Mechanics of extended masses in general relativity

Abraham I. Harte
Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut
Potsdam-Golm, Germany
E-mail: harte@aei.mpg.de

Abstract. The "external" or "bulk" motion of extended bodies is studied in general relativity. Material objects of arbitrary shape, spin, internal composition, and velocity are allowed as long as the metric remains near a vacuum solution (with a possible cosmological constant). Under this restriction, physically reasonable linear and angular momenta are proposed that evolve as though they were the momenta of an extended test body moving in an effective vacuum metric. This result holds to all multipole orders. The portion of the physical metric that does not directly affect the motion is a slightly generalized form of the Detweiler-Whiting S-field originally introduced in the context of self-force. This serves only to (finitely) renormalize the "bare" multipole moments of the object's stress-energy tensor. The MiSaTaQu expression for the gravitational self-force is recovered as a simple application. A gravitational self-torque is obtained as well. Lastly, a certain exact result is derived that may provide a basis for understanding self-interaction at higher orders.

1. Introduction

Newtonian celestial mechanics typically describes the motion of widely-separated masses using two types of parameters (see, e.g., [1, 2]). These concern either the behavior of each body as a whole or the details of their internal dynamics. Examples of the former type – often referred to as the "external" or "bulk" parameters – are the center of mass positions and spin angular momenta of the various masses. In typical applications, there is very little coupling between the internal and external descriptions. As a consequence, one can often compute the center of mass motions of an N-body system as though it were composed of point particles parameterized only by their positions and masses. Going beyond this approximation requires introducing additional parameters such as quadrupole moments. These depend on the internal dynamics, but in a relatively mild way that often lends itself to simple phenomenological models.

The external variables decouple from the internal ones in Newtonian gravity largely because there are no self-forces or self-torques in this theory. The evolution of a body's center of mass position and spin are only indirectly influenced by its self-field.

A priori, it is not clear that similar arguments can be applied to matter interacting with relativistic fields. These fields carry energy and momentum, so self-forces arise generically. This does not, however, preclude an internal-external split of the dynamics. It is not necessary that self-forces vanish entirely, but only that they do not depend in any essential way on the details of a body's internal structure.

To illustrate this point, consider the motion of a small electric charge in approximate internal equilibrium moving non-relativistically in flat spacetime. It has long been known that under suitable conditions, the center of mass acceleration $\vec{a}(t)$ of such a charge will very nearly satisfy‡

$$m\vec{a} = \vec{F}_{\text{ext}} + \frac{2}{3}q^2 \frac{\mathrm{d}\vec{a}}{\mathrm{d}t} - \delta m \vec{a}. \tag{1}$$

Here, $\vec{F}_{\text{ext}}$ is an externally-imposed force, q is the object's total charge and m its mass§. The last two terms on the right-hand side of this equation arise from interactions with the body's own electromagnetic field. The first of these is "simple" in that it depends only on bulk parameters already required to describe the motion of a charged test particle. δm, by contrast, has a very different character. It is related to the body's internal charge distribution in a nontrivial way. Despite this, It is clear that

$$\hat{m}\vec{a} = \vec{F}_{\text{ext}} + \frac{2}{3}q^2 \frac{\mathrm{d}\vec{a}}{\mathrm{d}t}, \tag{2}$$

where $\hat{m} := m + \delta m$ can be interpreted as an effective mass. The same assumptions that lead to the derivation of (1) can also be used to show that $\mathrm{d}\hat{m}/\mathrm{d}t = 0$.

Even though the self-force is significant in this example (and depends on the body's internal structure), the final equation of motion only involves parameters that are already needed to describe the motion of a charged test particle. To the extent that (1) can be trusted, this means that internal-external split remains useful in electromagnetism. The center of mass motion of an extended charge distribution can be treated as though it were the trajectory of a pointlike test charge parameterized only by its position, $\hat{m}$, and q. This charge moves in an effective electric field given by the external one plus $\frac{2}{3}q\mathrm{d}\vec{a}/\mathrm{d}t$ (at the particle's location). This field may be shown to be a restriction of one that satisfies the vacuum Maxwell equations [4, 7, 8, 9].

This result can be generalized considerably. For essentially any bounded charge-current distribution in flat spacetime, linear and angular momenta may be defined that evolve as though they were the momenta of an extended test charge (or a pointlike test charge "with structure") moving in a certain effective electromagnetic field [4]. This effective field satisfies the vacuum Maxwell equations near the charge. All effects of the self-force and self-torque are non-perturbatively absorbed into the definitions of the momenta and the effective field. Whether or not the internal structure is "effaced" therefore reduces to a question regarding the nature of the effective field. In all but the most extreme systems, this may be shown to depend only on bulk parameters like the total charge. Very similar results also hold in generic (but fixed) curved spacetimes. The only difference is that the quadrupole and higher multipole of the body's stress-energy tensor must be renormalized along with its momenta [10]. Analogous statements are known for matter interacting with linear scalar fields as well [10, 11].

‡ This equation has been established as a valid approximation only for the *acceleration* of a physical charge (see, e.g., [3, 4]). This does not mean that a trajectory with an acceleration satisfying (1) for all time is guaranteed to stay near the physical trajectory. Many such motions violate the conditions under which the equation was derived (even on short timescales), and must therefore be discarded. Additionally, there may be neglected terms which lead to qualitatively different behavior over long times. Better-behaved equations arise by "reducing order" [3, 5], which changes (1) only by an amount comparable to the error terms that are already present. This leads to an equation often attributed to Landau and Lifshitz in the relativistic case [6].

§ There is some subtlety in defining precisely what is meant by mass and center of mass in (1), but the details are not important here. See, e.g., [4].

Results of this type greatly expand the scope of – and provide a basis for – what has been referred to as the Detweiler-Whiting axiom [8, 12]. It is well-known that point particles are incompatible with, e.g., the standard formulation of Maxwell electrodynamics (and with general relativity [13]). Despite this, "point particle methods" can still be used if additional axioms are introduced into the theory. Suppose, for example, that a certain portion of the self-field associated with a point particle is assumed not to affect its motion. Detweiler and Whiting considered this possibility with a field sourced from a certain symmetric Green function [8]. Assuming that this does not affect the motion leaves a finite vacuum field at the location of the particle. This is easily calculated. Substituting it into the Lorentz force equation produces the standard Dewitt-Brehme result [12, 14] for the motion of a self-interacting charged particle in curved spacetime. Similar subtractions were also used to efficiently reproduce equations of motion that had previously been derived for self-interacting scalar charges as well as uncharged masses in general relativity.

The results of [4, 11] show that the ability to ignore what is referred to as the Detweiler-Whiting S-field can be *derived* from first principles for a very wide class of extended scalar and electromagnetic charge distributions. This paper uses similar methods to treat the gravitational problem. Specifically, it investigates whether the bulk dynamics of an uncharged mass in general relativity can be reduced to test body motion in an effective field (in a nontrivial sense‖). Related questions have been studied in various contexts using the post-Newtonian approximation [1, 15, 16, 17], where they are often referred to as "effacement principles" or demonstrations of the strong equivalence principle.

The work presented here is more concerned with the types of systems typically encountered in discussions of the gravitational self-force. These allow the body of interest to move at relativistic speeds in a strongly curved background spacetime, but restrict it to be small compared to all scales associated with that background. One also assumes that the internal structure of the body does not vary too rapidly. Under these conditions – made precise in [18, 19] – an equation of motion may be derived that does not depend on any details of the body's internal structure. At lowest order, it is just the geodesic equation associated with the background spacetime. The next approximation introduces forces due to both gravitational self-interaction and spin. The latter effect is the Papapetrou force long known to act on spinning test particles. The self-force component is typically referred to as the MiSaTaQuWa force after the authors who originally obtained it: Mino, Sasaki, Tanaka, Quinn, and Wald [20, 21]. Neglecting the Papapetrou term, the motion may be viewed as geodesic with respect to a certain effective metric [8].

We show that this is a special case of a much more general result. Certain definitions of linear and angular momenta are proposed for extended (but bounded) stress-energy distributions in general relativity. These are shown to satisfy the same evolution equations as the momenta of an extended test mass moving in a particular effective metric. A similar result also holds for a certain definition of the center of mass. The effective metric is computed by using the Detweiler-Whiting prescription to subtract an S-field from the physical metric. In the monopole approximation, equations for the center of mass reduce to the geodesic equation in the effective metric. The MiSaTaQuWa equation follows immediately if the physical metric is assumed to be

‖ Suppose that it is known, for example, that the acceleration of an electric charge satisfies $m\vec{a} = q\vec{E} + \vec{f}$. This is trivially equivalent to the motion of a test body in the field $\vec{E} + q^{-1}\vec{f}$. It is only in special cases, however, that such an identification is useful.

the retarded field. To lowest order, the spin vector is found to be parallel-propagated with respect to the effective metric.

These results apply to a spatially-compact uncharged mass moving through a spacetime containing no other matter (except perhaps dark energy). The presence of a nonzero cosmological constant adds no significant complication, so this is allowed. Under these conditions, an equation is derived that provides a sense in which the gravitational "Detweiler-Whiting axiom" is satisfied exactly. This result is, however, applied only in a regime where the metric inside the body can be adequately approximated by a solution to the linearized Einstein equation (at least for short times). This linearization is carried out about an arbitrary vacuum solution.

The assumptions adopted here are different from those found in other treatments of the gravitational self-force. Most importantly, the approaches of, e.g., [18, 19] require that the metric perturbation be small only in some intermediate vacuum region outside of the body of interest. They can therefore be applied even to the motion of black holes. This lies beyond the scope of the formalism developed here. Nevertheless, there are considerable benefits to our assumptions. They allow the analysis of objects that may be highly distorted and dynamical: Dixon's multipole expansions [2, 22] for the motion of extended test masses are generalized to all orders. Explicit formulae for the momenta are also provided in terms of the body's internal structure. There are aesthetic advantages as well. In regimes where they overlap, the method presented here requires far less computation than others in the literature. It also provides significantly more physical insight.

This paper starts by discussing Dixon's formalism [2, 22, 23] as it pertains to the motion of extended test masses. Although this is not new, many subsequent arguments require familiarity with these results. Next, Sect. 3 discusses various properties of gravitational Green functions in general relativity. Sect. 4 introduces the notion of a Detweiler-Whiting S-field for an extended mass and briefly discusses the evolution of the "bare momentum." The central results of this paper are contained in Sect. 5. Evolution equations for the momenta are first analyzed perturbatively and shown to have a multipole expansion like that arising in the theory of extended test bodies. This is used as a guide to derive a generalization that does not require linearizing Einstein's equation. Sect. 6 develops explicit multipole expansions for the force and torque. It also defines a center of mass and relates it to the momenta. The gravitational self-force and self-torque are obtained in a simple case. Appendix A provides a discussion of Einstein's equation in wave gauge. Appendix B defines certain "generalized Killing fields" used in the text.

The sign conventions used here are those of Wald [24]. Metrics therefore have signature $+1$ and the Riemann tensor satisfies $2\nabla_{[a}\nabla_{b]}\omega_c = R_{abc}{}^{d}\omega_d$ for any 1-form ω_a. The Ricci tensor is given by $R_{ab} = R_{acb}{}^{c}$. Multiple metrics are discussed in this paper, so indices are not raised and lowered unless indicated otherwise. In almost all cases, factors of the appropriate metric are displayed explicitly. There are four metrics that are used here: g_{ab} denotes the full physical metric, $\bar{g}_{ab}$ the background metric, and $\hat{g}_{ab}$ a certain effective metric, and $\tilde{g}_{ab}$ an alternative effective metric. Derivative operators and curvature tensors associated with the latter three geometries are distinguished with a bar, hat, or tilde as appropriate. Non-geometric quantities (like momenta) with hats or tildes denote renormalized or effective versions of their plainer counterparts. Abstract indices are written using the Latin alphabet, while Greek indices represent coordinate components. Units are used where $G = c = 1$.

2. Motion of extended test masses

The main goal of this paper is to describe, in some sense, the large-scale or bulk motion of extended masses in general relativity. This is done by analyzing quantities that may be interpreted as a body's net linear and angular momenta. Closely related to these is the notion of a center of mass.

The type of momentum considered here is similar to the one developed by Dixon [2, 22, 25]. Mathematically, Dixon's momenta are tensor fields defined non-perturbatively along a preferred worldline in the physical spacetime. They take as input this worldline and a timelike vector field prescribed along it. The linear or angular momentum of an extended body are computed by integrating its stress-energy tensor over a spacelike hypersurface in a particular way. The evolution of these quantities is strongly constrained by stress-energy conservation.

The only significant restriction to the use of Dixon's momenta is that the object's stress-energy tensor be bounded in spatial directions (and that this bound is not "extremely large" in a certain sense [22, 26]). Limitations on the metric are minimal. Despite this, most applications have been restricted to the test body regime (e.g., [27, 28]). While Dixon's momenta retain a number of interesting properties in a more a general context [2, 22, 29, 30], other characteristics are less satisfactory. For example, it has been shown that even in flat spacetime electromagnetism, the momenta do not behave as simply as might have been expected once self-interaction is taken into account [9]. This problem can be eliminated with a simple modification [4].

Similar changes are proposed here in order to obtain physically reasonable momenta that obey simple evolution equations in the presence of significant gravitational self-interaction (but without electromagnetic or other long-range non-gravitational fields). The basic strategy is to first postulate "bare" momenta that agree with Dixon's definitions in the test mass regime. More generally, there will be differences. The important point is that the evolution equations for the bare momenta include total time derivatives of certain terms involving part of the self-field. These derivatives are easily eliminated by redefining the momenta. The resulting variables obey simple evolution equations in a wide variety of contexts.

Before demonstrating this, it is important to review the theory of extended test mass motion in a curved spacetime $(\mathcal{M}, g_{ab})$. Once a certain result has been established – namely (65) below – many aspects of this formalism will be seen to carry through almost without modification to cases involving self-interaction.

