A quantum affine algebra for the deformed Hubbard chain

Niklas Beisert[1]**, Wellington Galleas**[1] **and Takuya Matsumoto**[1,2]

[1] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Potsdam, Germany
[2] Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan

E-mail: nbeisert@ethz.ch, wgalleas@unimelb.edu.au and tmatsumoto@usyd.edu.au

Abstract

The integrable structure of the one-dimensional Hubbard model is based on Shastry's R-matrix and the Yangian of a centrally extended $\mathfrak{sl}(2|2)$ superalgebra. Alcaraz and Bariev have shown that the model admits an integrable deformation whose R-matrix has recently been found. This R-matrix is of trigonometric type and here we derive its underlying exceptional quantum affine algebra. We also show how the algebra reduces to the above-mentioned Yangian and to the conventional quantum affine $\mathfrak{sl}(2|2)$ algebra in two special limits.

PACS numbers: 02.30.Ik, 02.20.Uw, 71.10.Fd
Mathematics Subject Classification: 17B37, 17B67, 81U15

1. Introduction and overview

The algebraic structures underlying integrable models have been intensively studied in the past few years and a variety of approaches have been formulated in order to systematically derive solutions of the Yang–Baxter equation [1, 2]. The solutions of the Yang–Baxter equation, also known as R-matrices, characterize the integrability of the model and a large number of solutions have been obtained through the quantum group framework making use of deformations of universal enveloping algebras. One of the most prominent applications of quantum groups, or more specifically quantum deformations $\mathcal{U}_q[\mathfrak{g}]$ in the case considered here, lies in the fact that if $\mathfrak{g}$ is finite dimensional, we can associate an operator $R \in \mathrm{End}(\mathcal{A} \otimes \mathcal{A})$ satisfying the quantum Yang–Baxter equation with any representation $\mathcal{A}$ of $\mathcal{U}_q[\mathfrak{g}]$. This fact was realized independently by Drinfel'd and Jimbo [3] who showed how to associate a family of Hopf algebras with any symmetrizable Kac–Moody algebra. Nevertheless, it is worth remarking here that the defining relations of the quantum deformed algebra $\mathcal{U}_q[\mathfrak{g}]$ first appeared in the work of Kulish and Reshetikhin on the quantum sine-Gordon model [4]. The definitions of $\mathcal{U}_q[\mathfrak{g}]$ can be extended to arbitrary Kac–Moody algebras, in particular, to the affine Lie (super)

algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$, and the distinction between a Lie (super) algebra and its affine extension has remarkable consequences.

It is well known that the Yang–Baxter equation has an intimate connection with Artin's braid group [5] when an *R*-matrix does not depend on spectral parameters [6]. The constant solutions of the Yang–Baxter equation are usually, though not always, prevenient from the non-affine Lie algebras $\mathfrak{g}$ and the introduction of the spectral parameter can be performed in two principal ways. The first one is the so-called Baxterization method developed by Jones [7]. This method makes use of the algebraic structures related to Artin's braid group as a starting point to derive spectral parameter-dependent solutions of the Yang–Baxter equation. The second method is based on the affine Lie algebras $\widehat{\mathfrak{g}}$, more specifically the quantum affine algebras $\mathcal{U}_q[\widehat{\mathfrak{g}}]$ or the Yangian algebras $\mathcal{Y}[\mathfrak{g}]$ as a special case. For the latter, the parameter of the evaluation representation lifting the representations of $\mathfrak{g}$ to $\widehat{\mathfrak{g}}$ becomes the spectral parameter of the *R*-matrix.

Within the quantum group framework, the *R*-matrix describing scattering on the string worldsheet in the context of the AdS/CFT correspondence (see [8] for reviews) can be obtained from a central extension of $\mathfrak{sl}(2|2)$ [9–11] and its Yangian algebra $\mathcal{Y}$ [12] (see also [13, 14]). Curiously, the spectral parameter-dependent *R*-matrix in the fundamental representation already follows from the non-affine algebra [9]. This property however does not carry over to higher representations where the Yangian most conveniently determines the *R*-matrix [15].

Interestingly enough, the fundamental *R*-matrix associated with the centrally extended $\mathfrak{sl}(2|2)$ superalgebra turns out to be equivalent [16] to Shastry's *R*-matrix [17] responsible for the integrable structure of the one-dimensional Hubbard model. The Hubbard model (see [18]) is the simplest generalization beyond the band theory description of metals and it has found applications in a variety of contexts. It can be used to describe the Mott metal–insulator transition [19], π electrons in the benzene molecule [20] as well as some higher loop planar anomalous dimensions of local operators in $\mathcal{N} = 4$ super Yang–Mills theory [21]. Now it is clear that the one-dimensional Hubbard model takes a solitary place among the spin chain models, not just phenomenologically, but also algebraically. This can be observed in the Lieb–Wu equations [19] which have a peculiar form which is unlike those for conventional spin chains based on a generic Lie (super) algebra $\mathfrak{g}$. Moreover, Shastry's *R*-matrix is non-standard in the sense that it depends non-trivially on *two spectral parameters*, rather than on their simple combination. On the algebraic level, these unique features can be traced to the exceptional nature of $\mathfrak{psl}(2|2)$ which is the only simple Lie superalgebra with a non-trivial *threefold central extension* [22]. Although the existence of such a large center allows more freedom in setting up the integrable structure, and it is thus ultimately responsible for the peculiar features of this model, these non-standard features have left scientists puzzled for a long time. Even now the algebraic structures underlying the integrability of the one-dimensional Hubbard model are far less developed than those for conventional spin chains, cf [18] and [14]. Merely the classical limit of the algebra and its classical *r*-matrix is reasonably well understood [23, 24].

The one-dimensional Hubbard Hamiltonian is also a paradigm in condensed matter physics, and together with the supersymmetric *t-J* model [25], it is the fundamental block for the study of non-perturbative effects in strongly correlated electron systems due to the fact that they are integrable. In [26], Alcaraz and Bariev proposed a Bethe ansatz solvable Hamiltonian interpolating between the Hubbard and the supersymmetric *t-J* models. Besides the hopping term (kinetic energy), this model contains not only a Coulomb interaction as in the case of the Hubbard model, but also a spin–spin interaction resembling the *t-J* Hamiltonian. It turns out that this Alcaraz–Bariev model can be viewed as a quantum deformation of the Hubbard model [27] in much the same way that the Heisenberg XXZ model is a quantum
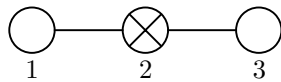
Figure 1. Dynkin diagram for $\mathfrak{sl}(2|2)$.

deformation of the XXX model. More precisely, the *R*-matrix of the Alcaraz–Bariev model is based on a quantum deformation $\mathcal{Q}$ of the extended $\mathfrak{sl}(2|2)$ algebra[3]. Although the *R*-matrix is not necessary in order to obtain the exact spectrum of the model, this knowledge still offers the possibility of studying thermodynamic properties in an efficient way through the quantum transfer matrix method [29].

Many of the same peculiar features of the Hubbard model apply to the Alcaraz–Bariev model and the associated quantum deformation $\mathcal{Q}$ of the centrally extended $\mathfrak{sl}(2|2)$ algebra [27]. However, with the caveat that quantum deformation makes some structures substantially more complicated to handle. Except for its classical limit [30], which already provides valuable insights into the expected structures, it is fair to say that our knowledge of the complete underlying algebra is still limited. With that in mind, the scenario described above thus asks for a formulation of the *quantum affine algebra* $\widehat{\mathcal{Q}}$ based on the extended $\mathfrak{sl}(2|2)$. Even though quantum deformations introduce additional complexity, they also bring about some new symmetries into the framework as compared to Yangians which are rather singular limits thereof. This may eventually help us to uncover the full structure of the Hopf algebra underlying integrability in the AdS/CFT correspondence.

This paper is organized as follows. We start in section 2 with a review of the quantum deformed extended $\mathfrak{sl}(2|2)$ algebra $\mathcal{Q}$ and its associated integrable structures. Next we use a special property of its affine Dynkin diagram to derive the affine extension $\widehat{\mathcal{Q}}_0$ in section 3. For the reader's convenience, we summarize the algebraic relations of $\widehat{\mathcal{Q}}_0$ in section 4. We go on by establishing the fundamental representation in section 5 which requires refining the algebra $\widehat{\mathcal{Q}}_0$ to $\widehat{\mathcal{Q}}$. In the remainder of the paper, we study two interesting limits of the algebra. One of them is the conventional quantum affine algebra $\mathcal{U}_q[\widehat{\mathfrak{sl}(2|2)}]$ described in section 6, followed by the extended $\mathfrak{sl}(2|2)$ Yangian $\mathcal{Y}$ discussed in section 7. Section 8 is left for conclusions and final remarks.

2. Quantum deformation of extended $\mathfrak{sl}(2|2)$

In the following, we shall briefly review the quantum deformed extended $\mathfrak{sl}(2|2)$ algebra $\mathcal{Q}$ introduced in [27].

Cartan matrix. We shall consider the $\mathfrak{sl}(2|2)$ Dynkin diagram in figure 1 such that the associated Cartan matrix *A* and normalization matrix *D* read

$$A = \begin{pmatrix} +2 & -1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +2 \end{pmatrix}, \qquad D = \mathrm{diag}(+1, -1, -1). \tag{2.1}$$

With the help of *D*, we obtain the following symmetric matrix which frequently appears in the defining relations:

$$DA = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}. \tag{2.2}$$

---

[3] The algebra has also been discussed in the Faddeev–Zamolodchikov framework in [28].

Generators. The algebra is conveniently presented in terms of Chevalley–Serre generators. The generators are the raising and lowering generators E_j and F_j as well as the exponentiated Cartan generators $K_j = q^{H_j}$ with $j = 1, 2, 3$. All of them are even generators of our superalgebra, except for the pair of odd generators E_2 and F_2, in accordance with the Dynkin diagram in figure 1. In addition, there are *two* central charges U and $V = q^C$. The algebra has two parameters: the deformation parameter q and the coupling parameter g. A third parameter α could be absorbed into a redefinition of the generators, and thus does not count as a parameter of the algebra. Nevertheless, it is convenient to keep it unspecified.

