_\theta]$. As mentioned in Section 3, V^λ has a decomposition into the orthogonal direct sum of irreducible $\mathbf{g}_\theta$-modules,

$$
V^\lambda = \bigoplus_i V^\lambda_{\gamma_i}(m_i)
$$

where $\dim(V_{\gamma_i}^{\lambda}(m_i)) = m_i + 1$ and the highest weight of $V_{\gamma_i}^{\lambda}(m_i)$ is $\gamma_i \in \Pi^{\lambda}$ so $m_i = \gamma_i(h_{\theta})$. Also recall from Section 3 that from the representation theory of sl$_2$, on each irreducible component we have the orthogonal decomposition

$$V_{\gamma_i}^{\lambda}(m_i) = \mathrm{Ker}\big(\pi_{\lambda}(f_{\theta})^p\big) \oplus \mathrm{Im}\big(\pi_{\lambda}(e_{\theta})^p\big)$$

into the p lowest h_{θ} weight spaces and the rest. So we also get the orthogonal decomposition

$$V^{\lambda} = \mathrm{Ker}\big(\pi_{\lambda}(f_{\theta})^p\big) \oplus \mathrm{Im}\big(\pi_{\lambda}(e_{\theta})^p\big).$$

Of course, in the first equation above we mean the kernel and image of those operators restricted to each irreducible component. This gives an orthogonal decomposition

$$V^{\lambda} \otimes V^{\mu} = \mathrm{Ker}\big(\pi_{\lambda}(f_{\theta})^p\big) \otimes V^{\mu} \oplus \mathrm{Im}\big(\pi_{\lambda}(e_{\theta})^p\big) \otimes V^{\mu}.$$

Lemma 3.3 applied to this decomposition of the tensor product gives the orthogonal direct sum decomposition of the subspace $\Psi(U_{-\nu})$ as stated.   □

Let $\{\Psi(L_1), \ldots, \Psi(L_{d_p})\}$ be a basis of the first summand $\Psi(U_{-\nu}(p))$ in the above decomposition of $\Psi(U_{-\nu})$, and let $\{\Psi(L_{d_p+1}), \ldots, \Psi(L_M)\}$ be a basis of the second summand, where $M = \mathrm{Mult}_{\lambda,\mu}^{\nu^*} = \dim(U_{-\nu}) = \dim(\Psi(U_{-\nu}))$. Then there is a basis, $\{g_1, \ldots, g_{d_p}, \ldots, g_M\}$ of $\mathcal{H} = \mathrm{Hom}_{\mathbf{g}}(V^{\lambda} \otimes V^{\mu}, V^{\nu^*})$ determined by the conditions $g_i(\Psi(L_j)) = \delta_{i,j} v_{-\nu}^{\nu^*}$ for $v_{-\nu}^{\nu^*}$ a lowest weight vector in V^{ν^*}. The subspace

$$\mathcal{H}(K_p) = \big\{g \in \mathcal{H} \mid g\big(\Psi\big(U_{-\nu}(p)\big)\big) = 0\big\}$$

of elements of $\mathcal{H}$ that vanish on the first summand, has basis $\{g_{d_p+1}, \ldots, g_M\}$ and the subspace

$$\mathcal{H}(I_p) = \big\{g \in \mathcal{H} \mid g\big(\mathrm{Proj}_{\Psi(U_{-\nu})}^{\lambda,\mu}\big(\mathrm{Im}\big(\pi_{\lambda}(e_{\theta})^p\big) \otimes V^{\mu}\big)\big) = 0\big\}$$

of elements of $\mathcal{H}$ that vanish on the second summand, has basis $\{g_1, \ldots, g_{d_p}\}$ so $d_p = \dim(\mathcal{H}(I_p))$. Remember that the dimension of the Walton space $W_k^-(\lambda, -\beta^*, \mu^*)$ is d_p when $p = k - \langle \nu, \theta \rangle + 1$. But in that case, $\mathcal{H}(I_p)$ equals the Frenkel–Zhu space

$$\mathrm{FZ}_k'(\lambda, \mu, \nu) = \big\{g \in \mathcal{H} \mid g\big(\mathrm{Proj}_{\Psi(U_{-\nu})}^{\lambda,\mu}\big(e_{\theta}^{k-\langle \nu,\theta\rangle+1} V^{\lambda} \otimes V^{\mu}\big)\big) = 0\big\}$$

so we have completed the proof of Theorem 6.2.

Acknowledgment

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