Suppose that the body of interest is associated with a stress-energy tensor $T^{ab}_{\mathrm{B}} = T^{(ab)}_{\mathrm{B}}$ and that its worldtube $W := \mathrm{supp}\ T^{ab}_{\mathrm{B}}$ is spatially compact. Now consider the scalar functional

$$P_\xi(\Sigma) := \int_\Sigma g_{ab} \xi^a T^{bc}_{\mathrm{B}} \mathrm{d}S_c. \tag{3}$$

This takes as input a hypersurface Σ assumed to bisect W and a vector field ξ^a that will be chosen later. $P_\xi(\Sigma)$ may be viewed as returning the component of momentum "conjugate" to ξ^a at a time defined by Σ. This interpretation is completely standard if ξ^a is a Killing vector: $P_\xi(\Sigma)$ is then conserved – i.e., independent of Σ – as long as $\nabla_a T^{ab}_{\mathrm{B}} = 0$.

In general, $P_\xi(\Sigma)$ will depend on the chosen hypersurface. Suppose that Σ_1 and Σ_2 are two non-intersecting hypersurfaces that completely cut through W. Assuming that

$$\nabla_a T^{ab}_{\mathrm{B}} = 0, \tag{4}$$

the difference in "ξ-momentum" at the two corresponding times is then

$$\delta P_\xi(\Sigma_1, \Sigma_2) := P_\xi(\Sigma_2) - P_\xi(\Sigma_1) = \frac{1}{2} \int_{\Omega(\Sigma_1, \Sigma_2)} T_{\mathrm{B}}^{ab} \mathcal{L}_\xi g_{ab} \mathrm{d}V. \tag{5}$$

Here, $\Omega(\Sigma_1, \Sigma_2)$ is defined to be the portion of W bounded by Σ_1 and Σ_2. $\mathrm{d}V$ denotes the natural volume element associated with g_{ab} and $\mathcal{L}_\xi$ the Lie derivative with respect to ξ^a.

Eq. (5) does not depend on any particular details of ξ^a or Σ. It describes a property shared by all functionals with the form (3). We now specialize to specific definitions that can be used to recover Dixon's definitions for the linear and angular momentum of an extended body. Using the terminology of [23], ξ^a will be assumed to be of the form¶

$$\xi^a = g^{ab} \Xi_b, \tag{6}$$

where Ξ_a is a Killing-type generalized affine collineation constructed using g_{ab}. This is defined precisely in Appendix B. Following [4, 10, 11], we will refer to the Ξ_a (or ξ^a) simply as generalized Killing fields (GKFs) with respect to g_{ab}.

Defining GKFs requires fixing not only a metric, but also a timelike worldline $\Gamma = \{\gamma_s | s \in \mathbb{R}\}$ and a timelike vector field $n_s^a \in T_{\gamma_s}\mathcal{M}$ along it. The worldline serves as an origin about which to compute multipole moments of T_{B}^{ab}. The n_s^a fix a family of spacelike hypersurfaces Σ_s that provide a time function $\Sigma_s \ni x \mapsto s$ inside the body's worldtube W. At fixed s, Σ_s is defined as the union of all geodesics that pass through γ_s and are orthogonal to n_s^a at that point. These geodesics are not to be extended so far that they intersect either with each other (except at γ_s) or with a neighboring hypersurface in the family. It is assumed that the body is sufficiently small that such restricted geodesics still form hypersurfaces Σ_s that foliate W. Under mild assumptions, γ_s and n_s^a can both be specified uniquely using center of mass conditions [26] (see also (23) and (24) below). For now, however, we continue to describe the general case where they are left free.

Once g_{ab}, Γ, and n_s^a are given, the allowed covector fields Ξ_a are fixed using the definition in Appendix B. They have a number of characteristics that are very similar to those of genuine Killing fields. First among these is the "rigidity property:" Given $\Xi_a(\gamma_s)$ and $\nabla_a \Xi_b(\gamma_s)$ at any single point $\gamma_s \in \Gamma$, $\Xi_a(x)$ is fixed for all x in the set $\mathcal{W} \supset W$ defined in Appendix B. The $\Xi_a(x)$ depend linearly on these "initial data." They are also "approximately Killing" near Γ, meaning that

$$\nabla_{(a} \Xi_{b)}|_\Gamma = \nabla_a \nabla_{(b} \Xi_{c)}|_\Gamma = 0, \tag{7}$$

or equivalently,

$$\mathcal{L}_\xi g_{ab}|_\Gamma = \nabla_a \mathcal{L}_\xi g_{bc}|_\Gamma = 0. \tag{8}$$

It follows from these statements that the generalized Killing fields form a ten-dimensional vector space in four spacetime dimensions (with fixed g_{ab}, Γ and n_s^a). This space includes any genuine Killing fields associated with g_{ab}: If ψ^a satisfies $\mathcal{L}_\psi g_{ab} = 0$ everywhere, it is also a generalized Killing field. In maximally-symmetric spacetimes, the space of generalized Killing fields coincides with the space of genuine

¶ The simpler notation $\xi^a = g^{ab}\xi_b$ is not used in order to avoid confusion when multiple metrics are introduced below.

Killing fields. The dependence on a preferred worldline and foliation disappears in this special case.

It is useful at this point to introduce linear and angular momenta $p^a(s)$ and $S^{ab} = S^{[ab]}(s)$ as tensor fields along Γ. It has already been noted that $\Xi_a(x)$ depends linearly on, say, $\Xi_a(\gamma_s)$ and $\nabla_a \Xi_b(\gamma_s) = \nabla_{[a}\Xi_{b]}(\gamma_s)$. It follows from (3) that $P_\xi(\Sigma_s)$ can be written as a linear combination of these parameters. The coefficients may be identified as momenta:

$$P_\xi(\Sigma_s) = p^a(s)\Xi_a(\gamma_s) + \frac{1}{2}S^{ab}(s)\nabla_a\Xi_b(\gamma_s). \tag{9}$$

If P_ξ is known for all possible Ξ_a, this equation determines p^a and S^{ab} completely. More explicitly [2, 22, 23, 25],

$$p^a(s) = \int_{\Sigma_s} K^a{}_{a'}(\gamma_s, x') T_{\mathrm{B}}^{a'b'}(x') \mathrm{d}S_{b'}, \tag{10}$$

and

$$S^{ab}(s) = 2\int_{\Sigma_s} g_{a'c'}(x') H^{[a|a'}(\gamma_s, x') g^{b]c}(\gamma_s) \nabla_c \sigma(\gamma_s, x') T_{\mathrm{B}}^{b'c'}(x') \mathrm{d}S_{b'}. \tag{11}$$

Here, $\sigma(x, x') = \sigma(x', x)$ is Synge's world function, $H^{aa'} := [-\nabla_a \nabla_{a'}\sigma]^{-1}$, and $K^a{}_{a'} := H^{ab'}\nabla_{a'}\nabla_{b'}\sigma$. These momenta coincide with standard textbook definitions in flat spacetime. In general, however, they are the momenta identified by Dixon as being particularly useful for the description of objects with conserved stress-energy tensors [2, 22, 25]. Various properties of Synge's function and related bitensors may be found in [12, 31, 32].

Differentiating (9) with respect to s while using (8) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}P_\xi = \left(\frac{\mathrm{D}p^a}{\mathrm{d}s} - \frac{1}{2}R_{bcd}{}^a S^{bc}\dot{\gamma}_s^d\right)\Xi_a + \frac{1}{2}\left(\frac{\mathrm{D}S^{ab}}{\mathrm{d}s} - 2p^{[a}\dot{\gamma}_s^{b]}\right)\nabla_a\Xi_b. \tag{12}$$

As is standard, the notation $\dot{\gamma}_s^a$ denotes the tangent vector to the curve γ_s. This equation provides a recipe for extracting $\mathrm{D}p^a/\mathrm{d}s$ and $\mathrm{D}S^{ab}/\mathrm{d}s$ from knowledge of $\mathrm{d}P_\xi(\Sigma_s)/\mathrm{d}s$ for all ξ^a. The differential form for changes in P_ξ follows immediately from (5):

$$\frac{\mathrm{d}}{\mathrm{d}s}P_\xi(\Sigma_s) = \frac{1}{2}\int_{\Sigma_s} T_{\mathrm{B}}^{ab}\mathcal{L}_\xi g_{ab}\mathrm{d}S. \tag{13}$$

Here, $\mathrm{d}S := t^a\mathrm{d}S_a$, where t^a is a time evolution vector field for the foliation $\{\Sigma_s\}$. Equating the right-hand side of (12) with the right-hand side of (13) provides evolution equations for p^a and S^{ab}.

Note that if $\mathrm{d}P_\xi/\mathrm{d}s = 0$, one recovers the Papapetrou equations typically used to model a spinning test particle. More generally, changes in P_ξ measure the deviation from these equations. In this formalism, Papapetrou terms in the laws of motion arise purely as a kinematic consequence of adopting (8) and (9).

The discussion up to this point has not made any strong assumptions regarding the nature of the metric. In particular, self-fields have not been excluded. We now assume, however, that in a Riemann normal coordinate system with origin γ_s, the metric components $g_{\mu\nu}$ may be accurately expanded throughout $\Sigma_s \cap W$ in a Taylor series about γ_s. This leads to a "multipole expansion" for $\mathrm{d}P_\xi(\Sigma_s)/\mathrm{d}s$ that is typically useful only in a test body limit. Using certain details of the GKFs, it may be shown that the result has the coordinate-independent form+ [10]

$$\frac{\mathrm{d}}{\mathrm{d}s} P_\xi(\Sigma_s) = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n!} I^{c_1 \cdots c_n ab}(s) \mathcal{L}_\xi g_{ab,c_1 \cdots c_n}(\gamma_s). \tag{14}$$

The coefficients $I^{c_1 \cdots c_n ab}(s)$ are interpreted as the 2^n-pole moments of T^{ab} at time s. $g_{ab,c_1 \cdots c_n}(\gamma_s)$ is the $n^{\rm th}$ tensor extension of g_{ab} at γ_s. Tensor extensions of the metric are also referred to as metric normal tensors. These objects require some explanation.

Briefly, $g_{ab,c_1 \cdots c_n}(x)$ is defined to be the (unique) tensor field satisfying $g_{\mu\nu,\lambda_1 \cdots \lambda_n}(x) = \partial_{\lambda_1} \cdots \partial_{\lambda_n} g_{\mu\nu}(x)$ in a Riemann normal coordinate system with origin x. The zeroth extension is the metric itself and the first extension vanishes. In general, it is clear that the $n^{\rm th}$ metric normal tensor is symmetric in both its first two and its last n indices. It may also be shown that [10]

$$g_{a(b,c_1 \cdots c_n)} = g_{(ab,c_1 \cdots c_{n-1})c_n} = 0 \tag{15}$$

for all $n \geq 2$. Keeping this restriction on n, all metric normal tensors can be written as polynomials in the Riemann tensor. To linear order [22],

$$g_{ab,c_1 \cdots c_n} = 2 \left(\frac{n-1}{n+1} \right) \nabla_{(c_3 \cdots c_n} (R_{|a|c_1 c_2)}{}^d g_{bd}) + O(R^2). \tag{16}$$

This equation is exact for $n = 2, 3$. For higher n, there will be additional terms that are nonlinear in $R_{abc}{}^d$ or its derivatives.

Without loss of generality, the symmetry properties of the metric normal tensors allow $I^{c_1 \cdots c_n ab}$ to be chosen such that it is separately symmetric in its first n and last two indices. It may also be taken to satisfy

$$I^{(c_1 \cdots c_n a)b} = I^{c_1 (c_2 \cdots c_n ab)} = 0. \tag{17}$$

A unique formula linking moments with these properties to $T^{ab}_{\rm B}$ may be derived using (13) and (14) [10] (see also [22]). Like (10) and (11), the result has the form of an integral over Σ_s that involves the stress-energy tensor and various bitensors constructed from σ.

The given index symmetries imply that $I^{c_1 \cdots c_n ab}$ has a total of

$$(n+3)(n+2)(n-1) \tag{18}$$

algebraically independent components. This far exceeds the number typically ascribed to multipole moments in other formalisms [33]. The reason for this is essentially that the $I^{\cdots}$ are "complete." Knowing all of them together with p^a and S^{ab} is completely equivalent to knowledge of $T^{ab}_{\rm B}$ [22]. Additionally, (14) does not make any use of Einstein's equation other than in assuming stress-energy conservation. If g_{ab} – now interpreted as a background metric – satisfies $R_{ab} = 0$, many components of the $I^{\cdots}$ decouple from the laws of motion. This may be seen by noting that certain traces of (16) vanish in this case. Use of (8) shows that these same traces still vanish if $R_{ab} = \Lambda g_{ab}$ for any constant Λ. Additional discussion of these points may be found in [10].