Algebra. The Chevalley–Serre generators satisfy the standard quantum deformed commutation relations ($j, k = 1, 2, 3$) [4],

$$K_j E_k = q^{DA_{jk}} E_k K_j, \qquad F_k K_j = q^{DA_{jk}} K_j F_k, \qquad [E_j, F_k] = D_{jj} \delta_{jk} \frac{K_j - K_j^{-1}}{q - q^{-1}}. \tag{2.3}$$

In addition, the following Serre relations hold ($j = 1, 3$):

$$[E_1, E_3] = \{E_2, E_2\} = [E_j, [E_j, E_2]] - (q - 2 + q^{-1})E_j E_2 E_j = 0$$
$$[F_1, F_3] = \{F_2, F_2\} = [F_j, [F_j, F_2]] - (q - 2 + q^{-1})F_j F_2 F_j = 0. \tag{2.4}$$

Center. The algebra defined by the above relations has three central elements,

$$C_1 = K_1 K_2^2 K_3$$
$$C_2 = \{[E_2, E_1], [E_2, E_3]\} - (q - 2 + q^{-1})E_2 E_1 E_3 E_2$$
$$C_3 = \{[F_2, F_1], [F_2, F_3]\} - (q - 2 + q^{-1})F_2 F_1 F_3 F_2. \tag{2.5}$$

The latter two are usually projected out by the Serre relations $C_2 = C_3 = 0$ of the superalgebra $\mathfrak{sl}(2|2)$. Furthermore, in $\mathfrak{psl}(2|2)$ the former is also projected out by the condition $C_1 = 1$. Here we keep them all, and thus our algebra is based on a central extension of $\mathfrak{psl}(2|2)$ or $\mathfrak{sl}(2|2)$. As shown in [11, 27], it turns out that we obtain a very interesting algebra if we impose one constraint on the central elements as follows:

$$C_1 = V^{-2}, \qquad C_2 = g\alpha(1 - U^2 V^2), \qquad C_3 = g\alpha^{-1}(V^{-2} - U^{-2}). \tag{2.6}$$

Coalgebra. All the above relations are compatible with the following coalgebra structure. The coproduct for all $X \in \{K_j, U, V\}$ is group-like, $\Delta(X) = X \otimes X$, while for E_j and F_j, it takes the standard form but with a twist induced by the central element U,

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U^{+\delta_{j,2}} \otimes E_j, \qquad \Delta(F_j) = F_j \otimes K_j + U^{-\delta_{j,2}} \otimes F_j. \tag{2.7}$$

The twist is based on the $\mathfrak{gl}(1)$ derivation in $\mathfrak{gl}(2|2)$ which applies only to the fermionic generators E_2 and F_2.

Fundamental representation. The algebra has a family of representations acting on the $(2|2)$-dimensional graded space $\mathbb{V}$. The raising and lowering generators are represented by the following $(2|2) \times (2|2)$ supermatrices:

$$E_1 \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E_2 \simeq \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E_3 \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

---

[4] Our K_2, K_3 are inverted compared to usual conventions to make the symmetric matrix *DA* appear in place of the Cartan matrix *A*; this makes D_{jj} appear in $[E_j, F_k]$.
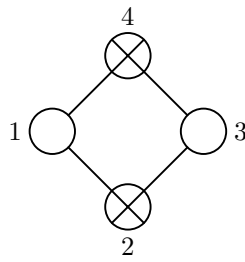
4

Figure 2. Dynkin diagram for $\widehat{\mathfrak{sl}}(2|2)$.

$$F_1 \simeq \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F_2 \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ \hline 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix} \quad F_3 \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

We shall not present here the supermatrix representations for K_j since they easily follow from the algebra relations (2.3). The central elements U and V are represented by uniform multiplication with U and V, respectively. In their turn, these central elements are related to the coefficients a, b, c and d through the constraints

$$ad = \frac{q^{1/2}V - q^{-1/2}V^{-1}}{q - q^{-1}}, \quad bc = \frac{q^{-1/2}V - q^{1/2}V^{-1}}{q - q^{-1}},$$
$$ab = g\alpha(1 - U^2V^2), \quad cd = g\alpha^{-1}(V^{-2} - U^{-2}). \tag{2.9}$$

The above constraints imply the following relation between U and V:

$$g^2(V^{-2} - U^{-2})(1 - U^2V^2) = \frac{(V - qV^{-1})(V - q^{-1}V^{-1})}{(q - q^{-1})^2}, \tag{2.10}$$

while one of the parameters a, b, c, d can be chosen freely. Altogether, we thus have a two-parameter family of representations.

Fundamental R-matrix.   In [27], the fundamental *R*-matrix for the above-described algebra has been explicitly derived. The *R*-matrix is a linear map $\mathcal{R} : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ which is a function of the variables parametrizing each one of the spaces $\mathbb{V}$. The form of the *R*-matrix was obtained by demanding that the cocommutativity condition

$$\mathcal{R}\Delta(X) = \widetilde{\Delta}(X)\mathcal{R} \tag{2.11}$$

hold for $X \in \{E_j, F_j, K_j, U, V\}$. Here $\widetilde{\Delta}(X)$ stands for the opposite coproduct defined through the permutation map

$$\widetilde{\Delta}(X) = \mathcal{P}\Delta(X)\mathcal{P}, \tag{2.12}$$

where $\mathcal{P}$ denotes the graded permutation operator. Relation (2.11) has proved to completely and consistently determine the fundamental *R*-matrix up to an overall scalar factor. The explicit form of $\mathcal{R}$ is lengthy and shall not be reproduced here since it was given in [27].

3. Derivation of the affine extension

Now we shall consider the affine extension of the algebra defined above. The affine extension for the Dynkin diagram in figure 1 is given in figure 2. The associated Cartan matrix *A* for

$\widehat{\mathfrak{sl}}(2|2)$ and the symmetric Cartan matrix DA with $D = \mathrm{diag}(+1, -1, -1, -1)$ now read

$$A = \begin{pmatrix} +2 & -1 & 0 & -1 \\ +1 & 0 & -1 & 0 \\ 0 & -1 & +2 & -1 \\ +1 & 0 & -1 & 0 \end{pmatrix}, \quad DA = \begin{pmatrix} +2 & -1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -2 & +1 \\ -1 & 0 & +1 & 0 \end{pmatrix}. \tag{3.1}$$

The crucial observation here is that the new fourth node of the Dynkin diagram is completely analogous to the second one. Consequently, the second and fourth rows and columns of the matrix DA coincide. In practice, this means that the associated Chevalley–Serre generators should obey analogous commutation relations. This observation will help us tremendously in completing this unusual affine algebra.

Doubling the fermionic node. We introduce the new set of generators $\{E_4, F_4, K_4\}$ and, as explained above, they should act as copies of the generators $\{E_2, F_2, K_2\}$. In their turn, the coupling constant g, the normalization α as well as the central elements U and V always appear in conjunction with the generators $\{E_2, F_2, K_2\}$. Thus, it makes sense to double those as well in such a way that we relabel $\{g, \alpha, U, V\}$ as $\{g_2, \alpha_2, U_2, V_2\}$, and introduce new constants and central generators $\{g_4, \alpha_4, U_4, V_4\}$.

The algebra relations and coproducts for the new generators $\{E_4, F_4, K_4\}$ will be direct copies of the ones for $\{E_2, F_2, K_2\}$ discussed in section 2. This almost guarantees that we get a consistent algebra and coalgebra structure. Now we merely have to take care of the relations of the quantum affine algebra $\widehat{\mathfrak{sl}}(2|2)$ mixing the two sets of generators, namely the anticommutators $\{E_2, F_4\}$, $\{E_4, F_2\}$, $\{E_2, E_4\}$ and $\{F_2, F_4\}$.

Compatibility. The anticommutators $\{E_2, F_4\}$ and $\{E_4, F_2\}$ commute with the Cartan subalgebra and thus they should belong to it as well. Fortunately, the coproducts for the generators involved are completely fixed at this stage and the compatibility between them imposes constraints on the algebra. In particular, we have

$$\Delta(E_2) = E_2 \otimes 1 + K_2^{-1}U_2 \otimes E_2, \quad \Delta(F_4) = F_4 \otimes K_4 + U_4^{-1} \otimes F_4, \tag{3.2}$$

and thus

$$\{\Delta(E_2), \Delta(F_4)\} = \{E_2, F_4\} \otimes K_4 + K_2^{-1}U_2U_4^{-1} \otimes \{E_2, F_4\}. \tag{3.3}$$

This suggests that $\{E_2, F_4\}$ should be composed by a linear combination of the group-like elements K_4 and $K_2^{-1}U_2U_4^{-1}$. Under these considerations, we can use an ansatz and easily obtain a solution for the compatibility condition $\{\Delta(E_2), \Delta(F_4)\} = \Delta(\{E_2, F_4\})$. By doing so, we find

$$\{E_2, F_4\} = -\tilde{g}\tilde{\alpha}^{-1}\left(K_4 - U_4^{-1}U_2K_2^{-1}\right) \tag{3.4}$$

and similarly

$$\{E_4, F_2\} = +\tilde{g}\tilde{\alpha}^{+1}\left(K_2 - U_2^{-1}U_4K_4^{-1}\right), \tag{3.5}$$

with two new constants $\tilde{g}$ and $\tilde{\alpha}$. In the standard quantum affine algebra $\widehat{\mathfrak{sl}}(2|2)$, the rhs of (3.4) and (3.5) vanishes and this is one of the main differences of our unusual affine algebra. It is worth remarking here that similar relations, though not equivalent, also appeared in [31].

The anticommutators $\{E_2, E_4\}$ and $\{F_2, F_4\}$ do not commute with the Cartan subalgebra and considerations on the coalgebra structure lead us to conclude that they must be trivial. Hence,

$$\{E_2, E_4\} = \{F_2, F_4\} = 0. \tag{3.6}$$

The question remains whether the above relations, in particular the mixed ones (3.4) and (3.5), define a consistent algebra: as we shall see later, the algebra admits at least

one representation. Using the coproduct, one can define further representations as tensor products. Hence, the relations consistently define an algebra with a non-trivial representation theory[5].

4. Hopf algebra structure

We shall call the above-derived quantum affine algebra $\widehat{\mathcal{Q}}_0$ and in what follows, we summarize its defining relations. Some of the constants will be refined later to give a more special algebra $\widehat{\mathcal{Q}}$.