It is important to note that the sum in (14) starts at $n = 2$. This corresponds to quadrupole order. It is a consequence of (8) and (13) that the monopole and dipole

+ Unless the components $g_{\mu\nu}$ are analytic, this series can only be expected to be asymptotic. This means that it should be truncated at finite n. An infinite upper limit is written here for simplicity (and in similar sums below).

moments of $T_{\rm B}^{ab}$ – essentially p^a and S^{ab} – do not directly contribute to $\mathrm{d}P_\xi/\mathrm{d}s$. These quantities *do* enter the laws of motion for $\mathrm{D}p^a/\mathrm{d}s$ and $\mathrm{D}S^{ab}/\mathrm{d}s$, but only due to the Papapetrou-like terms appearing in (12). Explicitly, define a net force $F^a(s)$ and a net torque $N^{ab} = N^{[ab]}(s)$ such that

$$\frac{\mathrm{D}p^a}{\mathrm{d}s} = \frac{1}{2} R_{bcd}{}^a S^{bc} \dot{\gamma}_s^d + F^a, \tag{19}$$

$$\frac{\mathrm{D}S^{ab}}{\mathrm{d}s} = 2p^{[a}\dot{\gamma}_s^{b]} + N^{ab}. \tag{20}$$

Comparison with (9) and (14) shows that

$$F^a(s) = \frac{1}{2} g^{ab}(\gamma_s) \sum_{n=2}^{\infty} \frac{1}{n!} I^{f_1 \cdots f_n cd}(s) \nabla_b g_{cd, f_1 \cdots f_n}(\gamma_s), \tag{21}$$

and

$$N^{ab}(s) = 2 \sum_{n=2}^{\infty} \frac{1}{n!} I^{c_1 \cdots c_n df}(s) \Big[g_{fh, c_1 \cdots c_n}(\gamma_s) \delta^{[a}_d + \frac{n}{2} g_{df, hc_1 \cdots c_{n-1}}(\gamma_s) \delta^{[a}_{c_n} \Big] g^{b]h}(\gamma_s). \tag{22}$$

The hope in writing these series is, of course, that adequate approximations may be obtained by truncating them at some small maximum n. This can only happen if Γ and $\{\Sigma_s\}$ are chosen appropriately (if it is possible at all for a given system). We now fix a particular worldline and foliation that is hopefully "appropriate" in this sense. This is done by imposing center of mass conditions as described in, e.g., [2, 25, 34]. First recall that Σ_s is constructed using geodesics that pass through γ_s and are orthogonal to n_s^a at that point. Now suppose that Γ and n_s^a are chosen such that

$$p^a(s) \propto n_s^a, \tag{23}$$

$$g_{ab}(\gamma_s) p^a(s) S^{bc}(s) = 0. \tag{24}$$

Under mild assumptions, the resulting Γ and n_s^a exist, are unique, and are timelike [26].

The center of mass velocity $\dot{\gamma}_s^a$ is not necessarily proportional to p^a. Relating these two quantities is simpler if the time parameter s is chosen such that $g_{ab}(\gamma_s)p^a(s)\dot{\gamma}_s^b = -m(s)$, where the mass $m(s)$ is given by

$$m := \sqrt{-g_{ab} p^a p^b}. \tag{25}$$

This means that in general, $\dot{\gamma}_s^a$ does not have unit norm. There is, however, no loss of generality in assuming that $g_{ab} n_s^a n_s^b = -1$. Hence,

$$p^a = m n_s^a. \tag{26}$$

With these conventions, (24) may be used to show that [34]

$$m \dot{\gamma}_s^a = p^a - N^{ab} g_{bc} n_s^c - \frac{S^{ab}[m g_{bc} F^c - \frac{1}{2} S^{cd}(p^f - N^{fh} g_{hr} n_s^r) R_{cdb}{}^l g_{fl}]}{m^2 + \frac{1}{4} S^{bc} S^{df} R_{bcd}{}^l g_{fl}}. \tag{27}$$

The complexity of this equation arises from the fact that $\dot{\gamma}_s^a$ has been completely eliminated from the right-hand side. This makes it clear that if the $I^{\cdots}(s)$ are known, (19)-(22) and (27) form a closed system of ordinary differential equations (ODEs) for γ_s, p^a, and S^{ab}. There is, however, no physical reason that the higher moments should be treated as given functions of s. Generically, their evolution depends on the details of the specific system under consideration.

To summarize, Dixon's momenta have now been described as they apply to a compact body with a conserved stress-energy tensor. The most important definitions are (3) for P_ξ, (9) for p^a and S^{ab}, and the specification of the generalized Killing fields described in Appendix B. Taken together, these are Dixon's momenta in a very general context. It is only in providing the multipole expansion (14) for $\mathrm{d}P_\xi/\mathrm{d}s$ that a significant assumption has been made regarding the metric. Roughly speaking, this series is useful only if the metric does not vary rapidly inside any cross-section $\Sigma_s \cap W$. This is a reasonable assumption for a test mass. It is not for cases involving significant self-gravity. A direct Taylor expansion of (12) fails to provide any simplification in general. Nevertheless, it is shown below that there *is* a sense in which useful multipole expansions can be performed even in the presence of a significant self-field.

A bare momentum for a self-gravitating mass

We are now in a position to postulate a bare momentum for self-gravitating matter distributions in general relativity. As mentioned above, this will be required to reduce to the definitions just provided in the test mass limit. More than this, the form (3) for $P_\xi(\Sigma)$ will be retained exactly as written. The volume element in the integral will be the one associated with the full physical metric g_{ab}. The vector fields ξ^a will, however, be left as undefined for now. Similarly, no properties of the hypersurfaces Σ_s or worldline Γ will be assumed at this point (other than that $\{\Sigma_s\}$ foliates the body's worldtube W). The form of $\mathrm{d}P_\xi/\mathrm{d}s$ derived below will suggest natural definitions for these quantities that lead to laws of motion essentially identical to those in the test mass regime.

We continue to assume that the body's stress-energy tensor is conserved. This is an immediate consequence of Einstein's equation in the absence of any other nearby matter (or non-gravitational fields). It follows that changes in P_ξ are given by (5). The metric will usually vary rapidly inside the body, so it is not useful to expand the right-hand side of this equation in a multipole series like (14). We instead analyze the behavior of g_{ab} assuming that it can be considered close to some known background metric $\bar{g}_{ab}$ that does not involve the body of interest. This first requires some discussion of perturbation theory in general relativity.

3. Perturbation theory and gravitational Green functions

Most of the new results presented in this paper may be obtained using stress-energy conservation in the physical spacetime together with the well-known linearization of Einstein's equation in Lorenz gauge. There are, however, several reasons to postpone introducing approximations for as long as possible. This helps to clarify the origin and validity of various results, and might also simplify attempts to generalize them. Additionally, the fully linearized ("non-relaxed") Einstein equation is not an interesting theory of gravity. It implies that the stress-energy tensor must be conserved with respect to the background spacetime. Matter therefore decouples from the metric perturbations. See, e.g., [18] and [35] for different perspectives on this problem. We avoid issues of this type by first obtaining exact expressions for the force and torque acting on an extended body. This results in integrals that can be approximated without significant difficulty.

Simplifications that are eventually applied require that the physical spacetime $(\mathcal{M}, g_{ab})$ be "close" to a known background. Physically, it is not important what happens in the distant past or future, nor very far from the body of interest. Indeed, it is unlikely that ordinary perturbation theory can be successfully applied in large regions. A correspondence between the physical and background spacetimes will therefore be assumed only on a submanifold $M \subseteq \mathcal{M}$. The spacetime $(M, g_{ab}|_M)$ is then to be compared with $(M, \bar{g}_{ab})$, where $\bar{g}_{ab}$ is a known solution of Einstein's equation. It is convenient to describe the difference between the two metrics using the variable $H^{ab} = H^{(ab)}$ defined by

$$H^{ab} := \bar{g}^{ab} - \left(\frac{\sqrt{-g}}{\sqrt{-\bar{g}}} \right) g^{ab}. \tag{28}$$

This field reduces to the trace-reversed metric perturbation when the full metric g_{ab} approaches $\bar{g}_{ab}$. It is sometimes written as $h^{\mu\nu}$ in post-Newtonian contexts where $\bar{g}^{ab} \to \eta^{\mu\nu} := \mathrm{diag}(-1,1,1,1)$. The notation $\sqrt{-g}/\sqrt{-\bar{g}}$ is covariantly defined as the proportionality factor linking the volume elements associated with g_{ab} and $\bar{g}_{ab}$. In coordinates, it is $\sqrt{-\det g_{\mu\nu}}/\sqrt{-\det \bar{g}_{\mu\nu}}$.

Appendix A recasts Einstein's equation (with a possible cosmological constant Λ)

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi g_{ac} g_{bd} T^{cd}, \tag{29}$$

into an equation for H^{ab}. This is done in a wave gauge [36], which generalizes both the Lorenz and harmonic gauges commonly used for various types of perturbation theory in general relativity. Details of this choice are described in Appendix A. It implies that $\bar{\nabla}_a H^{ab} = 0$, where $\bar{\nabla}_a$ is the derivative operator associated with the background metric. If g_{ab} and $\bar{g}_{ab}$ satisfy (29) with respective stress-energy tensors T^{ab} and $\bar{T}^{ab}$, H^{ab} must be a solution to the relaxed Einstein equation (A.12). Linearizing about $H^{ab} = 0$ in the case $\bar{R}_{ab} = \Lambda \bar{g}_{ab}$,

$$\bar{\Box} H^{ab} + 2 \bar{g}^{c(a} \bar{R}_{dcf}{}^{b)} H^{df} = -16\pi T^{ab} + O(H^2). \tag{30}$$

This is the familiar equation satisfied by trace-reversed metric perturbations in Lorenz gauge (typically derived only in the case $\Lambda = 0$; see, e.g., [24]). We eventually assume that $T^{ab} = T^{ab}_{\mathrm{B}}$.

It is useful to introduce a Green function $\bar{G}^{aba'b'}(x,x') = \bar{G}^{(ab)a'b'} = \bar{G}^{ab(a'b')}$ for (30). Let it satisfy

$$\bar{\Box} \bar{G}^{aba'b'} + 2\bar{g}^{c(a} \bar{R}_{dcf}{}^{b)} \bar{G}^{dfa'b'} = -4\pi \bar{g}^{a'c'} \bar{g}^{b'd'} \bar{g}^{(a}{}_{c'} \bar{g}^{b)}{}_{d'} \bar{\delta}(x,x'). \tag{31}$$

Here, $\bar{g}^{a}{}_{a'}(x,x')$ is the parallel propagator associated with the background spacetime. If $\mathrm{d}\bar{V}$ is used to denote the volume element associated with $\bar{g}_{ab}$, the distribution $\bar{\delta}(x,x')$ satisfies

$$\int_M \bar{\delta}(x,x') f(x') \mathrm{d}\bar{V}' = f(x) \tag{32}$$

for any test function $f(x)$ and any $x \in M$. It is the Dirac measure associated with the background.

Some Green functions satisfying (31) are also solutions to the adjoint equation

$$\bar{\Box}' \bar{G}^{aba'b'} + 2\bar{g}^{c'(a'} \bar{R}_{d'c'f'}{}^{b')} \bar{G}^{abd'f'} = -4\pi \bar{g}^{a'c'} \bar{g}^{b'd'} \bar{g}^{(a}{}_{c'} \bar{g}^{b)}{}_{d'} \bar{\delta}(x,x'). \tag{33}$$

It follows from the well-known reciprocity relation $\bar{G}^{aba'b'}_{\mathrm{ret}}(x,x') = \bar{G}^{a'b'ab}_{\mathrm{adv}}(x',x)$ that the advanced and retarded Green functions both have this property. These are not the only possibilities, however. For any $\bar{G}^{aba'b'}_{*}$ that is a solution to both (31) and (33), the linearized Einstein equation (30) may be rewritten entirely using volume and surface integrals linear in $\bar{G}_*^{aba'b'}$. A similar procedure may also be carried out for the full Einstein equation (A.12). Choosing a closed spacetime region $R \subset M$ and a point $x \in R$, this results in (assuming again that $\bar{R}_{ab} = \Lambda \bar{g}_{ab}$)

$$H^{ab}(x) = 4 \int_R \bar{g}_{a'c'} \bar{g}_{b'd'} \bar{G}_*^{aba'b'} [(g'/\bar{g}') T^{c'd'} + \tau^{c'd'}] \mathrm{d}\bar{V}' + \frac{1}{4\pi} \oint_{\partial R} \bar{g}_{a'd'} \bar{g}_{b'f'} \bar{g}^{c'h'} (\bar{G}_*^{aba'b'} \bar{\nabla}_{c'} H^{d'f'} - \bar{\nabla}_{c'} \bar{G}_*^{aba'b'} H^{d'f'}) \mathrm{d}\bar{S}_{h'}, \qquad (34)$$

where

$$\tau^{ab} := \frac{1}{16\pi} \Big\{ \lambda^{ab} - 2 H^{d(a} \bar{R}_{cdf}{}^{b)} H^{cf} - 2\Lambda \Big[(\sqrt{g/\bar{g}} + \frac{1}{2} \bar{g}_{cd} H^{cd} - 1)(\bar{g}^{ab} - H^{ab}) \Big] \Big\}, \quad (35)$$

and λ^{ab} is given by (A.13). Although this equation is exact, it is usually most useful when τ^{ab} can be ignored. In all cases, however, the surface integral on the second line of (34) is a homogeneous solution to the linearized (relaxed or gauge-reduced) Einstein equation (30).

As a word of caution, (34) has been derived assuming that the gauge condition is satisfied. This will only occur if $\nabla_a T^{ab} = 0$. Eq. (A.15) may therefore be replaced by (4) if $T^{ab} = T^{ab}_{\mathrm{B}}$. Metric perturbations computed via (A.12) or (34) using stress-energy tensors that are not conserved will result in a perturbed metric that fails to satisfy the full Einstein equation. This means, in particular, that $\bar{G}^{aba'b'}$ cannot be interpreted as an approximate solution to Einstein's equation with a pointlike source. Despite this, it remains a useful construction.

We now introduce the "singular" (or S-type) Detweiler-Whiting Green function $\bar{G}_{\mathrm{S}}^{aba'b'}(x,x')$ [8, 12]. This is defined at least in regions where its arguments can be connected by exactly one geodesic. It is symmetric, meaning that

$$\bar{G}_{\mathrm{S}}^{aba'b'}(x,x') = \bar{G}_{\mathrm{S}}^{a'b'ab}(x',x). \qquad (36)$$

It also satisfies (31) and (33), and is constrained to vanish whenever its arguments are timelike-separated. It may be shown that these properties define $\bar{G}_{\mathrm{S}}^{aba'b'}$ uniquely [12], at least in finite regions.

If x and x' are sufficiently close, it is known that $\bar{G}_{\mathrm{S}}^{aba'b'}(x,x')$ has the Hadamard form* [12]

$$\bar{G}_{\mathrm{S}}^{aba'b'} = \frac{1}{2} [\bar{g}^{ac} \bar{g}^{bd} \bar{g}^{(a'}{}_c \bar{g}^{b')}{}_d \bar{\Delta}^{1/2} \delta(\bar{\sigma}) - V^{aba'b'} \Theta(\bar{\sigma})]. \qquad (37)$$

Here, δ and Θ are the Dirac and Heaviside distributions, respectively. $\bar{\sigma}(x,x') = \bar{\sigma}(x',x)$ is the world function associated with the background, $\bar{\Delta}(x,x') = \bar{\Delta}(x',x)$ is the van Vleck determinant, and $\bar{V}^{aba'b'}(x,x') = \bar{V}^{a'b'ab}(x',x)$ is a certain homogeneous solution to (31). $\bar{\Delta}$ may be defined as the unique biscalar satisfying

$$\bar{g}^{ab} \bar{\nabla}_a \bar{\sigma} \bar{\nabla}_b \ln \bar{\Delta} = 4 - \bar{g}^{ab} \bar{\nabla}_a \bar{\nabla}_b \bar{\sigma} \qquad (38)$$

and $\bar{\Delta}(x,x) = 1$. This equation can be viewed as an ODE along the geodesic connecting x and x'. In coordinates, the solution is

$$\bar{\Delta}(x,x') = -\frac{\det(\bar{\nabla}_\mu \bar{\nabla}_{\mu'} \bar{\sigma})}{\sqrt{-\bar{g}}\sqrt{-\bar{g}'}}. \qquad (39)$$

* Ref. [12] only derives this result in cases where $\bar{R}_{ab} = 0$. The derivation is easily extended to the general case with no change in the conclusion.

It will be important to note that the trace (with respect to $\bar{g}_{ab}$) of $\bar{G}_{\rm S}^{aba'b'}$ has a particularly simple form. Any solution to (31) satisfies

$$\bar{\Box}(\bar{g}_{ab}\bar{G}^{aba'b'}) + 2\Lambda(\bar{g}_{ab}\bar{G}^{aba'b'}) = -4\pi\bar{g}^{a'b'}\bar{\delta}(x,x'). \tag{40}$$

It follows that

$$\bar{g}_{ab}\bar{G}_{\rm S}^{aba'b'} = \bar{g}^{a'b'}\bar{G}_{\rm S}, \tag{41}$$

where $\bar{G}_{\rm S}(x,x')$ is the S-type Detweiler-Whiting Green function associated with a massive minimally-coupled scalar wave equation. $\bar{G}_{\rm S}$ satisfies

$$\bar{\Box}\bar{G}_{\rm S} + 2\Lambda\bar{G}_{\rm S} = -4\pi\bar{\delta}(x,x'), \tag{42}$$

and is symmetric in its two arguments. It also vanishes when those arguments are timelike-separated with respect to $\bar{g}_{ab}$.

4. Self-fields

We now return to discussing the motion of a body with stress-energy tensor $T_{\rm B}^{ab}$ contained in a worldtube $W := \text{supp}\, T_{\rm B}^{ab} \cap M$♯. Define its Detweiler-Whiting S-field by

$$H_{\rm S}^{ab} := 4\int_W \bar{G}_{\rm S}^{aba'b'}\bar{g}_{a'c'}\bar{g}_{b'd'}T_{\rm B}^{c'd'}\,{\rm d}\bar{V}'. \tag{43}$$

This may be interpreted as the "bound" component of the self-field in the linearized regime. Various arguments in the literature lead to expectations that at least in some limits, $H_{\rm S}^{ab}$ should not directly contribute to the body's motion (except, perhaps, through various renormalizations) [8, 12]. It is shown below that this is indeed the case in a very general context.

It is therefore interesting to consider the difference field $\hat{H}^{ab} := H^{ab} - H_{\rm S}^{ab}$. Using (34) and setting $T^{ab} = T_{\rm B}^{ab}$,

$$\begin{aligned}\hat{H}^{ab} = \frac{1}{4\pi}\int_{\partial W} \bar{g}_{a'd'}\bar{g}_{b'f'}\bar{g}^{c'h'}(\bar{G}_{\rm S}^{aba'b'}\bar{\nabla}_{c'}H^{d'f'} - \bar{\nabla}_{c'}\bar{G}_{\rm S}^{aba'b'}H^{d'f'}){\rm d}\bar{S}_{h'} \\ + 4\int_W \bar{g}_{a'c'}\bar{g}_{b'd'}\bar{G}_{\rm S}^{aba'b'}\{[(g'/\bar{g}') - 1]T_{\rm B}^{c'd'} + \tau^{c'd'}\}{\rm d}\bar{V}'.\end{aligned} \tag{44}$$

The surface integral in the first line is a vacuum solution to the linearized (and relaxed) Einstein equation. It is essentially a type of surface-averaged self-field in that case, and often varies much more slowly than $H_{\rm S}^{ab}$. This can be seen more clearly by noting that

$$\int_{\partial W} \bar{g}_{a'd'}\bar{g}_{b'f'}\bar{g}^{c'h'}(\bar{G}_{\rm S}^{aba'b'}\bar{\nabla}_{c'}H_{\rm S}^{d'f'} - \bar{\nabla}_{c'}\bar{G}_{\rm S}^{aba'b'}H_{\rm S}^{d'f'}){\rm d}\bar{S}_{h'} = 0. \tag{45}$$

Instances of H^{ab} in the first line of (44) may therefore be freely replaced with $\hat{H}^{ab}$.

Using these definitions in the "force law" (5) requires writing Lie derivatives of the metric in terms of the perturbation variable H^{ab}. Use of (28) shows that

$$\begin{aligned}\mathcal{L}_\xi g_{ab} = \left[\sqrt{\bar{g}/g}(g_{af}g_{bh} - \frac{1}{2}g_{ab}g_{fh})\bar{g}^{cf}\bar{g}^{dh} + \frac{1}{2}g_{ab}\bar{g}^{cd}\right]\mathcal{L}_\xi\bar{g}_{cd} \\ + \sqrt{\bar{g}/g}(g_{ac}g_{bd} - \frac{1}{2}g_{ab}g_{cd})\mathcal{L}_\xi H^{cd}.\end{aligned} \tag{46}$$

♯ The stress-energy tensor is technically a field on $\mathcal{M}$, not $M \subseteq \mathcal{M}$. Any effects associated with the boundary of M (if it exists) will be ignored. Except where noted explicitly, this is consistent as long as M is chosen appropriately and we do not consider the body's behavior very near ∂M.

Hence,

$$\delta P_\xi = \frac{1}{2} \int_\Omega T_B^{ab} \left[i^{cd}{}_{ab} \mathcal{L}_\xi \bar{g}_{cd} + (g_{ac} g_{bd} - \frac{1}{2} g_{ab} g_{cd}) \mathcal{L}_\xi H^{cd} \right] \mathrm{d}\bar{V}, \qquad (47)$$

where

$$i^{cd}{}_{ab} := (g/\bar{g}) \delta^c_a \delta^d_b + 2\sqrt{g/\bar{g}} (\delta^c_a g_{bf} H^{df} - \frac{1}{4} g_{ab} H^{cd}) + (g_{af} g_{bh} - \frac{1}{2} g_{ab} g_{fh}) H^{cf} H^{dh}. \qquad (48)$$

The term involving $\mathcal{L}_\xi \bar{g}_{ab}$ in (47) is of relatively little importance. Temporarily suppose, for the purposes of motivation, that the ξ^a are GKFs constructed using the background metric. Comparison with Sect. 2 – particularly Eq. (8) – then shows that any term involving $\mathcal{L}_\xi \bar{g}_{ab}$ can only produce quadrupole and higher corrections to the laws of motion. Formulae for these moments in terms of T_{B}^{ab} will be renormalized somewhat by the presence of $i^{cd}{}_{ab}$, but this is relatively simple to take into account. All "interesting" gravitational self-force effects are therefore contained in the term involving $\mathcal{L}_\xi H^{ab}$. This is the focus of further simplifications below.

5. Free-fall

Changes in the momenta of a self-gravitating mass falling freely in a vacuum background are determined by (47). This equation is exact. We now introduce an approximation for the first time by assuming that the metric perturbations are "small." H^{ab} and its derivatives – including T_{B}^{ab} – will only be retained to quadratic order in δP_ξ. Eq. (47) then reduces to

$$\delta P_\xi = \frac{1}{2} \int_\Omega \left\{ \mathcal{T}^{ab} \mathcal{L}_\xi \bar{g}_{ab} + T_{\mathrm{B}}^{ab} \mathcal{L}_\xi \left[(\bar{g}_{ac} \bar{g}_{bd} - \frac{1}{2} \bar{g}_{ab} \bar{g}_{cd}) H^{cd} \right] \right\} \mathrm{d}\bar{V} + O(H^3), \qquad (49)$$

where

$$\mathcal{T}^{ab} := (1 - \frac{1}{2} \bar{g}_{cd} H^{cd}) T_{\mathrm{B}}^{ab}. \qquad (50)$$

The error here has been abbreviated as "$O(H^3)$." More explicitly, neglected terms in the integrand of (49) have the schematic forms $T_{\mathrm{B}} H H \mathcal{L}_\xi \bar{g}$ and $T_{\mathrm{B}} H \mathcal{L}_\xi H$.

Now consider the effect of H_{S}^{ab} on δP_ξ by splitting the metric perturbation via $H^{ab} = \hat{H}^{ab} + H_{\mathrm{S}}^{ab}$. Substituting (43) into (49) results in an expression involving a term of the form

$$\int_\Omega \mathrm{d}\bar{V} \int_W \mathrm{d}\bar{V}' \mathcal{F}_\xi(x, x'), \qquad (51)$$

where the "force density" $\mathcal{F}_\xi(x, x')$ at point x due to point x' is

$$\mathcal{F}_\xi(x, x') := 2 T_{\mathrm{B}}^{ab} T_{\mathrm{B}}^{a'b'} \mathcal{L}_\xi^{(x)} \left[(\bar{g}_{ac} \bar{g}_{bd} - \frac{1}{2} \bar{g}_{ab} \bar{g}_{cd}) \bar{g}_{a'c'} \bar{g}_{b'd'} \bar{G}_{\mathrm{S}}^{cdc'd'} \right]. \qquad (52)$$

Here, $\mathcal{L}_\xi^{(x)}$ denotes a "partial Lie derivative" that varies x but not x'. For any $\mathcal{F}_\xi$, it is straightforward to show that (assuming all integrals converge strongly enough that they can be commuted at will)

$$\int_\Omega \mathrm{d}\bar{V} \int_W \mathrm{d}\bar{V} \mathcal{F}_\xi(x, x') = \frac{1}{2} \int_\Omega \mathrm{d}\bar{V} \int_W \mathrm{d}\bar{V}' \left[\mathcal{F}_\xi(x, x') + \mathcal{F}_\xi(x', x) \right] + \frac{1}{2} \int_\Omega \mathrm{d}\bar{V} \int_{W \setminus \Omega} \mathrm{d}\bar{V}' \left[\mathcal{F}_\xi(x, x') - \mathcal{F}_\xi(x', x) \right]. \qquad (53)$$

The first line of this equation may be physically interpreted as averaging the force on matter at x due to matter at x' and vice-versa. In this sense, it measures the failure of Newton's 3rd law. Using (36), (41), and (52),

$$\frac{1}{2}[\mathcal{F}_\xi(x,x')+\mathcal{F}_\xi(x',x)] = T_{\rm B}^{ab}T_{\rm B}^{a'b'}\mathcal{L}_\xi(\bar{g}_{ac}\bar{g}_{bd}\bar{g}_{a'c'}\bar{g}_{b'd'}\bar{G}_{\rm S}^{cdc'd'} - \frac{1}{2}\bar{g}_{ab}\bar{g}_{a'b'}\bar{G}_{\rm S}). \quad (54)$$

The Lie derivative appearing here is the ordinary one: It acts on both points appearing in its argument. One has, e.g., $\mathcal{L}_\xi \bar{G}_{\rm S}(x,x') = \xi^a(x)\nabla_a \bar{G}_{\rm S}(x,x') + \xi^{a'}(x')\nabla_{a'}\bar{G}_{\rm S}(x,x')$. If ξ^a is an exact Killing vector associated with $\bar{g}_{ab}$, Eq. (54) vanishes exactly. In many other cases of interest, it is very small.

The second line of (53) effectively renormalizes P_ξ. To see this, first note that

$$\frac{1}{2}\int_{\Omega(\Sigma_1,\Sigma_2)} \mathrm{d}\bar{V} \int_{W\backslash\Omega(\Sigma_1,\Sigma_2)} \mathrm{d}\bar{V}'\,[\mathcal{F}_\xi(x,x')-\mathcal{F}_\xi(x',x)] = \mathcal{E}_\xi(\Sigma_1)-\mathcal{E}_\xi(\Sigma_2), \quad (55)$$

where

$$\mathcal{E}_\xi(\Sigma) := \frac{1}{2}\int_{\Sigma^+}\mathrm{d}\bar{V}\int_{\Sigma^-}\mathrm{d}\bar{V}'\,[\mathcal{F}_\xi(x,x')-\mathcal{F}_\xi(x',x)]. \quad (56)$$

Any hypersurface Σ used in this equation is assumed, as usual, to bisect W. The (four-dimensional) portion of W in the future of Σ is denoted by Σ^+, while the portion in its past is denoted by Σ^-. An explicit formula for the shift $\mathcal{E}_\xi$ in the ξ-momentum is easily obtained by combining (52) and (56). Using the notation

$$\bar{h}^{\rm S}_{ab}[R] := 4(\bar{g}_{ac}\bar{g}_{bd} - \frac{1}{2}\bar{g}_{ab}\bar{g}_{cd})\int_R \bar{g}_{a'c'}\bar{g}_{b'd'}T_{\rm B}^{c'd'}\bar{G}_{\rm S}^{cda'b'}\mathrm{d}\bar{V}' \quad (57)$$

for any spacetime volume R,

$$\mathcal{E}_\xi(\Sigma) = \frac{1}{4}\left(\int_{\Sigma^+} T_{\rm B}^{ab}\mathcal{L}_\xi \bar{h}^{\rm S}_{ab}[\Sigma^-]\mathrm{d}\bar{V} - \int_{\Sigma^-} T_{\rm B}^{ab}\mathcal{L}_\xi \bar{h}^{\rm S}_{ab}[\Sigma^+]\mathrm{d}\bar{V}\right). \quad (58)$$

Despite appearances, this depends on the behavior of the body only in a *finite* four-dimensional spacetime volume around Σ. Recalling that $\bar{G}_{\rm S}^{aba'b'}(x,x')=0$ when x and x' are timelike-separated with respect to $\bar{g}_{ab}$, this is the portion of W that cannot be connected to every part of $\Sigma\cap W$ via timelike curves. In simple cases, it is essentially a ball with radius of order the body's light-crossing time. Further discussion may be found in Sect. 5.2 below.