Algebra.   The algebra $\widehat{\mathcal{Q}}_0$ consists of a deformed extension of the quantum affine algebra $\widehat{\mathfrak{sl}}(2|2)$. It is generated by the corresponding Chevalley–Serre generators K_j, E_j, F_j ($i, j = 1, 2, 3, 4$) and central elements U_k and V_k ($k = 2, 4$). It is also useful to recall here the symmetric matrix DA and the normalization matrix D associated with the Cartan matrix A for $\widehat{\mathfrak{sl}}(2|2)$:

$$DA = \begin{pmatrix} +2 & -1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -2 & +1 \\ -1 & 0 & +1 & 0 \end{pmatrix}, \quad D = \mathrm{diag}(+1, -1, -1, -1). \tag{4.1}$$

The algebra has a set of group-like elements $X, Y \in \{1, K_j, U_k, V_k\}$ which are invertible and commutative,

$$XX^{-1} = 1, \quad XY = YX. \tag{4.2}$$

The Chevalley–Serre raising and lowering generators E_j and F_j satisfy the usual relations, except for the two mixed anticommutators given in (3.4) and (3.5),

$$K_i E_j K_i^{-1} = q^{DA_{ij}} E_j, \qquad\qquad K_i F_j K_i^{-1} = q^{-DA_{ij}} F_j,$$
$$\{E_2, F_4\} = -\tilde{g}\tilde{\alpha}^{-1}(K_4 - U_2 U_4^{-1} K_2^{-1}), \qquad \{E_4, F_2\} = \tilde{g}\tilde{\alpha}(K_2 - U_4 U_2^{-1} K_4^{-1}),$$
$$[E_j, F_j] = D_{jj}\frac{K_j - K_j^{-1}}{q - q^{-1}}, \qquad\qquad [E_i, F_j] = 0 \quad \text{for } i \neq j, i + j \neq 6. \tag{4.3}$$

In addition to relations (4.3), the algebra $\widehat{\mathcal{Q}}_0$ also satisfies the following Serre relations ($j = 1, 3$):

$$\begin{aligned}
[E_1, E_3] = E_2 E_2 = E_4 E_4 = \{E_2, E_4\} &= 0 \\
[F_1, F_3] = F_2 F_2 = F_4 F_4 = \{F_2, F_4\} &= 0 \\
[E_j, [E_j, E_k]] - (q - 2 + q^{-1})E_j E_k E_j &= 0 \\
[F_j, [F_j, F_k]] - (q - 2 + q^{-1})F_j F_k F_j &= 0.
\end{aligned} \tag{4.4}$$

The quartic Serre relations of the superalgebra $\widehat{\mathfrak{sl}}(2|2)$ are deformed by the central elements U_k and V_k as follows:

$$\begin{aligned}
\{[E_1, E_k], [E_3, E_k]\} - (q - 2 + q^{-1})E_k E_1 E_3 E_k &= g_k \alpha_k (1 - V_k^2 U_k^2) \\
\{[F_1, F_k], [F_3, F_k]\} - (q - 2 + q^{-1})F_k F_1 F_3 F_k &= g_k \alpha_k^{-1}(V_k^{-2} - U_k^{-2}),
\end{aligned} \tag{4.5}$$

and the remaining central elements of the superalgebra $\widehat{\mathfrak{sl}}(2|2)$ are then related to V_k through

$$K_1^{-1} K_k^{-2} K_3^{-1} = V_k^2. \tag{4.6}$$

In summary, the above quantum affine algebra $\widehat{\mathcal{Q}}_0$ has five parameters: $q, g_k, \tilde{g}$ and $\tilde{\alpha}$. The two normalizations α_k merely originate from our choice of basis.

---

[5] It is conceivable though that the above relations imply further simple relations, such as $U_2 U_4 = 1$ and $V_2 V_4 = 1$ which hold on the representation in section 5.

Algebra automorphism. The quantum affine algebra $\widehat{\mathcal{Q}}_0$ has been constructed by making use of the similarity between nodes 2 and 4 of the Dynkin diagram in figure 2. In fact this similarity leads to an algebra automorphism flipping nodes 2 and 4 if the coupling constants are related by

$$g_2 = g_4, \quad \alpha_4 = \zeta^2 \tilde{\alpha}^2 \alpha_2, \tag{4.7}$$

where $\zeta^4 = 1$. Thus, the following map is an algebra automorphism:

$$
\begin{aligned}
E_2 &\to \zeta \tilde{\alpha}^{-1} E_4, & E_4 &\to -\zeta \tilde{\alpha} E_2, \\
F_2 &\to \zeta^{-1} \tilde{\alpha} F_4, & F_4 &\to -\zeta^{-1} \tilde{\alpha}^{-1} F_2, \\
U_2 &\to U_4, & U_4 &\to U_2, \\
K_2 &\to K_4, & K_4 &\to K_2.
\end{aligned}
\tag{4.8}
$$

Coalgebra, antipode and counit. For the group-like elements $X \in \{1, K_j, U_k, V_k\}$ ($j = 1, 2, 3, 4$ and $k = 2, 4$), the coproduct Δ, the antipode S and the counit ε are defined as usual,

$$\Delta(X) = X \otimes X, \quad S(X) = X^{-1}, \quad \varepsilon(X) = 1, \tag{4.9}$$

while for the remaining Chevalley–Serre generators, they are deformed by the central elements U_k as follows ($j = 1, 2, 3, 4$):

$$
\begin{aligned}
\Delta(E_j) &= E_j \otimes 1 + K_j^{-1} U_2^{+\delta_{j,2}} U_4^{+\delta_{j,4}} \otimes E_j, & S(E_j) &= -U_2^{-\delta_{j,2}} U_4^{-\delta_{j,4}} K_j E_j, & \varepsilon(E_j) &= 0, \\
\Delta(F_j) &= F_j \otimes K_j + U_2^{-\delta_{j,2}} U_4^{-\delta_{j,4}} \otimes F_j, & S(F_j) &= -U_2^{+\delta_{j,2}} U_4^{+\delta_{j,4}} F_j K_j^{-1}, & \varepsilon(F_j) &= 0.
\end{aligned}
\tag{4.10}
$$

The above relations characterize our quantum affine algebra $\widehat{\mathcal{Q}}_0$ as a Hopf algebra. We have verified explicitly the compatibility between the algebra and the coalgebra (as well as the antipode relations). In other words, $\Delta(XY) = \Delta(X)\Delta(Y)$ is compatible with all algebra relations. In particular, the unusual algebra relations (2.5), (2.6), (3.4) and (3.5) were derived in order to obtain a consistent Hopf algebra structure. In the following section, we shall discuss the algebra's fundamental representation, upon which a large class of finite-dimensional representations can be constructed by means of the coalgebra.

5. Fundamental representation

Now we would like to lift the four-dimensional fundamental representation given in (2.8) to a representation of the affine algebra. The representation theory of affine algebras has been discussed in [32]. In particular, it was shown in [33] that any finite-dimensional irreducible representation of $\widehat{\mathfrak{g}}$ extended from $\mathfrak{g}$ is isomorphic to an evaluation representation. In the quantum case, there also exists an evaluation homomorphism $ev : \mathcal{U}_q[\mathfrak{g}] \to \mathcal{U}_q[\widehat{\mathfrak{g}}]$ defined by Jimbo in [34, 2] which reduces to the usual evaluation in the classical limit $q \to 1$. Moreover, when $\mathfrak{g} \cong \mathfrak{sl}$, it was shown in [35] that any extension of a representation from $\mathcal{U}_q[\mathfrak{g}]$ to $\mathcal{U}_q[\widehat{\mathfrak{g}}]$ on the same space is isomorphic to an evaluation representation.

Due to the non-standard nature of our extended quantum affine algebra $\widehat{\mathcal{Q}}_0$, it is not clear if this whole scenario of evaluation representations applies to our case. Nevertheless, we find here that the set of generators $\{K_4, E_4, F_4, U_4, V_4\}$ satisfying (4.3) can be obtained as copies of the generators $\{K_2, E_2, F_2, U_2, V_2\}$ with modified coefficients.

Doubling ansatz.   As before, we assume the generators E_4 and F_4 to act respectively as copies of E_2 and F_2 but with different coefficients. Hence,

$$E_k \simeq \begin{pmatrix} 0 & 0 & 0 & b_k \\ 0 & 0 & 0 & 0 \\ 0 & a_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F_k \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d_k & 0 \\ 0 & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad k = 2, 4. \tag{5.1}$$

By doing so, we obtain two sets of four constraints from (2.9). Furthermore, the mixed relations (3.4) and (3.5) yield another set of four constraints, namely

$$a_2 d_4 = \tilde{g}\tilde{\alpha}^{-1}\big(q^{1/2}U_2 U_4^{-1}V_2 - q^{-1/2}V_4^{-1}\big), \quad b_2 c_4 = \tilde{g}\tilde{\alpha}^{-1}\big(q^{-1/2}U_2 U_4^{-1}V_2 - q^{1/2}V_4^{-1}\big),$$
$$c_2 b_4 = \tilde{g}\tilde{\alpha}\big(q^{1/2}V_2^{-1} - q^{-1/2}U_2^{-1}U_4 V_4\big), \quad d_2 a_4 = \tilde{g}\tilde{\alpha}\big(q^{-1/2}V_2^{-1} - q^{1/2}U_2^{-1}U_4 V_4\big). \tag{5.2}$$

In total, we have 12 constraints for 12 parameters ($a_k, b_k, c_k, d_k, U_k, V_k$). Thus, the solution of the constraints completely fixes all the parameters and leaves just a discrete set of four-dimensional representations.

Constrained parameters.   The seven constants $g_k, \alpha_k, \tilde{g}, \tilde{\alpha}, q$ can be chosen in a special way in order to solve two of the constraints. One suitable choice[6] expressed in terms of the four parameters $g, q, \alpha, \tilde{\alpha}$ reads[7]

$$g_2 = g_4 = g, \quad \alpha_2 = \alpha_4 \tilde{\alpha}^{-2} = \alpha, \quad \tilde{g}^2 = \frac{g^2}{1 - g^2(q - q^{-1})^2}. \tag{5.3}$$

In fact there is a convenient replacement for g in terms of a new parameter $\tilde{q}$ which also allows us to parametrize the quadratic relation for $\tilde{g}$ as

$$g = \frac{\tilde{q} - \tilde{q}^{-1}}{2\mathrm{i}(q - q^{-1})}, \quad \tilde{g} = \frac{\mathrm{i}(\tilde{q} - \tilde{q}^{-1})}{(q - q^{-1})(\tilde{q} + \tilde{q}^{-1})}. \tag{5.4}$$

We shall be mainly concerned with the above choice of parameters in this paper. Thus, we shall denote the algebra $\widehat{\mathcal{Q}}_0$ obeying constraints (5.3) by $\widehat{\mathcal{Q}}_{g,q}$ or $\widehat{\mathcal{Q}}$ for short. It depends on two parameters, g and q, and it is expressed using two normalization constants α and $\tilde{\alpha}$. Nevertheless, we shall also use the original parameters $g_k, \alpha_k, \tilde{g}, \tilde{\alpha}, q$ with the above relations implied.