Changes in P_ξ are computed by combining (53)-(55) with (49). Supposing, as above, that a 1-parameter family of spacelike hypersurfaces Σ_s are available that bisect W, the result may be written in differential (rather than difference) form:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[P_\xi(\Sigma_s)+\mathcal{E}_\xi(\Sigma_s)] = \frac{1}{2}\int_{\Sigma_s}\mathrm{d}\bar{S}\,\mathcal{T}^{ab}\Big\{\mathcal{L}_\xi\Big[\bar{g}_{ab} + (\bar{g}_{ac}\bar{g}_{bd} - \frac{1}{2}\bar{g}_{ab}\bar{g}_{cd})\hat{H}^{cd}\Big] \\ + 2\int_W \mathrm{d}\bar{V}'\,\mathcal{T}^{a'b'}\mathcal{L}_\xi\Big(\bar{g}_{ac}\bar{g}_{bd}\bar{g}_{a'c'}\bar{g}_{b'd'}\bar{G}_{\rm S}^{cdc'd'} - \frac{1}{2}\bar{g}_{ab}\bar{g}_{a'b'}\bar{G}_{\rm S}\Big)\Big\} + O(H^3). \quad (59)\end{aligned}$$

The field

$$\hat{g}_{ab} := \bar{g}_{cd} + (\bar{g}_{ac}\bar{g}_{bd} - \frac{1}{2}\bar{g}_{ab}\bar{g}_{cd})\hat{H}^{cd} \quad (60)$$

appearing in the first line of (59) is essentially the physical metric with the portion due to $H^{ab}_{\rm S}$ subtracted out. Recalling from (44) that

$$\hat{H}^{ab} = \frac{1}{4\pi}\int_{\partial W}\bar{g}_{a'd'}\bar{g}_{b'f'}\bar{g}^{c'h'}(\bar{G}_{\rm S}^{aba'b'}\bar{\nabla}_{c'}\hat{H}^{d'f'} - \bar{\nabla}_{c'}\bar{G}_{\rm S}^{aba'b'}\hat{H}^{d'f'})\mathrm{d}\bar{S}_{h'} + O(H^2), \quad (61)$$

it satisfies the vacuum Einstein equation (with a possible cosmological constant) up to terms of order H^2. We refer to $\hat{g}_{ab}$ as the effective metric.

Now note that $\bar{G}_{\mathrm{S}}^{aba'b'}$ is a purely geometric object associated with the background spacetime. This means that $\mathcal{L}_\xi \bar{G}_{\mathrm{S}}^{aba'b'}$ can always be written as a (nonlocal) linear functional of $\mathcal{L}_\xi \bar{g}_{ab}$. Any such instances of $\mathcal{L}_\xi \bar{g}_{ab}$ appearing in (59) may be replaced by $\mathcal{L}_\xi \hat{g}_{ab}$ without changing the size of the error already present in that equation. The evolution equation for $P_\xi(\Sigma_s) + \mathcal{E}_\xi(\Sigma_s)$ can therefore be written entirely in terms of integrals linear in $\mathcal{L}_\xi \hat{g}_{ab}$ and confined to regions near Σ_s.

Given that $\hat{g}_{ab}$ is an approximate solution to the vacuum Einstein equation, one might expect that it often varies slowly inside the body. This suggests that it can be useful to expand (59) in a multipole series analogous to (14). As in the test body case, the result is simple only if the vector fields ξ^a are chosen carefully. By analogy with the discussion in Sect. 2, set

$$\xi^a = \hat{g}^{ab} \Xi_b, \tag{62}$$

where Ξ_a is a generalized Killing field constructed as in Appendix B, but using the effective metric $\hat{g}_{ab}$ instead of g_{ab}. The $\hat{g}^{ab}$ appearing here is the inverse of $\hat{g}_{ab}$. It is clear from (8) that for GKFs of this type,

$$\mathcal{L}_\xi \hat{g}_{ab}|_\Gamma = \hat{\nabla}_a \mathcal{L}_\xi \hat{g}_{bc}|_\Gamma = 0 \tag{63}$$

when $\Gamma = \{\gamma_s | s \in I \subseteq \mathbb{R}\}$ is the $\hat{g}$-timelike worldline used to construct the GKFs and $\hat{\nabla}_a$ is the derivative operator associated with $\hat{g}_{ab}$.

If the components $\hat{g}_{\mu\nu}$ are sufficiently well-behaved in Riemann normal coordinates constructed with the origin γ_s and metric $\hat{g}_{ab}$, it is now possible to show that [10]

$$\mathcal{L}_\xi \hat{g}_{ab}(x) = \sum_{n=2}^{\infty} (\cdots)^{abc_1\cdots c_n} \mathcal{L}_\xi \hat{g}_{ab,c_1\cdots c_n}(\gamma_s), \tag{64}$$

where the usual caveats regarding asymptotic series apply. $\hat{g}_{ab,c_1\cdots c_n}$ denotes the n^{th} tensor extension of $\hat{g}_{ab}$ (see Sect. 2). The omitted coefficients in this equation are complicated but calculable. Details of their properties are discussed in [10]. Regardless, it is now clear that there exist tensors $\hat{I}^{c_1\cdots c_n ab}(s)$ such that (59) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}s} \hat{P}_\xi(\Sigma_s) = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n!} \hat{I}^{c_1\cdots c_n ab}(s) \mathcal{L}_\xi \hat{g}_{ab,c_1\cdots c_n}(\gamma_s) + O(H^3), \tag{65}$$

where

$$\hat{P}_\xi := P_\xi + \mathcal{E}_\xi. \tag{66}$$

Eq. (65) is identical to the test mass expression (14) for $\mathrm{d}P_\xi/\mathrm{d}s$ under the replacements

$$g_{ab} \to \hat{g}_{ab}, \qquad P_\xi \to \hat{P}_\xi, \qquad I^{c_1\cdots c_n ab} \to \hat{I}^{c_1\cdots c_n ab}. \tag{67}$$

This means that in general relativity linearized about a vacuum background – possibly with $\Lambda \neq 0$ – *the momenta $\hat{P}_\xi$ associated with a self-gravitating compact body behave as though they were the momenta of a test mass with multipole moments $\hat{I}^{c_1\cdots c_n ab}$ moving in a vacuum metric $\hat{g}_{ab}$*. All direct effects of H_{S}^{ab} have been absorbed into the definitions of $\hat{P}_\xi$ and $\hat{I}^{c_1\cdots c_n ab}$. This is made more explicit in Sect. 6 below, where laws of motion are obtained for effective linear and angular momenta $\hat{p}^a$ and $\hat{S}^{ab}$. The arguments are essentially identical to the ones given in Sect. 2 for extended test masses.

5.1. A non-perturbative formulation: eliminating the background metric

The central result obtained so far is the expansion (65) for $\mathrm{d}\hat{P}_\xi/\mathrm{d}s$. This provides asymptotic series for changes in the linear and angular momenta of a large class of extended (but bounded) masses in general relativity. The main assumption is that the metric near the body be "close" to a particular vacuum metric $\bar{g}_{ab}$. In actuality, it is not necessary to specify an independent background at all. An appropriate $\bar{g}_{ab}$ may be *constructed* by self-consistently identifying it with the effective metric.

To see this, it is first useful to recount the main steps in the derivation that has already been presented. Briefly, combining stress-energy conservation with the definitions (3) and (28) for P_ξ and H^{ab} yields (49). Substituting the definition (43) for H^{ab}_{S} into this equation and following Eqs. (51)-(56) leads to (59). From this point, essentially no calculation is required to recover the final form (65) for $\mathrm{d}\hat{P}_\xi/\mathrm{d}s$.

Note that the error estimate associated with, e.g., (59) does not change if the S-type Green function $\bar{G}^{aba'b'}_{\mathrm{S}}$ is modified by terms linear in the metric perturbation. This freedom may be used to consider slightly different definitions from those introduced above. Introduce a new effective metric $\tilde{g}_{ab}$ and a Green function $\tilde{G}^{aba'b'}_{\mathrm{S}}(x,x') = \tilde{G}^{a'b'ab}_{\mathrm{S}}(x',x)$ satisfying (41) with the replacements $\bar{g}_{ab} \to \tilde{g}_{ab}$, $\bar{G}_{\mathrm{S}}(x,x') \to \tilde{G}_{\mathrm{S}}(x,x') = \tilde{G}_{\mathrm{S}}(x',x)$, etc. Also suppose that $\tilde{G}^{aba'b'}_{\mathrm{S}}(x,x') = \tilde{G}_{\mathrm{S}}(x,x') = 0$ if x and x' are timelike-separated with respect to $\tilde{g}_{ab}$. $G^{aba'b'}_{\mathrm{S}}$ and $\tilde{G}_{\mathrm{S}}$ can be solutions to (31) and (42), but other equations may be used instead.

In any case, define

$$\tilde{H}^{ab}_{\mathrm{S}} := 4\int_W \tilde{G}^{aba'b'}_{\mathrm{S}} \tilde{g}_{a'c'}\tilde{g}_{b'd'} T^{c'd'}_{\mathrm{B}} \mathrm{d}V', \tag{68}$$

and

$$\tilde{g}_{ab} = g_{ab} - (\tilde{g}_{ac}\tilde{g}_{bd} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}_{cd})\tilde{H}^{cd}_{\mathrm{S}}. \tag{69}$$

This last equation is to be interpreted as an implicit definition for $\tilde{g}_{ab}$ in terms of the physical metric and the stress-energy tensor. There is no reason to suppose that an exact solution exists or is unique. We assume, however, that there are no such difficulties.

In practice, one might have physical reason to suppose that g_{ab} is "near" a known vacuum solution $\bar{g}_{ab}$. This can be used as an initial guess for $\tilde{g}_{ab}$ in order to compute an approximate $\tilde{H}^{ab}_{\mathrm{S}}$. Substituting the result into (69) provides an improved approximation for $\tilde{g}_{ab}$. The prescription (60) for $\hat{g}_{ab}$ provided above may therefore be interpreted as essentially the first iteration in a recursive sequence that (hopefully) converges to $\tilde{g}_{ab}$.

The advantage of this construction is that it is possible to derive an exact analog of (59). Suppose that the definition (3) for P_ξ is retained precisely as written (with no particular assumption regarding the nature of the ξ^a). Repeating the same types of calculations as before, one finds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[P_\xi(\Sigma_s) + \tilde{\mathcal{E}}_\xi(\Sigma_s)] = \frac{1}{2}\int_{\Sigma_s} \mathrm{d}S\, T^{ab}\bigg[\mathcal{L}_\xi \tilde{g}_{ab} + 2\int_W \mathrm{d}V'\, T^{a'b'} \\ \times \mathcal{L}_\xi\Big(\tilde{g}_{ac}\tilde{g}_{bd}\tilde{g}_{a'c'}\tilde{g}_{b'd'}\tilde{G}^{cdc'd'}_{\mathrm{S}} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}_{a'b'}\tilde{G}_{\mathrm{S}}\Big)\bigg],\end{aligned} \tag{70}$$

where

$$\tilde{\mathcal{E}}_\xi(\Sigma) := \frac{1}{4}\left(\int_{\Sigma^+} T^{ab}_{\mathrm{B}} \mathcal{L}_\xi \tilde{h}^{\mathrm{S}}_{ab}[\Sigma^-]\mathrm{d}V - \int_{\Sigma^-} T^{ab}_{\mathrm{B}} \mathcal{L}_\xi \tilde{h}^{\mathrm{S}}_{ab}[\Sigma^+]\mathrm{d}V\right), \tag{71}$$

and

$$\tilde{h}^{\rm S}_{ab}[R] := 4(\tilde{g}_{ac}\tilde{g}_{bd} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}_{cd}) \int_R \tilde{g}_{a'c'}\tilde{g}_{b'd'} T^{c'd'}_{\rm B} \tilde{G}^{cda'b'}_{\rm S} {\rm d}V'. \tag{72}$$

No approximations have been made in these equations. If, however, $\tilde{g}_{ab} = \bar{g}_{ab} + O(H)$ for some known metric $\bar{g}_{ab}$, $\tilde{g}_{ab} = \hat{g}_{ab} + O(H^2)$ and $\tilde{\mathcal{E}}_\xi = \mathcal{E}_\xi + O(H^3)$.

Although this result is very simple, it is not clear if it is significantly more powerful than the perturbative equation (59). The new effective metric $\tilde{g}_{ab}$ is not a vacuum solution to the exact Einstein equation. Despite this, one might still suppose that $\tilde{g}_{\mu\nu}$ can be expanded in a Taylor series near Σ_s in a Riemann normal coordinate system constructed with the metric $\tilde{g}_{ab}$ and origin γ_s. If this is possible – and if an accurate approximation can be obtained by truncating the series after a small number of terms – Eq. (64) may be used to obtain a multipole expansion for the evolution of $P_\xi + \tilde{\mathcal{E}}_\xi$. This is most conveniently done using GKFs $\Xi_a = \tilde{g}_{ab}\xi^b$ constructed using the effective metric $\tilde{g}_{ab}$. In this case,

$$\frac{\rm d}{{\rm d}s}(P_\xi + \tilde{\mathcal{E}}_\xi) = \frac{1}{2}\sum_{n=2}^\infty \frac{1}{n!}\tilde{I}^{c_1\cdots c_n ab}\mathcal{L}_\xi \tilde{g}_{ab,c_1\cdots c_n}. \tag{73}$$

To the extent that this equation may be trusted, it generalizes (65). In the perturbative regime, $\tilde{I}^{\cdots} = \hat{I}^{\cdots} + O(H^3)$ and $\tilde{g}_{\dots} = \hat{g}_{\dots} + O(H^2)$.

The development here provides a sense in which the gravitational Detweiler-Whiting axiom is exact: The field $\tilde{H}^{ab}_{\rm S}$ can be entirely removed from the laws of the motion using finite renormalizations. All references to perturbative expansions and background metrics have been eliminated. Unfortunately, the effective metric in which the body is found to move does not satisfy the vacuum Einstein equation exactly. It will be left to future work to decide whether or not this result is useful for computing, e.g., nonlinear corrections to the gravitational self-force and self-torque. The remainder of this paper focuses on the implications of the perturbative viewpoint. This is equivalent to the exact treatment for sufficiently small metric perturbations.

5.2. *An interlude: effective multipole moments*

Before continuing to describe the motion of an extended body, we first discuss the renormalized multipole moments appearing (65). The monopole and dipole moments are contained in $\hat{P}_\xi$, which differs from P_ξ by the $\mathcal{E}_\xi$ given in (58). The physical meaning of this expression is not immediately clear. It is made considerably more transparent by specializing to the case of a stationary system in a Minkowski background $\bar{g}_{ab} = \eta_{ab}$. Suppose that $T^{ab}_{\rm B}$ is invariant under the action of a particular time-translation vector field $\partial/\partial t$, where t is a globally-inertial time coordinate for η_{ab}. Neglecting terms of $O(H^3)$ and higher, the ξ^a (technically constructed using $\hat{g}_{ab}$) may be replaced with Minkowski Killing fields. This may be used to show that

$$\mathcal{E}_\xi(\Sigma) \to -\frac{1}{4}\int_\Sigma T^{ab}_{\rm B} h^{\rm S}_{ab}\xi^c {\rm d}\bar{S}_c + O(H^3). \tag{74}$$

Now specialize further so that the only significant component of $T^{ab}_{\rm B}$ is proportional to $(\partial/\partial t)\otimes(\partial/\partial t)$. Also take Σ to be a hypersurface of constant t that is unbounded in every direction. Suppose as well that the only significant metric perturbation is $H^{ab}_{\rm S}$. Then

$$\hat{P}_\xi(\Sigma) = P_\xi(\Sigma) + \mathcal{E}_\xi(\Sigma) \to \int_\Sigma (T^{ab}_{\rm B} + t^{ab})\eta_{bc}\xi^c {\rm d}\bar{S}_a + O(H^3), \tag{75}$$

where t^{ab} is the Landau-Lifshitz tensor (A.14). If the ξ^a were exactly Minkowski Killing fields, the linear and angular momenta extracted from this using a formula like (9) would be exactly those typically identified as the total momenta of the system in post-Newtonian theory.

Returning to the general case of a dynamical system evolving on a curved background, consider the quadrupole and higher effective moments $\hat{I}^{\cdots}$ appearing in (65). It is more difficult to find formulae for these objects than for $\hat{P}_\xi$, although they may still be determined (at least approximately) using the techniques developed in [10]. This is not difficult conceptually, but requires a great deal of tedious calculation. We merely note that like $\hat{P}_\xi(\Sigma_s)$, the $\hat{I}^{\cdots}(s)$ depend on details of the system in a finite four-dimensional region around Σ_s.

The form of (65) and the discussion in Sect. 2 also makes it clear that the effective moments may be taken to have the same index symmetries as their bare counterparts. This means that $\hat{I}^{c_1\cdots c_n ab}$ is separately symmetric in its first n and last two indices. It also satisfies (17) with the substitution $I^{\cdots} \to \hat{I}^{\cdots}$. Additionally, recall that the effective metric $\hat{g}_{ab}$ is a solution to the linearized vacuum Einstein equation. There is therefore no loss of generality in setting certain traces of the multipole moments to zero as discussed in Sect. 2 and in [10].

6. Center of mass motion

At this point, effective linear and angular momenta $\hat{p}^a(s)$ and $\hat{S}^{ab} = \hat{S}^{[ab]}(s)$ may be introduced as tensor fields on the worldline Γ used to define the GKFs. By analogy with the test mass equation (9), suppose that

$$\hat{P}_\xi(\Sigma_s) = \hat{p}^a(s)\Xi_a(\gamma_s) + \frac{1}{2}\hat{S}^{ab}(s)\hat{\nabla}_a\Xi_b(\gamma_s) \tag{76}$$

for all GKFs $\Xi_a(x)$ defined using the metric $\hat{g}_{ab}$, the worldline Γ, and the vector field n_s^a. The ξ^a appearing on the left-hand side of this equation is related to Ξ_a via (62). Differentiating (76) while using (63) gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{P}_\xi = \left(\frac{\hat{\mathrm{D}}\hat{p}^a}{\mathrm{d}s} - \frac{1}{2}\hat{R}_{bcd}{}^a\hat{S}^{bc}\dot{\gamma}_s^d\right)\Xi_a + \frac{1}{2}\left(\frac{\hat{\mathrm{D}}\hat{S}^{ab}}{\mathrm{d}s} - 2\hat{p}^{[a}\dot{\gamma}_s^{b]}\right)\hat{\nabla}_a\Xi_b. \tag{77}$$

$\hat{\mathrm{D}}/\mathrm{d}s$ is the covariant path derivative compatible with $\hat{g}_{ab}$ and $\hat{R}_{bcd}{}^d$ is the Riemann tensor associated with this metric. Eq. (77) is directly analogous to the test mass relation (12). Combining it with (65) yields multipole expansions for $\hat{\mathrm{D}}\hat{p}^a/\mathrm{d}s$ and $\hat{\mathrm{D}}\hat{S}^{ab}/\mathrm{d}s$ identical to (19)-(22) after the replacements

$$\begin{gathered} p^a \to \hat{p}^a, \quad S^{ab} \to \hat{S}^{ab}, \quad I^{c_1\cdots c_n ab} \to \hat{I}^{c_1\cdots c_n ab} \\ g^{ab} \to \hat{g}^{ab}, \quad R_{abc}{}^d \to \hat{R}_{abc}{}^d, \quad g_{ab,c_1\cdots c_n} \to \hat{g}_{ab,c_1\cdots c_n} \\ \nabla_a \to \hat{\nabla}_a, \quad \mathrm{D}/\mathrm{d}s \to \hat{\mathrm{D}}/\mathrm{d}s, \quad F^a \to \hat{F}^a, \quad N^{ab} \to \hat{N}^{ab}. \end{gathered} \tag{78}$$

We refer to the resulting equations as the "hatted forms" of their counterparts in the theory of extended test bodies.

The worldline and foliation used to construct the GKFs may now be fixed by choosing them such that

$$\hat{p}^a(s) \propto n_s^a, \tag{79}$$

$$\hat{g}_{ab}(\gamma_s)\hat{p}^a(s)\hat{S}^{bc}(s) = 0. \tag{80}$$

These are directly analogous to the center of mass conditions (23) and (24). Unlike in that case, however, there exists no proof that these equations have well-behaved solutions. We assume, however, that they do.

As in Sect. 2, it is useful to choose the parameter s such that $\hat{g}_{ab}\hat{p}^a\dot{\gamma}_s^b = -\hat{m}$, where

$$\hat{m} := \sqrt{-\hat{g}_{ab}\hat{p}^a\hat{p}^b}. \tag{81}$$

We also set $\hat{g}_{ab}n_s^a n_s^b = -1$, so $\hat{p}^a = \hat{m}n_s^a$. The center of mass velocity is then given by (27) with the replacements (78) and $m \to \hat{m}$. Together, the hatted versions of (19)-(22) and (27) strongly constrain the evolution of the body's linear and angular momenta as well as its center of mass. They do not determine it completely. As in the test body case, the evolution of the quadrupole and higher moments must be specified using other methods. Additionally, the effective metric $\hat{g}_{ab}$ couples to the motion in a nontrivial way. This is the main complication in practical computations involving the gravitational self-force.

It can be useful to define a spin 1-form $\hat{S}_a$ via

$$\hat{S}_a := -\frac{1}{2}\hat{\epsilon}_{abcd}n_s^b\hat{S}^{cd}. \tag{82}$$

The center of mass condition (80) guarantees that all information contained in $\hat{S}^{ab}$ is also contained in $\hat{S}_a$. This means that (82) is invertible:

$$\hat{S}^{ab} = \hat{\epsilon}^{abcd}\hat{g}_{cf}n_s^f\hat{S}_d. \tag{83}$$

Note that $\hat{p}^a\hat{S}_a = 0$. The hatted form of (20) implies that

$$\frac{\hat{\mathrm{D}}\hat{S}_a}{\mathrm{d}s} = n_s^b\left(m^{-1}\hat{g}_{ab}\hat{S}_c\frac{\hat{\mathrm{D}}\hat{p}^c}{\mathrm{d}s} - \frac{1}{2}\hat{\epsilon}_{abcd}\hat{N}^{cd}\right). \tag{84}$$

The $\hat{\mathrm{D}}\hat{p}^c/\mathrm{d}s$ appearing on the right-hand side of this equation may be eliminated using the hatted form of (19). By not doing so, one may interpret the first term in (84) as being responsible for a kind of Thomas precession. It arises from the requirement that $\hat{p}^a$ and $\hat{S}_a$ remain orthogonal.

There is nothing that prevents the effective mass $\hat{m}$ from varying. It immediately follows from its definition (81) that

$$\frac{\mathrm{d}\hat{m}}{\mathrm{d}s} = -\hat{g}_{ab}n_s^a\frac{\hat{\mathrm{D}}\hat{p}^b}{\mathrm{d}s}. \tag{85}$$

Substituting the hatted form of (19) into this equation and simplifying with the hatted form of (27) leads to a (large) equation that does not explicitly involve $\dot{\gamma}_s^a$. Another useful form is [25]

$$\frac{\mathrm{d}\hat{m}}{\mathrm{d}s} = \hat{g}_{ab}\left(-\dot{\gamma}_s^a\hat{F}^b + \hat{m}^{-1}n_s^a\hat{N}^{bc}\hat{g}_{cd}\frac{\hat{\mathrm{D}}\hat{p}^d}{\mathrm{d}s}\right). \tag{86}$$

which follows from (80) as well as the hatted versions of (19) and (20). Additional manipulations to the right-hand side of this equation may be used to bring it into a form involving total s-derivatives and "induction terms" that depend on derivatives of the moments in a certain non-rotating reference frame [2, 25]. Regardless, it is clear that $\hat{m}$ couples only to the quadrupole and higher moments $\hat{I}^{\cdots}$.

6.1. Monopole and dipole approximations

As a simple application of the laws of motion just derived, consider truncating them at dipole order. This means that the quadrupole and higher moments $\hat{I}^{\cdots}$ are to be ignored. As a consequence, $\hat{F}^a = \hat{N}^{ab} = 0$. It also follows from (86) that the effective mass $\hat{m}$ remains fixed in this approximation. All of the $\hat{P}_\xi$ are also constant. The linear and angular momenta evolve via the Papapetrou equations in the effective metric:

$$\frac{\hat{\mathrm{D}}\hat{p}^a}{\mathrm{d}s} = \frac{1}{2}\hat{R}_{bcd}{}^a \hat{S}^{bc}\dot{\gamma}_s^d, \tag{87}$$

$$\frac{\hat{\mathrm{D}}\hat{S}^{ab}}{\mathrm{d}s} = 2\hat{p}^{[a}\dot{\gamma}_s^{b]}. \tag{88}$$

The center of mass velocity satisfies

$$\hat{m}\dot{\gamma}_s^a = \hat{p}^a + \frac{1}{2}\left(\frac{\hat{S}^{ab}\hat{S}^{cd}p^f\hat{R}_{cdb}{}^l\hat{g}_{fl}}{m^2 + \frac{1}{4}\hat{S}^{bc}\hat{S}^{df}\hat{R}_{bcd}{}^l\hat{g}_{fl}}\right), \tag{89}$$

which may be used to eliminate $\dot{\gamma}_s^a$ from the right-hand sides of (87) and (88). The result is a coupled set of ODEs for $\hat{p}^a$, $\hat{S}^{ab}$, and γ_s. Alternatively, one may replace the evolution equation for $\hat{S}^{ab}$ with one for $\hat{S}_a$ using (82) and (84).

Now consider a non-spinning body. It follows from (88) and (89) that $\hat{S}^{ab} = 0$ is a valid solution to the dipole equations for all time. In this case, the object's center of mass moves on a geodesic of the effective metric:

$$\frac{\hat{\mathrm{D}}^2\gamma_s^a}{\mathrm{d}s^2} = 0. \tag{90}$$

It is common in the literature to write equations of this sort in terms of the background derivative operator $\bar{\mathrm{D}}/\mathrm{d}s$. For any vector $v^a(s)$ or covector $w_a(s)$,

$$\frac{\hat{\mathrm{D}}v^a}{\mathrm{d}s} = \frac{\bar{\mathrm{D}}v^a}{\mathrm{d}s} + \hat{C}^a_{bc}\dot{\gamma}_s^b v^c; \qquad \frac{\hat{\mathrm{D}}w_a}{\mathrm{d}s} = \frac{\bar{\mathrm{D}}w_a}{\mathrm{d}s} - \hat{C}^c_{ab}\dot{\gamma}_s^b w_c, \tag{91}$$

where $\hat{C}^a_{bc}$ is given by (A.8) with the substitution $g_{ab} \to \hat{g}_{ab}$. To linear order in the metric perturbation, (90) is therefore equivalent to

$$\frac{\bar{\mathrm{D}}^2\gamma_s^a}{\mathrm{d}s^2} = \frac{1}{2}\bar{g}^{ad}(\bar{\nabla}_d\hat{g}_{bc} - 2\bar{\nabla}_b\hat{g}_{cd})\dot{\gamma}_s^b\dot{\gamma}_s^c. \tag{92}$$

Note that there is no projection operator on the right-hand side of this equation. In general, the background acceleration need not be orthogonal to the 4-velocity (with respect to $\bar{g}_{ab}$). This is because the parameter s has been chosen such that $\dot{\gamma}_s^a$ has unit norm with respect to $\hat{g}_{ab}$. It is more typical to normalize the 4-velocity with respect to the background metric. Introduce a new time parameter $\bar{s}(s)$ such that $u^a := (\mathrm{d}s/\mathrm{d}\bar{s})\dot{\gamma}_s^a$ satisfies $\bar{g}_{ab}u^a u^b = -1$. Eq. (92) then becomes the more familiar

$$\frac{\bar{\mathrm{D}}u^a}{\mathrm{d}\bar{s}} = \frac{1}{2}(\bar{g}^{ad} + u^a u^d)(\bar{\nabla}_d\hat{h}_{bc} - 2\bar{\nabla}_b\hat{h}_{cd})u^b u^c. \tag{93}$$

This background acceleration (multiplied by $\hat{m}$) is what is typically referred to as the gravitational self-force.