Two-parameter family.   The solution of the remaining constraints for the fundamental representation leaves us with

$$U_4 = \pm U_2^{-1} \quad V_4 = \pm V_2^{-1}. \tag{5.5}$$

Relations (2.10) between the U_k and the V_k then automatically coincide. Furthermore, one of the coefficients a_k, b_k, c_k, d_k can be chosen freely. Altogether, this amounts to a two-parameter family of representations which is thus a unique lift of the fundamental representation to the quantum affine algebra.

It is interesting to observe here that the representations of E_2 and F_2 are respectively related to the representations of E_4 and F_4 by the simple map given in (5.5). In fact, this map also appears when considering the transpose representation.

---

[6] Another choice that will not be discussed here is $\tilde{g}^2 = -1/(q - q^{-1})^2$ and $\alpha_4 = -\alpha_2 \tilde{\alpha}^2 (g_2/g_4)^{\pm 1}$.
[7] It would be interesting to see what implications these relations might have on the algebra relations defined in section 4 as they change the representation theory substantially.

The $x^{\pm}$-parametrization.   Above we have obtained constraints for the coefficients a_k, b_k, c_k, d_k ($k = 2, 4$) characterizing the fundamental representation of the quantum affine algebra $\widehat{\mathcal{Q}}$. In particular, instead of solving constraints (2.9) in favor of U_k and V_k, as was done in (2.10), we could also have solved them in favor of the coefficients a_k, b_k, c_k, d_k. In that case, we would be left with the relation ($k = 2, 4$)

$$(a_k d_k - q b_k c_k)(a_k d_k - q^{-1} b_k c_k) = 1. \tag{5.6}$$

A convenient novel parametrization of this constraint uses a pair of variables x^+ and x^- related by $q^{-1} \zeta(x^+) = q \zeta(x^-)$ with

$$\zeta(x) = -\frac{x + 1/x + \xi + 1/\xi}{\xi - 1/\xi}, \quad \xi = -i\tilde{g}(q - q^{-1}). \tag{5.7}$$

Note that in order to simplify our results, we consider a convention for $x^{\pm}$ different from that used in [27]. More precisely, the convention used here can be obtained from that of [27] by performing the transformation $x_{BK}^{\pm} = g\tilde{g}^{-1}(x_{here}^{\pm} + \xi)$.[8]

The a_k, b_k, c_k, d_k can now be parametrized in terms of the variables $x_k^{\pm}$ and γ_k as follows:

$$a_k = \sqrt{g}\gamma_k, \qquad\qquad b_k = \frac{\sqrt{g}\alpha_k}{\gamma_k} \frac{x_k^- - x_k^+}{x_k^-},$$

$$c_k = \frac{\sqrt{g}\gamma_k}{\alpha_k} \frac{i\sqrt{q\tilde{g}}}{V_k g(x_k^+ + \xi)}, \quad d_k = \frac{\sqrt{g}}{\gamma_k} \frac{V_k \tilde{g}\sqrt{q}(x_k^+ - x_k^-)}{ig(\xi x_k^+ + 1)}, \tag{5.8}$$

while U_k and V_k read

$$U_k^2 = q^{-1} \frac{x_k^+ + \xi}{x_k^- + \xi} = q \frac{x_k^+}{x_k^-} \frac{\xi x_k^- + 1}{\xi x_k^+ + 1}, \qquad V_k^2 = q^{-1} \frac{\xi x_k^+ + 1}{\xi x_k^- + 1} = q \frac{x_k^+}{x_k^-} \frac{x_k^- + \xi}{x_k^+ + \xi}. \tag{5.9}$$

Now the mixed constraints (5.2) impose a relation between $(x_2^{\pm}, \gamma_2)$ and $(x_4^{\pm}, \gamma_4)$ which is then solved by

$$\begin{aligned} x_2^{\pm} &= x^{\pm}, \quad \gamma_2 = \gamma, \\ x_4^{\pm} &= \frac{1}{x^{\pm}}, \quad \gamma_4 = \frac{i\tilde{\alpha}\gamma}{x^+}, \end{aligned} \tag{5.10}$$

where the normalization coefficients α_2 and α_4 are related by (5.3).

A convenient multiplicative evaluation parameter z for our quantum affine algebra turns out to be

$$z = q^{-1}\zeta(x^+) = q\zeta(x^-). \tag{5.11}$$

Cocommutativity.   The R-matrix of the quantum deformed Hubbard model derived in [27] is in fact invariant under the full quantum affine algebra $\widehat{\mathcal{Q}}$ defined by relations (4.1)–(4.10). More precisely, the cocommutativity relation

$$\mathcal{R}\Delta(X_4) = \tilde{\Delta}(X_4)\mathcal{R} \tag{5.12}$$

is also fulfilled for $X_4 \in \{K_4, E_4, F_4, U_4, V_4\}$ in addition to those in (2.11).

In order to see that, it is convenient to work with the parametrization in terms of the variables $x^{\pm}$ and γ. Interesting enough, relations (5.10) and (5.1) state that the fundamental representation of X_4 can be obtained respectively as copies of $X_2 \in \{K_2, E_2, F_2, U_2, V_2\}$ under the mapping

$$x^{\pm} \mapsto \frac{1}{x^{\pm}}, \qquad \gamma \mapsto \frac{i\tilde{\alpha}\gamma}{x^+}, \qquad \alpha \mapsto \alpha\tilde{\alpha}^2, \qquad \tilde{\alpha} \mapsto -\frac{1}{\tilde{\alpha}}. \tag{5.13}$$

---

[8] Fortunately, the R-matrix in [27] is only mildly affected by this affine transformation: A, D, G, H, K, L do not change; in B, E substitute $s(x) = 1/x$; only C, F require more care.

10

Now considering the fundamental *R*-matrix given in [27],[9] a straightforward computation reveals that $\mathcal{R}$ is invariant under this map up to an overall scalar factor. More precisely, $\mathcal{R} \mapsto f\mathcal{R}$ with some irrelevant scalar factor $f = f(x_1^\pm, x_2^\pm)$. The cocommutativity condition for X_2 in (2.11), $\mathcal{R}\Delta(X_2) = \tilde{\Delta}(X_2)\mathcal{R}$, then directly maps to that for X_4 (5.12). This proves the invariance of $\mathcal{R}$ under the full quantum affine algebra $\widehat{\mathcal{Q}}$.

6. Conventional quantum affine limit

In this section, we aim at investigating the quantum affine algebra $\widehat{\mathcal{Q}}$ and its fundamental representation in the limit $g \to 0$. We shall show that it reduces to the standard $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ algebra up to a Reshetikhin twist [36] and a gauge transformation [6]. This limit corresponds to the case 'T(conv)' in the analysis of the classical algebra [30].

Algebra.   The affine algebra $\widehat{\mathcal{Q}}$ differs significantly from the standard $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ by the fact that the anticommutators $\{E_2, F_4\}$ and $\{E_4, F_2\}$ do not vanish. Nevertheless, one can readily see from (4.3) and (5.3) that the above-mentioned anticommutators vanish when $g \to 0$, as well as the central elements deforming the quartic Serre relations (4.5). Moreover, in the limit $g \to 0$, relations (4.3)–(4.10) almost reproduce the standard products, coproducts, antipodes and counits of the quantum affine algebra $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$.

Merely the Hopf algebra structure described in (4.10) requires a more elaborate analysis. The coproducts $\Delta(E_k)$ and $\Delta(F_k)$ with $k = 2, 4$ appear twisted by the central elements U_k,

$$\Delta(E_k) = E_k \otimes 1 + K_k^{-1}U_2^{+\delta_{j,2}}U_4^{+\delta_{k,4}} \otimes E_k$$

$$\Delta(F_k) = F_k \otimes K_k + U_2^{-\delta_{j,2}}U_4^{-\delta_{k,4}} \otimes F_k. \tag{6.1}$$

We recover the standard Hopf algebra structure of the $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ by the following similarity transformation of the coproduct[10]:

$$\bar{\Delta}(X) = (U_2 \otimes 1)^{-1 \otimes B_2}(U_4 \otimes 1)^{-1 \otimes B_4}\Delta(X)(U_2 \otimes 1)^{1 \otimes B_2}(U_4 \otimes 1)^{1 \otimes B_4}, \tag{6.2}$$

where B_k are two continuous automorphisms of $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ defined by

$$[B_k, E_j] = +\delta_{j,k}E_j, \quad [B_k, K_j] = 0, \quad [B_k, F_j] = -\delta_{j,k}F_j. \tag{6.3}$$

This clearly removes the central elements U_k from the above coproducts (6.1).

The above transformation can be viewed as composed from a Reshetikhin twist [36] and a change of basis. The operator

$$\mathcal{F} = (1 \otimes U_2)^{-B_2 \otimes 1/2}(U_2 \otimes 1)^{1 \otimes B_2/2}(1 \otimes U_4)^{-B_4 \otimes 1/2}(U_4 \otimes 1)^{1 \otimes B_4/2} \tag{6.4}$$

satisfies the relations $\mathcal{F}_{12}\mathcal{F}_{21} = 1$ and $\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}$. As demonstrated in [36], $\Delta^{(\mathcal{F})}(X)$ and $\mathcal{R}^{(\mathcal{F})}$ also form a Hopf algebra with

$$\Delta^{(\mathcal{F})}(X) = \mathcal{F}^{-1}\Delta(X)\mathcal{F}, \quad \mathcal{R}^{(\mathcal{F})} = \mathcal{F}\mathcal{R}\mathcal{F}. \tag{6.5}$$

The coproduct $\Delta^{(\mathcal{F})}$ is already equivalent to the standard coproduct $\bar{\Delta}$. This can be seen upon conjugating the basis $X' = U_2^{-B_2/2}U_4^{-B_4/2}XU_2^{B_2/2}U_4^{B_4/2}$ which effectively conjugates the coproduct by

$$(1 \otimes U_2)^{B_2 \otimes 1/2}(U_2 \otimes 1)^{1 \otimes B_2/2}(1 \otimes U_4)^{B_4 \otimes 1/2}(U_4 \otimes 1)^{1 \otimes B_4/2}. \tag{6.6}$$

---

[9] We use a convention for $x^\pm$ which differs slightly from the one used in [27], as explained above.