One might also be interested in defining a mass $\bar{m} := \sqrt{-\bar{g}_{ab}\hat{p}^a\hat{p}^b}$ using the background metric. While it has already been noted that the $\hat{m}$ defined by (81) remains constant in the current approximation, the same is not true of $\bar{m}$. The "background mass" evolves via

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\bar{m}\left(1 - \frac{1}{2}\hat{h}_{ab}u^a u^b\right)\right] = 0. \tag{94}$$

If the spin is "small" rather than exactly zero – meaning that only linear terms in $\hat{S}_a$ are to be retained in the various evolution equations – it is parallel-propagated along Γ with respect to $\hat{g}_{ab}$:

$$\frac{\hat{\mathrm{D}}\hat{S}_a}{\mathrm{d}s} = 0. \tag{95}$$

This may be interpreted as being equivalent to a self-torque in the background spacetime:

$$\frac{\bar{\mathrm{D}}\hat{S}_a}{\mathrm{d}s} = -\frac{1}{2}\bar{g}^{cd}(\bar{\nabla}_d \hat{h}_{ab} - 2\bar{\nabla}_{(a}\hat{h}_{b)d})u^b \hat{S}_c. \tag{96}$$

It is unclear how consistent this equation is over long times. It does not preserve the constraint $\hat{p}^a \hat{S}_a = 0$ if coupled to an evolution equation for $\hat{p}^a$ that involves any spin coupling at all (even when that coupling only involves the background curvature). If $\hat{S}^{ab}$ is evolved instead of $\hat{S}_a$, the center of mass condition (80) would fail to remain preserved over time. These complications do not arise if the more complicated equations (87)-(89) are retained in full.

6.2. A small mass

Self-force problems are usually considered in cases where the object of interest is very small compared to any background length scales. It is also assumed (at least implicitly) that the internal dynamics occur on timescales that are long compared to the body's size. This may be formalized by fixing the background and considering certain one-parameter families of stress-energy tensors that shrink in an approximately self-similar manner (see, e.g., [3, 18]). Unfortunately, it is difficult to obtain significant gravitational self-forces and self-torques without leaving the regime of linearized gravity. If self-interaction effects are to be significantly larger than the neglected nonlinearities, it would appear that the body must be able to support significant tensile and shear stress if it is to avoid tidal disruption.

The $O(H^3)$ error in (65) suggests that forces computed using that equation could be incorrect by terms of order (mass/radius)^3. The neglected torques are perhaps of order (mass)^3/(radius)^2. These are very conservative estimates. It is likely that most of the self-field will not directly affect the motion at higher orders, so the actual errors are probably significantly smaller (at least if the momenta are renormalized again). It might be possible to establish this using the exact formalism in Sect. 5.1, but we have not done so. It will therefore not be assumed.

Regardless, the spin and some of the higher moments generically have a larger effect on the motion than the self-force. This is relatively innocuous. It has already been established that corrections to the laws of motion involving the higher multipole moments are exactly those already known from the test body regime. We shall therefore focus only on computing the lowest order self-force and self-torque as these quantities appear in (93) and (96). Additional corrections independent of the self-interaction are easily added if necessary for the specific system under consideration. The arguments we now present can easily be made more rigorous by adopting scaling assumptions like those in, e.g., [3]. The details would not be particularly interesting, so we instead present what is essentially a plausibility argument.

The key physical assumption needed to recover equations already derived by other methods is that the metric perturbation essentially be the body's retarded field. It is not likely, however, that the linearized theory is valid in the distant past. For this reason, the submanifold M on which we're working should not be extended that far. A retarded field therefore cannot be produced by integrating $T^{ab}_{\rm B}$ against the retarded Green function $\bar{G}^{aba'b'}_{\rm ret}$ into the infinite past [19]. Instead, Cauchy data should be prescribed on some initial hypersurface $\Sigma_{s_0} \subset M$. Applying (34) then produces a field of the form

$$H^{ab} = 4 \int_{W^+} \bar{g}_{a'c'} \bar{g}_{b'd'} \bar{G}^{aba'b'}_{\rm ret} T^{c'd'}_{\rm B} {\rm d}\bar{V} + \mathcal{H}^{ab} + O(H^2), \tag{97}$$

where $\mathcal{H}^{ab}$ is some homogeneous solution of (30) and $W^+ := \Sigma^+_{s_0} \cap W$ is the portion of W to the future of Σ_{s_0}.

Far outside of W, H^{ab} will appear nearly indistinguishable from a retarded solution of the linearized Einstein equation with a point particle source (see, e.g., [12, 18, 19]). To a first approximation, this "effective particle" can be taken to have the worldline Γ, mass $\hat{m}$, and no higher moments. This means that

$$H^{ab}(x) \to 4\hat{m} \int_{s_0}^{\infty} \bar{G}^{aba'b'}_{\rm ret}(x, \gamma_{s'}) \bar{g}_{a'c'}(\gamma_{s'}) \bar{g}_{b'd'}(\gamma_{s'}) \dot{\gamma}^{c'}_{s'} \dot{\gamma}^{d'}_{s'} {\rm d}s' + \mathcal{H}^{ab} \tag{98}$$

at large distances. Now, the motion is determined by the effective field $\hat{H}^{ab}$ inside W. Using (61), this may be written in terms of a surface integral involving H^{ab}. While the surface used in that equation is ∂W, this may be changed to a very large timelike tube T that surrounds W. Integrating over T results in

$$\hat{H}^{ab} = 4\hat{m} \int_{s_0}^{\infty} \bar{G}^{aba'b'}_{\rm R} \bar{g}_{a'c'} \bar{g}_{b'd'} u^{c'} u^{d'} {\rm d}s' + \mathcal{H}^{ab}, \tag{99}$$

where $\bar{G}^{aba'b'}_{\rm R} := \bar{G}^{aba'b'}_{\rm ret} - \bar{G}^{aba'b'}_{\rm S}$ is typically referred to as the R-type Detweiler-Whiting "Green function" (even though it satisfies a homogeneous wave equation). As in Sect. 6.1, $u^a \propto \dot{\gamma}^a_s$ has unit norm with respect to $\bar{g}_{ab}$. The conclusion of this argument is that for a sufficiently small particle with slow internal dynamics, the effective field inside W is essentially that of a point particle. A similar comment cannot be made for the retarded field.

The Hadamard form (37) for $\bar{G}^{aba'b'}_{\rm S}$ (and the equivalent for $\bar{G}^{aba'b'}_{\rm ret}$) may now be used to compute $\hat{H}^{ab}$ explicitly. As can be seen from (93), we need only the first derivative of this field on Γ. This has already been computed in, e.g., [12] for the case where $\bar{R}_{ab} = 0$ and $\bar{\rm D}u^a/{\rm d}s = 0$. We shall continue to assume that the acceleration is zero, as any terms involving it will be negligibly small. We do, however, generalize the Ricci tensor to be $\Lambda \bar{g}_{ab}$. Then

$$\begin{aligned} \bar{\nabla}_c \hat{H}^{ab}(\gamma_s) = 4\hat{m} \left[\left(\bar{R}_{cdf}{}^{(a} u^{b)} - u_c \bar{R}_d{}^{(a}{}_f{}^{b)} \right) u^d u^f - \frac{1}{3} \Lambda u_c u^a u^b \right] \\ + H_c{}^{ab} + \bar{\nabla}_c \mathcal{H}^{ab}, \end{aligned} \tag{100}$$

where

$$H_c{}^{ab} := 4\hat{m} \lim_{\epsilon \to 0} \int_{s_0}^{s-\epsilon} \bar{\nabla}_c \bar{G}^{aba'b'}_{\rm ret} u_{a'} u_{b'} {\rm d}s'. \tag{101}$$

For simplicity, indices in these equations have been raised and lowered with the background metric. Also note that the limiting process used to define $H^{ab}{}_c$ avoids the singularity in the retarded Green function.

Now suppose for simplicity that $\bar{\nabla}_c \mathcal{H}^{bc}$ is negligible, as can be arranged if linear perturbation theory may be trusted sufficiently far in the past. Substituting (100) into (93) then yields

$$\frac{\bar{\mathrm{D}} u^a}{\mathrm{d}s} = \frac{1}{2}(\bar{g}^{ad} + u^a u^d) u^b u^c (H_{dbc} - 2H_{bcd}) + \ldots . \tag{102}$$

This is the MiSaTaQuWa equation as it is usually written (at least if $s_0 \to -\infty$) [12, 18, 19, 20, 21]. An equation for the spin evolution is easily obtained by substituting (100) into (96):

$$\frac{\bar{\mathrm{D}} \hat{S}_a}{\mathrm{d}s} = -2\hat{m} u^b u^c \bar{R}_{abc}{}^d \hat{S}_d - \frac{1}{2} u^b \hat{S}^c (H_{cab} - 2H_{(ab)c}) + \ldots . \tag{103}$$

As we have already noted, these are not the full laws of motion. Most objects in which the current analysis is valid will also be significantly affected by a number of terms involving higher multipole moments. These are ordinary test body effects, and have nothing to do with self-force. They are easily added as needed using the results obtained above.

7. Discussion

We have shown that the Detweiler-Whiting S-field does not directly affect the bulk motion of an uncharged mass in what is essentially linearized general relativity. It does shift definitions for the effective multipole moments of a body's stress-energy tensor. This can be viewed as providing a justification for the original Detweiler-Whiting axiom [8] that a "point mass" moves on a geodesic in an effective metric produced by subtracting the S-field from the physical metric (if "point mass" is replaced by "mass with small but finite size"). The validity of this type of statement has also been extended considerably. It applies to all multipole orders and also to the evolution of a body's angular momentum. This joins similar results that have recently been established for objects with scalar or electromagnetic charge moving in fixed background spacetimes [4, 11].

The forms of the evolution equations obtained here are formally the same as the multipole expansions provided by Dixon [2, 22, 34] for extended test masses. Dixon's series were originally derived using an extremely restrictive assumption regarding the deviation of the physical metric from the background. This work shows that their basic structure remains valid in a much wider context (after suitable renormalizations and a shift in the effective metric).

Various generalizations of this work could be attempted. Most obviously, it would be interesting to understand how the nonlinearities in Einstein's equation affect the motion of an uncharged mass distribution. A promising starting point for addressing this question is the development in Sect. 5.1. This shows that there is a sense in which the Detweiler-Whiting S-field may be exactly renormalized out of the laws of motion. This leaves an effective metric which is not quite a vacuum solution, so it is not clear to what extent it would be useful in deriving higher-order self-force effects.

Appendix A. Einstein's equation in wave gauge

This paper considers the behavior of a material body in a spacetime $(\mathcal{M}, g_{ab})$. This is compared to a known spacetime $(\bar{\mathcal{M}}, \bar{g}_{\bar{a}\bar{b}})$. A reasonable correspondence cannot be expected globally, so restrict attention to the submanifolds $M \subseteq \mathcal{M}$ and $\bar{M} \subseteq \bar{\mathcal{M}}$. These two regions are assumed to be diffeomorphic. An explicit diffeomorphism between them – the gauge map – may be used to identify "equivalent points" in a perturbative expansion. We make use of the wave gauge. This identifies points in the two spacetimes using a diffeomorphism $\phi : M \to \bar{M}$ that satisfies the Euler-Lagrange equations obtained by varying ϕ in the action [36]

$$S := \frac{1}{2} \int_M g^{ab} [g_{ab} - (\phi^* \bar{g})_{ab}] \mathrm{d}V, \tag{A.1}$$

where $(\phi^* \bar{g})_{ab}$ denotes the pullback of $\bar{g}_{\bar{a}\bar{b}}|_{\bar{M}}$ to M via ϕ. In the Riemannian case, such maps are termed harmonic. In the Lorentzian case considered here, they are called wave maps.

This action has two simple interpretations. The first arises from noting that the quantity in brackets in (A.1) is the metric perturbation as it would typically be defined in M:

$$h_{ab} := g_{ab} - (\phi^* \bar{g})_{ab}. \tag{A.2}$$

S is therefore the average trace of the metric perturbation (multiplied by one-half of the volume of M). Alternatively, introduce coordinates x^μ in M and $\bar{x}^{\bar{\mu}}$ in $\bar{M}$. The gauge action then takes the explicit form

$$S = 2\mathrm{vol}(M) - \frac{1}{2} \int_M g^{\mu\nu}(x) \frac{\partial \phi^{\bar{\mu}}}{\partial x^\mu} \frac{\partial \phi^{\bar{\nu}}}{\partial x^\nu} \bar{g}_{\bar{\mu}\bar{\nu}}(\phi(x)) \sqrt{-g(x)} \mathrm{d}^4 x. \tag{A.3}$$

The first term here is an irrelevant constant, while the second has the form of an energy integral for the map ϕ (or $\phi^{\bar{\mu}}(x^\mu)$ as it appears in coordinates). Although it is tempting to think of the ϕ we obtain as extremizing these quantities, it only does so if $\delta\phi = 0$ on ∂M. This is a physically strong restriction if M is a compact submanifold of $\mathcal{M}$.