[10] We define exponents with coproducts as $(U_2 \otimes 1)^{1 \otimes B_2} = \exp((\log U_2) \otimes B_2)$.

Fundamental representation. To understand the limit $g \to 0$, it is also convenient to consider the fundamental representation of $\widehat{\mathcal{Q}}$ given in terms of the variables $x^{\pm}$ and γ. Since the variables x^+ and x^- are constrained by relation (5.11), we first need to introduce an appropriate expansion for them in the proposed limit. Direct inspection of relation (5.11) leads us to the following expansion:

$$x^{\pm} = \frac{\mathrm{i}}{g}\frac{q^{\pm 1}z - 1}{q - q^{-1}} + \mathcal{O}(g) \quad \text{and} \quad \gamma = \frac{\bar{\gamma}}{\sqrt{g}}, \tag{6.7}$$

where $\bar{\gamma}$ emerges from a rescaling of γ required to obtain finite results.

Taking into account expansion (6.7), in the limit $g \to 0$, we find that the coefficients a_k, b_k, c_k and d_k defined in (5.8) assume the following values:

$$a_2 = \bar{\gamma}, \quad b_2 = 0, \quad a_4 = 0, \quad b_4 = \alpha\tilde{\alpha}\frac{z}{\bar{\gamma}},$$

$$c_2 = 0, \quad d_2 = \frac{1}{\bar{\gamma}}, \quad c_4 = -\frac{1}{\alpha\tilde{\alpha}}\frac{\bar{\gamma}}{z}, \quad d_4 = 0. \tag{6.8}$$

Up to some factors, these define the canonical representations of E_k, F_k in $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$. In their turn, the central element eigenvalues U_k and V_k are then given by

$$U^2 = U_2^2 = U_4^{-2} = \frac{1 - zq}{q - z}, \quad V^2 = V_2^2 = V_4^{-2} = q. \tag{6.9}$$

Moreover, we find

$$K_4 \simeq K_1^{-1}K_2^{-1}K_3^{-1}, \quad E_4 \simeq \alpha\tilde{\alpha}z[[F_3, F_2], F_1], \quad F_4 \simeq -\alpha^{-1}\tilde{\alpha}^{-1}z^{-1}[[E_3, E_2], E_1], \tag{6.10}$$

which corresponds to the standard evaluation representation of the quantum affine algebra $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ up to a conventional rescaling of the generators E_4 and F_4. This observation supports z as the evaluation parameter of the quantum affine algebra $\widehat{\mathcal{Q}}$.

Fundamental R-matrix. Next we would like to obtain the limit of the fundamental R-matrix. In order to proceed, we need to apply the Reshetikhin twist (6.4), (6.5) to the fundamental R-matrix. On one hand, we have to note that the automorphisms B_2 and B_4 have no fundamental representation. On the other hand, we are saved by the fact that they appear only in a combination which is represented by the fermion number operator

$$B = B_2 - B_4 \simeq \mathrm{diag}(0, 0, 1, 1) \tag{6.11}$$

due to the relation $U_2 \simeq U_4^{-1}$. Hence, the operator $\mathcal{F}$ in (6.4) becomes[11]

$$\mathcal{F} \simeq U_2^{-B/2} \otimes U_1^{+B/2}. \tag{6.12}$$

The matrix elements of $\mathcal{R}^{(\mathcal{F})}$ still contain the factors $\bar{\gamma}_i$ remaining from the normalization between the bosonic/fermionic states, cf (6.8), as well as some factors of U_i. These can be removed by a spectral parameter-dependent gauge transformation [6]

$$\bar{\mathcal{R}} = (\mathcal{G}_1 \otimes \mathcal{G}_2)\mathcal{R}^{(\mathcal{F})}(\mathcal{G}_1 \otimes \mathcal{G}_2)^{-1} \quad \text{with} \quad \mathcal{G}_i = U_i^{B/2}\bar{\gamma}_i^{-B}. \tag{6.13}$$

Altogether, the transformation reads

$$\bar{\mathcal{R}} = \left[(\sqrt{U_1/U_2}/\gamma_1)^B \otimes (\sqrt{U_1 U_2}/\gamma_2)^B\right]\mathcal{R}\left[(\gamma_1/\sqrt{U_1 U_2})^B \otimes (\gamma_2\sqrt{U_1/U_2})^B\right]. \tag{6.14}$$

Although we shall not present the explicit form of the R-matrix, we find that $\bar{\mathcal{R}}$ equals the R-matrix of the Perk–Schultz model $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ [37] up to an overall factor. Moreover, the matrix $\bar{\mathcal{R}}_{ij}$ depends only on the ratio z_i/z_j and as expected, it satisfies the Yang–Baxter equation in the usual trigonometric form

$$\bar{\mathcal{R}}_{12}(z_1/z_2)\bar{\mathcal{R}}_{13}(z_1/z_3)\bar{\mathcal{R}}_{23}(z_2/z_3) = \bar{\mathcal{R}}_{23}(z_2/z_3)\bar{\mathcal{R}}_{13}(z_1/z_3)\bar{\mathcal{R}}_{12}(z_1/z_2). \tag{6.15}$$

---

[11] In the following, U_i denotes the eigenvalue of $U_2 \simeq U_4^{-1}$ on site i.

7. Yangian limit

In the previous sections, we have found that the (trigonometric) *R*-matrix of [27] has a quantum affine symmetry. On the other hand, it is known that the undeformed (rational) *R*-matrix enjoys Yangian symmetry [12]. Since the quantum deformed fundamental *R*-matrix (in $x^{\pm}$ parametrization) trivially reduces to the undeformed one by taking the deformation parameter *q* to 1, one of the natural questions is how the quantum affine symmetry is related to the Yangian symmetry in this limit. This is not only an important consistency check of our quantum affine algebra but also it might serve the possibility of investigating the Yangian structure in the AdS/CFT correspondence from the viewpoint of the quantum affine algebra $\widehat{Q}$. This limit corresponds to the case 'R(full)' in the analysis of the classical algebra [30]. However, in comparison with the limit of the *R*-matrix itself, the Yangian limit of the quantum affine algebra is not straightforward. For instance, if we take the parameter *q* to 1 naively, the quantum affine algebra does not reduce to the Yangian algebra but just gives the undeformed universal enveloping algebra. Since the Yangian algebra is generated by the level-0 (non-affine) and (at least one) level-1 generators, we need to find a non-trivial limit to obtain the Yangian algebra.

In this section, we show that the AdS/CFT Yangian symmetries [12] are actually reproduced from our quantum affine algebra $\widehat{Q}$. The limit is analogous to the Yangian limit of the quantum affine $\mathfrak{gl}(n)$ outlined in appendix A. There is however a subtlety related to an extra generator of the Yangian $\mathcal{Y}$, which was called secret symmetry in [38].

Fundamental representation. The difficulty of the Yangian limit in our case is that the affine generators E_4 and F_4 in (5.1) do not obey the standard evaluation representation. The evaluation representation is helpful to find the algebraic identification between the quantum affine algebra and Yangian. However, we have found that it is possible to take the $q \to 1$ limit. In order to see this, we would like to start with investigating the analytic properties of the parameters a_2, b_2, c_2, d_2 and a_4, b_4, c_4, d_4. As an important fact, the two sets of parameters are related as follows:

$$MT_4 = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} T_2 \begin{pmatrix} w^{-1} & 0 \\ 0 & wz \end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix} 0 & \alpha\tilde{\alpha} \\ -\alpha^{-1}\tilde{\alpha}^{-1} & 0 \end{pmatrix}, \quad T_k = \begin{pmatrix} a_k & -b_k \\ -c_k & d_k \end{pmatrix}, \tag{7.1}$$

where the evaluation parameter z (cf (5.11) in $x^{\pm}$ variables) and w are given by

$$z = \frac{VU - V^{-1}U^{-1}}{V^{-1}U - VU^{-1}}, \qquad w = \frac{\tilde{g}}{g}\frac{q^{1/2}U - q^{-1/2}U^{-1}}{VU - V^{-1}U^{-1}} = \frac{g}{\tilde{g}}\frac{V^{-1}U - VU^{-1}}{q^{-1/2}U - q^{1/2}U^{-1}}. \tag{7.2}$$

The limit $q \to 1$ can be taken in different ways. For the Yangian limit, we assume *U* to remain finite and arbitrary, as expected from [9]. Relation (2.10) between *U* and *V* implies that $V \to 1$. More precisely, as $q = 1 + h$ for $h \to 0$,

$$V = 1 + hC + \mathcal{O}(h^2) \quad \text{with} \quad C^2 = \tfrac{1}{4} - g^2(U - U^{-1})^2. \tag{7.3}$$

The latter constraint between the central charges *U* and *C* agrees with [9]. The parameters $x^{\pm}$ remain finite and they obey the constraint[12]

$$(x^+ - x^-)(1 - 1/x^+x^-) = ig^{-1}. \tag{7.4}$$

Using these, the central charge eigenvalues take the familiar form

$$U^2 = \frac{x^+}{x^-}, \quad C = \frac{1}{2}\frac{1 + 1/x^+x^-}{1 - 1/x^+x^-}. \tag{7.5}$$

---

[12] Even though our $x^{\pm}$ parametrization is slightly different from [27], it has the same $q \to 1$ limit.

It is easy to see that the parameters z and w in (7.2) can be expanded as

$$z = 1 - 2higu + \mathcal{O}(h^2), \quad w = 1 + hig(u - v) + \mathcal{O}(h^2). \tag{7.6}$$

The rational evaluation parameters u [12] and v are given by

$$u = ig^{-1}C\frac{U + U^{-1}}{U - U^{-1}} = \frac{1}{2}(x^+ + x^-)(1 + 1/x^+x^-)$$

$$v = ig^{-1}\frac{1}{2}\frac{U + U^{-1}}{U - U^{-1}} = \frac{1}{2}(x^+ + x^-)(1 - 1/x^+x^-). \tag{7.7}$$

Note that $-\alpha\tilde{\alpha}(c_4, d_4) \to (a_2, b_2)$ and $\alpha^{-1}\tilde{\alpha}^{-1}(a_4, b_4) \to (c_2, d_2)$ and hence in the limit $q \to 1$, we find $-\alpha\tilde{\alpha}F_4 \simeq E_{321}$ and $\alpha^{-1}\tilde{\alpha}^{-1}E_4 \simeq F_{321}$ with

$$E_{321} = [[E_3, E_2], E_1], \quad F_{321} = [[F_3, F_2], F_1]. \tag{7.8}$$

That is, the limits of F_4 and E_4 are not independent and the generators should be replaced by the rescaled differences $(\alpha\tilde{\alpha}F_4 + E_{321})/(q-1)$ and $(\alpha^{-1}\tilde{\alpha}^{-1}E_4 - F_{321})/(q-1)$. Consequently, what matters in the Yangian limit is

$$\lim_{q\to 1}\frac{MT_4 - T_2}{ig(q-1)} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}T_2 + T_2\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} = NT_2, \tag{7.9}$$

where we have introduced the following matrix:

$$N = \begin{pmatrix} 2u & -i\alpha(1 + U^2) \\ -i\alpha^{-1}(1 + U^{-2}) & -2u \end{pmatrix}. \tag{7.10}$$

Algebra. Relation (7.9) with matrices (7.10) leads us to the following identification between the quantum affine algebra $\widehat{\mathcal{Q}}$ and its associated Yangian algebra:

$$\lim_{q\to 1}\frac{-\alpha\tilde{\alpha}F_4 - E_{321}}{ig(q-1)} = 2\widehat{E}_{321} + i\alpha(1 + U^2)F_2$$

$$\lim_{q\to 1}\frac{\alpha^{-1}\tilde{\alpha}^{-1}E_4 - F_{321}}{ig(q-1)} = -2\widehat{F}_{321} + i\alpha^{-1}(1 + U^{-2})E_2, \tag{7.11}$$

with the Yangian evaluation representation

$$\widehat{E}_{321} \simeq uE_{321}, \quad \widehat{F}_{321} \simeq uF_{321}. \tag{7.12}$$

Since the generator E_{321} (F_{321}) is the highest (lowest) weight in the adjoint of $\mathfrak{psl}(2|2)$, it is sufficient to obtain the other Yangian generators. In fact, we have listed all level-1 generators in appendix B. In comparison with the standard case (A.11), the left-hand sides of (7.11) have the same structure but on the right-hand sides, we need some additional terms.

The point is that these relations (7.11) are actually compatible with the coalgebra structure. In other words, the limit of the coproduct on the left-hand side of (7.11) induces the Yangian coproducts on the right-hand side,

$$\lim_{q\to 1}\frac{-\alpha\tilde{\alpha}\Delta F_4 - \Delta E_{321}}{ig(q-1)} = (2\widehat{E}_{321} + i\alpha(1 + U^2)F_2 - ig^{-1}kE_{321}) \otimes 1$$

$$+ U \otimes (2\widehat{E}_{321} + i\alpha(1 + U^2)F_2)$$

$$- ig^{-1}[-E_{321} \otimes (H_3 + H_2 + H_1) + (H_3 + H_2 + H_1)U \otimes E_{321}$$

$$+ E_{32} \otimes E_1 - E_1U \otimes E_{32} - E_3U \otimes E_{21} + E_{21} \otimes E_3]$$

$$= \Delta(2\widehat{E}_{321} + i\alpha(1 + U^2)F_2),$$

$$\lim_{q \to 1} \frac{\alpha^{-1}\tilde{\alpha}^{-1}\Delta E_4 - \Delta F_{321}}{ig(q-1)} = (-2\widehat{F}_{321} + i\alpha^{-1}(1 + U^{-2})E_2) \otimes 1$$
$$+ U^{-1} \otimes (-2\widehat{F}_{321} + i\alpha^{-1}(1 + U^{-2})E_2 + ig^{-1}kF_{321})$$
$$+ ig^{-1}[F_{321} \otimes (H_3 + H_2 + H_1) - (H_3 + H_2 + H_1)U^{-1} \otimes F_{321}$$
$$+ F_{32} \otimes F_1 - F_1 U^{-1} \otimes F_{32} - F_3 U^{-1} \otimes F_{21} + F_{21} \otimes F_3]$$
$$= \Delta(-2\widehat{F}_{321} + i\alpha^{-1}(1 + U^{-2})E_2). \tag{7.13}$$

Here k is the affine central element given by $k = H_1 + H_2 + H_3 + H_4$ whose eigenvalue k vanishes on evaluation representations. These coproducts coincide with the results in [12] where k was projected out, cf appendix B for a translation of the generator notation. Note that the additional terms $i\alpha(1 + U^2)F_2$ and $i\alpha^{-1}(1 + U^{-2})E_2$ on the right-hand side of (7.13) are required to cancel certain contributions from the higher central charges in the original Yangian symmetries $\Delta\widehat{E}_{321}$ and $\Delta\widehat{F}_{321}$ [12].

The deeper meaning of the additional terms in (7.11) is not clear to us. It is nevertheless interesting to interpret them as a contribution of an extra Yangian generator $\widehat{B}$ called secret symmetry [38]. The fundamental representation of this generator is given by

$$\widehat{B} \simeq \frac{v}{2}\text{diag}(1, 1, -1, -1) \tag{7.14}$$

with the parameter v in (7.7). The relevant two of its commutators read [24]

$$[\widehat{B}, E_{321}] = -\widehat{E}_{321} - i\alpha(1 + U^2)F_2, \quad [\widehat{B}, F_{321}] = \widehat{F}_{321} - i\alpha^{-1}(1 + U^{-2})E_2. \tag{7.15}$$

These are indeed compatible with their coproducts; therefore, the equivalent replacement in (7.13) is valid as well. Using this secret symmetry, we can rewrite the Yangian limit (7.11) as

$$\lim_{q \to 1} \frac{-\alpha\tilde{\alpha}F_4 - E_{321}}{ig(q-1)} = \widehat{E}_{321} - [\widehat{B}, E_{321}], \quad \lim_{q \to 1} \frac{\alpha^{-1}\tilde{\alpha}^{-1}E_4 - F_{321}}{ig(q-1)} = -\widehat{F}_{321} - [\widehat{B}, F_{321}]. \tag{7.16}$$

8. Conclusions

In this work, we have derived a novel quantum affine algebra $\widehat{\mathcal{Q}}$ based on a central extension of the $\mathfrak{sl}(2|2)$ Lie superalgebra. As a matter of fact, this algebra emerges naturally from compatibility requirements with the *R*-matrix of the deformed Hubbard chain [27] also known as the Alcaraz–Bariev model [26]. In this sense, the formulation of this algebra sheds some new light into a more complete understanding of the integrable structure underlying the Hubbard model and its deformed counterpart.

The construction of the quantum affine algebra $\widehat{\mathcal{Q}}$ was immensely guided by the Dynkin diagram of the $\widehat{\mathfrak{sl}}(2|2)$ algebra. More precisely, the similarity between the fermionic nodes 2 and 4 of the Dynkin diagram given in figure 2 suggests for instance that the generators associated with node 4 should act as copies of those associated with node 2. This observation has played a fundamental role not only for the establishment of the commutation relations (4.3), but also for the construction of the fundamental representation.

The quantum affine algebra $\widehat{\mathcal{Q}}$ possesses fundamentally a deformation parameter q originated from the deformation of the universal enveloping algebra of $\mathfrak{sl}(2|2)$, as well as a coupling parameter g introduced by the central extensions. Here we have also shown that the algebra $\widehat{\mathcal{Q}}$ reduces to the standard quantum affine algebra $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ in the limit $g \to 0$, which unveils a relation between the Alcaraz–Bariev model and the Perk–Schultz model $\mathcal{U}_q[\widehat{\mathfrak{sl}}(2|2)]$ in this particular limit. We have furthermore investigated the limit $q \to 1$ where we have found that the affine algebra $\widehat{\mathcal{Q}}$ reproduces the Yangian $\mathcal{Y}$ of a centrally extended $\mathfrak{sl}(2|2)$ algebra.

This Yangian $\mathcal{Y}$ corresponds to the same algebra underlying Shastry's *R*-matrix which also plays an important role for integrability in the context of the AdS/CFT correspondence. In this way, as quantum affine algebras offer a more uniform description in comparison to Yangians, this limit procedure might help us address integrability in the AdS/CFT correspondence.

In the analysis of the classical algebra performed in [30], the conventional quantum affine and Yangian limits reduce respectively to the cases 'T(conv)' and 'R(full)'. However, in the classical limit, a whole cascade of algebras has been presented in [30] which makes us wonder if all the cases indeed possess a quantum counterpart.

Furthermore, it would be worthwhile to investigate higher representations of the algebra, cf [39], which are likely to be direct analogs of the undeformed case studied in [16, 40, 15]. Finally, the formulation of Drinfel'd's second realization for this algebra would constitute a valuable step toward the universal *R*-matrix, cf [41].

Acknowledgments

Appendix A. Yangian limit of quantum affine $\mathfrak{gl}(n)$

As we have mentioned in the beginning of section 7, the Yangian limit is not so much trivial. Therefore, it is convenient to review the generic example of the $\mathfrak{gl}(n)$ case ($n \geqslant 3$) [42]. That is the limit from $\mathcal{U}_q[\widehat{\mathfrak{gl}(n)}]$ to $\mathcal{Y}[\mathfrak{gl}(n)]$, which enables us to make the logic clear.

The generators of the Lie algebra $\mathfrak{gl}(n)$ are given by $\mathrm{J}^i{}_j$ with $i, j = 1, \ldots, n$ and they satisfy the standard commutation relations

$$\left[\mathrm{J}^i{}_j, \mathrm{J}^k{}_l\right] = \delta^k_j \mathrm{J}^i{}_l - \delta^i_l \mathrm{J}^j{}_k. \tag{A.1}$$

In order to describe its quantum deformation $\mathcal{U}_q[\mathfrak{gl}(n)]$, it is convenient to introduce the corresponding Chevalley–Serre simple roots $\mathrm{E}_i, \mathrm{F}_i, \mathrm{H}_i$ with $i, j = 1, \ldots, n-1$, which are related as

$$\mathrm{E}_i = \mathrm{J}^i{}_{i+1}, \quad \mathrm{F}_i = \mathrm{J}^{i+1}{}_i, \quad \mathrm{H}_i = \mathrm{J}^i{}_i - \mathrm{J}^{i+1}{}_{i+1}. \tag{A.2}$$

Their commutation relations are given by

$$[\mathrm{H}_i, \mathrm{E}_j] = +A_{ij}\mathrm{E}_j, \quad [\mathrm{H}_i, \mathrm{F}_j] = -A_{ij}\mathrm{F}_j, \quad [\mathrm{E}_i, \mathrm{F}_j] = \delta_{ij}\frac{q^{\mathrm{H}_i} - q^{-\mathrm{H}_i}}{q - q^{-1}} \tag{A.3}$$

with the Cartan matrix *A* defined by

$$A_{ij} = \begin{cases} +2 & \text{for } i = j \\ -1 & \text{for } |i - j| = 1 \\ 0 & \text{for } |i - j| \geqslant 2. \end{cases} \tag{A.4}$$

Furthermore, the following Serre relations hold for $|i - j| = 1$:

$$[\mathrm{E}_i, [\mathrm{E}_i, \mathrm{E}_j]] = (q - 2 + q^{-1})\mathrm{E}_i\mathrm{E}_j\mathrm{E}_i$$
$$[\mathrm{F}_i, [\mathrm{F}_i, \mathrm{F}_j]] = (q - 2 + q^{-1})\mathrm{F}_i\mathrm{F}_j\mathrm{F}_i \tag{A.5}$$

16

and for $|i - j| \geqslant 2$

$$[E_i, E_j] = [F_i, F_j] = 0. \tag{A.6}$$

The affine extension $\mathcal{U}_q[\widehat{\mathfrak{gl}}(n)]$ to $\mathcal{U}_q[\mathfrak{gl}(n)]$ is obtained by adding the affine generators E_n, F_n, H_n and extending the Cartan matrix to $n \times n$. The relations are almost the same as the above but the indices in (A.3)–(A.6) are considered modulo n. It is noted that the summation of the Cartan generators $H_1 + \cdots + H_n = k$ turns out to be the affine central element.

The quantum affine algebra has a Hopf algebra structure. For the Chevalley–Serre generators, the coproducts, antipodes and counits are given by

$$\Delta(E_i) = E_i \otimes 1 + q^{-H_i} \otimes E_i, \quad S(E_i) = -q^{H_i} E_i, \quad \varepsilon(E_i) = 0,$$
$$\Delta(F_i) = F_i \otimes q^{H_i} + 1 \otimes F_i, \quad S(F_i) = -F_i q^{-H_i}, \quad \varepsilon(F_i) = 0,$$
$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad S(H_i) = -H_i, \quad \varepsilon(H_i) = 0. \tag{A.7}$$

One of the important representations of the algebra is the evaluation representation, in which the affine generators are expressed as

$$E_n \simeq z^{-1} q^{J^1{}_1 + J^n{}_n} F_{n-1\cdots1}, \quad F_n \simeq z q^{-J^1{}_1 - J^n{}_n} E_{n-1\cdots1}, \quad H_n \simeq -H_{n-1} - \cdots - H_1, \tag{A.8}$$

with the evaluation parameter z. Here we have used the following abbreviations:

$$E_{n-1\cdots1} = [[[E_{n-1}, E_{n-2}]_q, \ldots]_q, E_1]_q, \quad F_{n-1\cdots1} = [[[F_{n-1}, F_{n-2}]_{q^{-1}}, \ldots]_{q^{-1}}, F_1]_{q^{-1}}, \tag{A.9}$$

where the q-deformed commutators are defined by

$$[A, B]_{q^{\pm1}} = AB - q^{\pm1} BA. \tag{A.10}$$

Note that the affine central element k vanishes in this representation.

The Yangian limit is taken by the following identification:

$$\lim_{q \to 1} \frac{F_n - q^{-J^1{}_1 - J^n{}_n} E_{n-1\cdots1}}{q - 1} = \widehat{E}_{n-1\cdots1}, \quad \lim_{q \to 1} \frac{q^{J^1{}_1 + J^n{}_n} F_{n-1\cdots1} - E_n}{q - 1} = \widehat{F}_{n-1\cdots1}. \tag{A.11}$$

The left-hand sides of the above relations are the $q \to 1$ limit of the quantum affine generators and the right-hand sides are the level-1 Yangian generators. This identification (A.11) has two good properties. The first one is the consistency with the Yangian evaluation representation,

$$\widehat{E}_{n-1\cdots1} \simeq u E_{n-1\cdots1}, \quad \widehat{F}_{n-1\cdots1} \simeq u F_{n-1\cdots1}, \tag{A.12}$$

where the Yangian evaluation parameter u is related to the quantum one in (A.8) as $z = q^u$ and the book-keeping notations (A.9) are replaced by $q = 1$. The second one is the compatibility with the coproducts. In other words, the following Yangian coproducts are automatically derived from the quantum affine algebra from relations (A.11) up to the affine central element k,

$$\Delta \widehat{E}_{n-1\cdots1} = (\widehat{E}_{n-1\cdots1} + kE_{n-1\cdots1}) \otimes 1 + 1 \otimes \widehat{E}_{n-1\cdots1}$$
$$+ 2 \left[E_{n-1\cdots1} \otimes J^n{}_n + J^1{}_1 \otimes E_{n-1\cdots1} + \sum_{k=1}^{n-2} E_{n-1\cdots k+1} \otimes E_{k\cdots1} \right]$$
$$\Delta \widehat{F}_{n-1\cdots1} = \widehat{F}_{n-1\cdots1} \otimes 1 + 1 \otimes (\widehat{F}_{n-1\cdots1} + kF_{n-1\cdots1})$$
$$+ 2 \left[F_{n-1\cdots1} \otimes J^1{}_1 + J^n{}_n \otimes F_{n-1\cdots1} - \sum_{k=1}^{n-2} F_{1\cdots k} \otimes F_{k+1\cdots n-1} \right]. \tag{A.13}$$

In fact, the defining relations of the Yangian algebra $\mathcal{Y}[\mathfrak{gl}(n)]$ stem from those of the quantum affine algebra $\mathcal{U}_q[\widehat{\mathfrak{gl}}(n)]$ via identification (A.11).

17

Appendix B. Yangian limits for all generators

In this appendix, we would like to list the Yangian limits for all the generators in the quantum affine algebra $\mathcal{Q}$ for completeness. In order to do that, it is convenient to introduce some notations $Q^{a\alpha} = \epsilon^{ab}Q^{\alpha}{}_{b}$ and $S^{a\alpha} = \epsilon^{\alpha\beta}S^{a}{}_{\beta}$ ($a, \alpha = 1, 2$) for the fermionic generators [9, 16]. These generators are defined by the Chevalley–Serre basis as

$$
\begin{aligned}
Q^{11} &= E_{32}, & Q^{12} &= E_2, & Q^{21} &= -E_{321}, & Q^{22} &= -E_{21}, \\
S^{11} &= -F_{21}, & S^{12} &= -F_{321}, & S^{21} &= F_2, & S^{22} &= F_{32}.
\end{aligned}
\tag{B.1}
$$

We also denote another set of fermionic generators which include the affine generators E_4, F_4 as $\overline{Q}^{\alpha}{}_{a}$, $\overline{S}^{a}{}_{\alpha}$. They are given by replacing E_2, F_2 with E_4, F_4 in (B.1), respectively,

$$
\begin{aligned}
\overline{Q}^{11} &= E_{34}, & \overline{Q}^{12} &= E_4, & \overline{Q}^{21} &= -E_{341}, & \overline{Q}^{22} &= -E_{41}, \\
\overline{S}^{11} &= -F_{41}, & \overline{S}^{12} &= -F_{341}, & \overline{S}^{21} &= F_4, & \overline{S}^{22} &= F_{34}.
\end{aligned}
\tag{B.2}
$$

This notation allows us to express the Yangian limits in synthesized forms. The Yangian limits for the fermionic generators are now given by

$$
\lim_{q\to 1} \frac{\alpha\tilde{\alpha}\overline{S}^{a\alpha} - Q^{a\alpha}}{ig(q-1)} = 2\widehat{Q}^{a\alpha} - i\alpha(1 + U^2)S^{a\alpha} = \widehat{Q}^{a\alpha} - [\widehat{B}, Q^{a\alpha}]
$$

$$
\lim_{q\to 1} \frac{\alpha^{-1}\tilde{\alpha}^{-1}\overline{Q}^{a\alpha} + S^{a\alpha}}{ig(q-1)} = 2\widehat{S}^{a\alpha} + i\alpha^{-1}(1 + U^{-2})Q^{a\alpha} = \widehat{S}^{a\alpha} + [\widehat{B}, S^{a\alpha}].
\tag{B.3}
$$

The other Yangian limits for the bosonic generators, which are defined by

$$
\begin{aligned}
R^{11} &= -F_1, & R^{12} = R^{21} &= -\tfrac{1}{2}H_1, & R^{22} &= E_1, \\
L^{11} &= -E_3, & L^{12} = L^{21} &= -\tfrac{1}{2}H_3, & L^{22} &= F_3, \\
C &= -\tfrac{1}{2}H_1 - H_2 - \tfrac{1}{2}H_3, & P = \{[E_1, E_2], & [E_3, E_2]\}, & K = \{[F_1, F_2], [F_3, F_2]\},
\end{aligned}
\tag{B.4}
$$

are inductively obtained by computing suitable commutation relations from (B.3) as

$$
\lim_{q\to 1} \frac{\alpha\tilde{\alpha}\{\overline{S}^{a\alpha}, Q^{b\beta}\} - \epsilon^{ab}\epsilon^{\alpha\beta}P}{2ig(q-1)} = \epsilon^{ab}\epsilon^{\alpha\beta}\widehat{P} + \frac{i}{2}\alpha(1 + U^2)(\epsilon^{\alpha\beta}R^{ab} - \epsilon^{ab}L^{\alpha\beta} + \epsilon^{ab}\epsilon^{\alpha\beta}C)
$$

$$
\lim_{q\to 1} \frac{\{\overline{Q}^{a\alpha}, S^{b\beta}\} + \alpha\tilde{\alpha}\epsilon^{ab}\epsilon^{\alpha\beta}K}{2\alpha\tilde{\alpha}ig(q-1)} = \epsilon^{ab}\epsilon^{\alpha\beta}\widehat{K} + \frac{i}{2}\alpha^{-1}(1 + U^{-2})(\epsilon^{\alpha\beta}R^{ab} - \epsilon^{ab}L^{\alpha\beta} - \epsilon^{ab}\epsilon^{\alpha\beta}C)
$$

$$
\lim_{q\to 1} \frac{\alpha^{-1}\tilde{\alpha}^{-1}\{\overline{Q}^{a\alpha}, Q^{b\beta}\} + \alpha\tilde{\alpha}\{S^{a\alpha}, \overline{S}^{b\beta}\}}{4ig(q-1)}
$$

$$
= -\epsilon^{\alpha\beta}\widehat{R}^{ab} + \epsilon^{ab}\widehat{L}^{\alpha\beta} - \epsilon^{ab}\epsilon^{\alpha\beta}\widehat{C} - \frac{i}{2}g\epsilon^{ab}\epsilon^{\alpha\beta}(U^2 - U^{-2}).
\tag{B.5}
$$

The above limits (B.3) and (B.5) give the same coproducts presented by [12, 38] and the symmetries of the undeformed *R*-matrix [9].

References

[1] Bazhanov V V 1985 Trigonometric solution of triangle equations and classical Lie algebras *Phys. Lett.* B **159** 321

Bazhanov V V and Shadrikov A G 1987 Trigonometric solutions of triangle equations—simple Lie superalgebras *Theor. Math. Phys.* **73** 1302

Galleas W and Martins M J 2006 New *R*-matrices from representations of braid-monoid algebras based on superalgebras *Nucl. Phys.* B **732** 444 (arXiv:nlin/0509014)

Ge M L, Wu Y S and Xue K 1991 Explicit trigonometric Yang-Baxterization *Int. J. Mod. Phys.* A **6** 3735

[2] Jimbo M 1986 Quantum *r* matrix for the generalized Toda system *Commun. Math. Phys.* **102** 537

[3] Drinfel'd V G 1985 Hopf algebras and the quantum Yang–Baxter equation *Sov. Math. Dokl.* **32** 254

Jimbo M 1985 A q difference analog of U(g) and the Yang–Baxter equation *Lett. Math. Phys.* **10** 63

[4] Kulish P P and Reshetikhin N Y 1983 Quantum linear problem for the sine-Gordon and higher representations *J. Sov. Math.* **23** 2435

[5] Artin E 1947 Theory of braids *Ann. Math.* **48** 101

[6] Wadati M, Deguchi T and Akutsu Y 1989 Exactly solvable models and knot theory *Phys. Rep.* **180** 247

[7] Jones V F R 1991 Baxterization *Int. J. Mod. Phys.* A **6** 2035

[8] Arutyunov G and Frolov S 2009 Foundations of the AdS$_5$ × S^5 superstring: part I *J. Phys. A: Math. Theor.* **42** 254003 (arXiv:0901.4937)
Beisert N *et al* 2012 Review of AdS/CFT integrability: an overview *Lett. Math. Phys.* **99** 3 (arXiv:1012.3982)
Ahn C and Nepomechie R I 2012 Review of AdS/CFT integrability: chapter III.2. Exact world-sheet *S*-matrix *Lett. Math. Phys.* **99** 209 (arXiv:1012.3991)

[9] Beisert N 2008 The su(2/2) dynamic *S*-matrix *Adv. Theor. Math. Phys.* **12** 945 (arXiv:hep-th/0511082)

[10] Gómez C and Hernández R 2006 The magnon kinematics of the AdS/CFT correspondence *J. High Energy Phys.* JHEP11(2006)021 (arXiv:hep-th/0608029)

[11] Plefka J, Spill F and Torrielli A 2006 On the Hopf algebra structure of the AdS/CFT *S*-matrix *Phys. Rev.* D **74** 066008 (arXiv:hep-th/0608038)

[12] Beisert N 2007 The *S*-matrix of AdS/CFT and Yangian symmetry *PoS Solvay* 002 (arXiv:0704.0400)

[13] de Leeuw M The *S*-matrix of the AdS$_5$ × S^5 superstring *PhD Thesis* Utrecht University (arXiv:1007.4931)

[14] Torrielli A 2012 Review of AdS/CFT integrability: chapter VI.2. Yangian algebra *Lett. Math. Phys.* **99** 547 (arXiv:1012.4005)

[15] Arutyunov G, de Leeuw M and Torrielli A 2010 On Yangian and long representations of the centrally extended su(2/2) superalgebra *J. High Energy Phys.* JHEP06(2010)033 (arXiv:0912.0209)

[16] Beisert N 2007 The analytic Bethe ansatz for a chain with centrally extended su(2/2) symmetry *J. Stat. Mech.* **07** P01017 (arXiv:nlin.SI/0610017)

[17] Shastry B S 1986 Exact integrability of the one-dimensional Hubbard model *Phys. Rev. Lett.* **56** 2453

[18] Essler F H L, Frahm H, Göhmann F, Klümper A and Korepin V E 2005 *The One-Dimensional Hubbard Model* (Cambridge: Cambridge University Press)

[19] Lieb E H and Wu F Y 1968 Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension *Phys. Rev. Lett.* **20** 1445

[20] Heilmann O J and Lieb E H 1971 Violation of the noncrossing rule: the Hubbard Hamiltonian for benzene *Ann. New York Acad. Sci.* **172** 584

[21] Rej A, Serban D and Staudacher M 2006 Planar $\mathcal{N}$ = 4 gauge theory and the Hubbard model *J. High Energy Phys.* JHEP03(2006)018 (arXiv:hep-th/0512077)

[22] Nahm W 1978 Supersymmetries and their representations *Nucl. Phys.* B **135** 149
Iohara K and Koga Y 2001 Central extensions of Lie superalgebras *Comment. Math. Helv.* **76** 110

[23] Torrielli A 2007 Classical *r*-matrix of the su(2/2) SYM spin-chain *Phys. Rev.* D **75** 105020 (arXiv:hep-th/0701281)

[24] Beisert N and Spill F 2009 The classical *r*-matrix of AdS/CFT and its Lie bialgebra structure *Commun. Math. Phys.* **285** 537 (arXiv:0708.1762)

[25] Brinkman W F and Rice T M 1970 Single-particle excitations in magnetic insulators *Phys. Rev.* B **2** 1324
Lai C K 1974 Lattice gas with nearest-neighbor interaction in one dimension with arbitrary statistics *J. Math. Phys.* **15** 1675
Sutherland B 1975 Model for a multicomponent quantum system *Phys. Rev.* B **12** 3795
Essler F H L and Korepin V E 1992 Higher conservation laws and algebraic Bethe ansatz for the supersymmetric *t-J* model *Phys. Rev.* B **46** 9147

[26] Alcaraz F C and Bariev R Z 1999 Interpolation between Hubbard and supersymmetric t-j models: two-parameter integrable models of correlated electrons *J. Phys. A: Math. Gen.* **32** L483 (arXiv:cond-mat/9908265)

[27] Beisert N and Koroteev P 2008 Quantum deformations of the one-dimensional Hubbard model *J. Phys. A: Math. Theor.* **41** 255204 (arXiv:0802.0777)

[28] Murgan R and Nepomechie R I 2008 Q-deformed su(2/2) boundary *S*-matrices via the ZF algebra *J. High Energy Phys.* JHEP06(2008)096 (arXiv:0805.3142)

[29] Juttner G, Klümper A and Suzuki J 1998 The Hubbard chain at finite temperatures: *ab initio* calculations of Tomonaga–Luttinger liquid properties *Nucl. Phys.* B **522** 471 (arXiv:cond-mat/9711310)

[30] Beisert N 2011 The classical trigonometric *r*-matrix for the quantum-deformed Hubbard chain *J. Phys. A: Math. Theor.* **44** 265202 (arXiv:1002.1097)

[31] Yamane H 2003 A central extension of $U_q(sl(22)^{(1)})$ and *R*-matrices with a new parameter *J. Math. Phys.* **11** 5450 (arXiv:math.QA/0304406)

[32] Chari V and Pressley A 1995 Quantum affine algebras and their representations *Can. Math. Soc. Conf. Proc.* **16** 59 (arXiv:hep-th/9411145)
    Chari V and Pressley A 1991 Quantum affine algebras *Commun. Math. Phys.* **142** 261
[33] Chari V and Pressley A 1986 New unitary representations of loop groups *Math. Ann.* **275** 87
[34] Jimbo M 1986 A *q* analog of $u(Gl(n+1))$, Hecke algebra and the Yang–Baxter equation *Lett. Math. Phys.* **11** 247
[35] Chari V and Pressley A 1994 Small representations of quantum affine algebras *Lett. Math. Phys.* **30** 131
[36] Reshetikhin N 1990 Multiparameter quantum groups and twisted quasitriangular Hopf algebras *Lett. Math. Phys.* **20** 331
[37] Perk J H H and Schultz C L 1981 New families of commuting transfer matrices in *q*-state vertex models *Phys. Lett.* A **84** 407
    Schultz C L 1981 Solvable *q*-state models in lattice statistics and quantum field theory *Phys. Rev. Lett.* **46** 629
    de Vega H J and Lopes E 1991 Exact solution of the Perk–Schultz model *Phys. Rev. Lett.* **67** 489
[38] Matsumoto T, Moriyama S and Torrielli A 2007 A secret symmetry of the AdS/CFT *S*-matrix *J. High Energy Phys.* JHEP09(2007)099 (arXiv:0708.1285)
[39] Zhang Y-Z and Gould M D 2005 A unified and complete construction of all finite dimensional irreducible representations of gl(2/2) *J. Math. Phys.* **46** 013505 (arXiv:math/0405043)
[40] Chen H-Y, Dorey N and Okamura K 2007 The asymptotic spectrum of the $\mathcal{N}=4$ super Yang–Mills spin chain *J. High Energy Phys.* JHEP03(2007)005 (arXiv:hep-th/0610295)
    Arutyunov G, de Leeuw M and Torrielli A 2009 The bound state *S*-matrix for $AdS_5 \times S^5$ superstring *Nucl. Phys.* B **819** 319 (arXiv:0902.0183)
[41] Spill F and Torrielli A 2009 On Drinfeld's second realization of the AdS/CFT su(2/2) Yangian *J. Geom. Phys.* **59** 489 (arXiv:0803.3194)
[42] Tolstoy V N 2004 From quantum affine Kac–Moody algebras to drinfeldians and Yangians *Contemp. Math.* **343** 349 (arXiv:math.QA/0212370)
    Tolstoy V N 1998 Drinfeldians arXiv:math.QA/9803008
    Tolstoy V N 2001 Super-drinfeldians and super-Yangians of Lie superalgebras of type A(n/m) *Phys. At. Nucl.* **64** 2179

20