Regardless, varying (A.3) with respect to the gauge map yields

$$g^{\mu\nu}(x) \left(\frac{\partial^2 \phi^{\bar{\mu}}}{\partial x^\mu \partial x^\nu} + \frac{\partial \phi^{\bar{\rho}}}{\partial x^\mu} \frac{\partial \phi^{\bar{\nu}}}{\partial x^\nu} \bar{\Gamma}^{\bar{\mu}}_{\bar{\rho}\bar{\nu}}(\phi(x)) - \frac{\partial \phi^{\bar{\mu}}}{\partial x^\rho} \Gamma^\rho_{\mu\nu}(x) \right) = 0. \tag{A.4}$$

The Christoffel symbols $\Gamma^\mu_{\nu\rho}$ and $\bar{\Gamma}^{\bar{\mu}}_{\bar{\nu}\bar{\rho}}$ appearing here are computed from $g_{\mu\nu}$ and $\bar{g}_{\bar{\mu}\bar{\nu}}$ in the usual way. Given the covariant nature of the action (A.1), it should not be surprising that (A.4) may be written in a form that is completely free of coordinates [36]. This is somewhat nontrivial, however, so we do not describe it here. The important point is that any solution to (A.4) in one pair of coordinates remains a solution under any pair of coordinate transformations. The resulting ϕ is known as a wave map.

Now suppose that $M = \bar{M}$, and that the two coordinate systems considered above are identical. We then say that g_{ab} is in a wave gauge with respect to $\bar{g}_{ab}$ if the identity $\phi(x) = x$ is a wave map from M to itself. Substituting into (A.4) shows that this occurs if

$$g^{\mu\nu}(\Gamma^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\mu\nu}) = 0. \tag{A.5}$$

The term in parentheses here is the difference between two connections, which is a tensor field on M. In coordinate-free notation, the gauge condition has the equivalent forms

$$0 = g^{ab}(\bar{\nabla}_a g_{bc} - \frac{1}{2} \bar{\nabla}_c g_{ab}) \tag{A.6}$$

$$= g^{ab} \nabla_a (\bar{g}_{bc} - \frac{1}{2} g_{bc} g^{df} \bar{g}_{df}), \tag{A.7}$$

where ∇_a and $\bar{\nabla}_a$ are the usual Levi-Civita connections associated with g_{ab} and $\bar{g}_{ab}$, respectively.

Note that (A.5) is exactly the harmonic coordinate condition when $g^{\mu\nu}\bar{\Gamma}^\lambda_{\mu\nu} = 0$ (as occurs if $\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, for example). Eq. (A.7) is, in a sense, an "inverted Lorenz" condition. It specifies the ordinary Lorenz gauge condition if our interpretation of g_{ab} and $\bar{g}_{ab}$ is reversed so that g_{ab} is considered to be the background metric. In this sense, choosing ϕ to map from the full spacetime to the background gives a harmonic-like gauge condition. Choosing the opposite orientation for this map recovers the Lorenz gauge condition. Both of these cases coincide in the linearized theory.

It is straightforward to write down Einstein's equation in wave gauge. First define

$$C^c_{ab} := \frac{1}{2} g^{cd}(2\bar{\nabla}_{(a} g_{b)d} - \bar{\nabla}_d g_{ab}). \tag{A.8}$$

In any coordinate system (and in any gauge), $C^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\mu\nu}$. A similar expression also exists for the difference between the Riemann tensors associated with g_{ab} and $\bar{g}_{ab}$ [37]:

$$R_{abc}{}^d - \bar{R}_{abc}{}^d = 2(\bar{\nabla}_{[b} C^d_{a]c} - C^f_{c[b} C^d_{a]f}). \tag{A.9}$$

Contracting the second and fourth indices immediately gives

$$R_{ab} - \bar{R}_{ab} = 2(\bar{\nabla}_{[c} C^c_{a]b} - C^d_{b[c} C^c_{a]d}). \tag{A.10}$$

These equations are all exact.

Now introduce a stress-energy tensor $T^{ab} = T^{(ab)}$ via Einstein's equation (29). Do the same for the background stress-energy tensor $\bar{T}^{ab}$. Applying the wave gauge condition (A.5) to (A.10), one then finds that

$$\begin{aligned} g^{cd}\bar{\nabla}_c\bar{\nabla}_d g_{ab} + 2(g^{cd} g_{f(a}\bar{R}_{b)cd}{}^f + \bar{R}_{ab}) = -16\pi[(g_{ac}g_{bd} - \frac{1}{2} g_{ab}g_{cd})T^{cd} \\ - (\bar{g}_{ac}\bar{g}_{bd} - \frac{1}{2}\bar{g}_{ab}\bar{g}_{cd})\bar{T}^{cd}] - 2g^{cd}g^{fh}(\frac{1}{4}\bar{\nabla}_a g_{cf}\bar{\nabla}_b g_{dh} \\ + \bar{\nabla}_c g_{af}\bar{\nabla}_{[h} g_{d]b} - \bar{\nabla}_{(a|} g_{cf}\bar{\nabla}_d g_{|b)h}) - 2\Lambda(g_{ab} - \bar{g}_{ab}), \end{aligned} \tag{A.11}$$

The metric itself (or $h_{ab} = g_{ab} - \bar{g}_{ab}$) is not necessarily the simplest variable to use here. Another possibility is to introduce the H^{ab} defined by (28). This reduces to the trace-reversed metric perturbation $\bar{g}^{ac}\bar{g}^{bd}(h_{cd} - \frac{1}{2}\bar{g}_{cd}\bar{g}^{fh}h_{fh})$ when linearized. Also note that $\sqrt{-g}/\sqrt{-\bar{g}}$ is shorthand for the proportionality factor between the volume elements associated with the two metrics. In terms of H^{ab}, Einstein's equation takes the form

$$\begin{aligned} \bar{\Box} H^{ab} + 2(\bar{g}^{c(a} - H^{c(a})\bar{R}_{dcf}{}^{b)} H^{df} + [2\bar{g}^{c(a} H^{b)d} - (\bar{g}^{ab} - H^{ab})H^{cd}]\bar{R}_{cd} - H^{ab}\bar{R} \\ = -16\pi[(g/\bar{g})T^{ab} - \bar{T}^{ab} + (16\pi)^{-1}\lambda^{ab}] \\ + 2\Lambda[(\sqrt{g/\bar{g}} - 1)\bar{g}^{ab} - \sqrt{g/\bar{g}} H^{ab}] \end{aligned} \tag{A.12}$$

where $\bar{\Box} := \bar{g}^{cd}\bar{\nabla}_c\bar{\nabla}_d$,

$$\lambda^{ab} := 16\pi(g/\bar{g})t^{ab} + (\bar{\nabla}_d H^{ac}\bar{\nabla}_c H^{bd} - H^{cd}\bar{\nabla}_c\bar{\nabla}_d H^{ab}), \tag{A.13}$$

and

$$\begin{aligned} 16\pi(g/\bar{g})t^{ab} := g_{cd}g^{fh}\bar{\nabla}_f H^{ac}\bar{\nabla}_h H^{bd} + \frac{1}{2} g_{cd}g^{ab}\bar{\nabla}_f H^{ch}\bar{\nabla}_h H^{df} \\ - 2g_{cd}g^{f(a}\bar{\nabla}_h H^{b)c}\bar{\nabla}_f H^{dh} + \frac{1}{2}(g^{ac}g^{bd} - \frac{1}{2}g^{ab}g^{cd}) \\ \times (g_{fp}g_{hq} - \frac{1}{2}g_{fh}g_{pq})\bar{\nabla}_c H^{fh}\bar{\nabla}_d H^{pq}. \end{aligned} \tag{A.14}$$

The gauge condition (A.6) may also be written as

$$\bar{\nabla}_a H^{ab} = 0. \tag{A.15}$$

If the background metric is flat and coordinates are introduced such that $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, (A.12) and (A.15) reduce to the Landau-Lifshitz form of Einstein's equation. $t^{\mu\nu}$ also coincides with the Landau-Lifshitz pseudotensor in this case.

Appendix B. Generalized Killing fields

The notion of a generalized Killing field (GKF) used in this paper was developed in [23], where such objects were referred to as Killing-type generalized affine collineations. Their main properties are are summarized in Sect. 2. For completeness, this appendix provides explicit definitions. It is heavily based on the description in [38].

Everything in this appendix is formulated on a spacetime (M, g_{ab}). Note, however, that the main text discusses GKFs constructed using different metrics. Besides the geometry, a generalized Killing field Ξ_a also requires for its construction a smooth timelike worldline $\Gamma = \{\gamma_s | s \in I \subseteq \mathbb{R}\}$ and a future-directed timelike vector field $n^a(s) \in T_{\gamma_s}M$ defined along Γ. Note that s is not required to be proper time and n^a_s needn't lie tangent to Γ.

A specific generalized Killing field may be fixed by choosing a time s_0 together with tensors $\mathcal{A}_a(s_0)$ and $\mathcal{B}_{ab} = \mathcal{B}_{[ab]}(s_0)$ at γ_{s_0}. The Killing transport equations

$$\frac{\mathrm{D}}{\mathrm{d}s}\mathcal{A}_a(s) - \dot{\gamma}^b_s \mathcal{B}_{ba}(s) = 0 \tag{B.1}$$

$$\frac{\mathrm{D}}{\mathrm{d}s}\mathcal{B}_{ab}(s) + R_{abc}{}^d(\gamma_s)\dot{\gamma}^c_s \mathcal{A}_d(s) = 0 \tag{B.2}$$

are used to uniquely extend these tensors to all of Γ. Note that the skew symmetry of $\mathcal{B}_{ab}$ is preserved by this prescription††.

Now consider all pairs (γ_s, v^a), where $v^a \in T_{\gamma_s}M$ is orthogonal to n^a_s. This forms a subset $T_\perp\Gamma$ of the tangent bundle TM. For any element of $T_\perp\Gamma$, one may associate an affinely-parameterized geodesic $y(w)$ whose initial point is $y(0) = \gamma_s$ and whose initial tangent is v^a. As long as these geodesics can be extended sufficiently far, the map $(\gamma_s, v^a) \to y(1)$ will be a smooth function from $T_\perp\Gamma$ to M. Its Jacobian is clearly invertible at (least at) every point $(\gamma_s, 0)$, so it follows from the inverse function theorem that the given map defines a diffeomorphism on some neighborhood $\mathcal{W}$ of Γ. This will be the region in which the GKFs are to be defined. It is assumed in the main text that the body whose motion is being studied always lies inside this region: $W \subset \mathcal{W}$.

We now define the GKF $\Xi_a(x)$ associated with a choice of $\mathcal{A}_a(s_0)$ and $\mathcal{B}_{ab}(s_0)$. The diffeomorphism just described may be used to uniquely associate x with $(\gamma_\tau, v^a) \in T_\perp\Gamma$. Use this pair to construct a geodesic $y(w)$ as before. The GKF is then be computed along $y(w)$ by solving the Jacobi (or geodesic deviation) equation

$$\frac{\mathrm{D}^2\Xi_a}{\mathrm{d}w^2} - R_{abc}{}^d \dot{y}^b \dot{y}^c \Xi_d = 0, \tag{B.3}$$

with initial data

$$\Xi_a(\gamma_\tau) \quad = \mathcal{A}_a(\tau), \tag{B.4}$$

$$\frac{\mathrm{D}\Xi_a(\gamma_\tau)}{\mathrm{d}w} = v^b \mathcal{B}_{ba}(\tau). \tag{B.5}$$

†† It is possible to use initial data for which $\mathcal{B}_{ab}$ is not skew. The vector field that eventually results generalizes a homothety or other non-Killing affine collineation [23].

As is standard, $\dot{y}^a$ denotes the tangent to $y(w)$. The given equations uniquely define $\Xi_a(x)$ throughout $\mathcal{W}$ once $\mathcal{A}_a$ and $\mathcal{B}_{ab}$ are given at any one point on Γ. More detailed discussions may be found in [10, 23, 38].

Acknowledgments

I am grateful for helpful comments and discussions with Robert Wald, Samuel Gralla, and Barry Wardell. This work was partially supported by NSF grant PHY04-56619 to the University of Chicago.

References

[1] Damour T 1987 in *300 Years of Gravitation* eds Hawking S W and Israel W (Cambridge: Cambridge University Press)
[2] Dixon W G 1979 in *Isolated Systems in General Relativity* ed Ehlers J (Amserdam: North-Holland)
[3] Gralla S E, Harte A I, and Wald R M 2009 *Phys. Rev.* D **80** 024031
[4] Harte A I 2009 *Class. Quantum Grav.* **26** 155015
[5] Spohn H 2004 *Dynamics of Charged Particles and their Radiation Field* (Cambridge: Cambridge University Press)
[6] Landau L D and Lifshitz E M 1962 *The Classical Theory of Fields* (Oxford: Pergamon Press)
[7] Dirac P A M 1938 *Proc. R. Soc.* A **167** 148
[8] Detweiler S and Whiting B F 2003 *Phys. Rev.* D
[9] Harte A I 2006 *Phys. Rev.* D **73** 065006
[10] Harte A I 2010 *Class. Quantum Grav.* **27** 135002
[11] Harte A I 2008 *Class. Quantum Grav.* **25** 235020
[12] Poisson E 2004 *Living Rev. Relativity* **7** 6
[13] Geroch R and Traschen J 1987 *Phys. Rev.* D **36** 1017
[14] DeWitt B S and Brehme R W 1960 *Ann. Phys. (NY)* **9** 220
[15] Levi-Civita T 1937 *Am. J. Math.* **59** 9
[16] Damour T 1983 in *Gravitational Radiation* eds Dereulle N and Piran T (Amsterdam: North-Holland)
[17] Mitchell T and Will C M 2007 *Phys. Rev.* D **75** 124025
[18] Gralla S E and Wald R M 2008 *Class. Quantum Grav.* **25** 205009
[19] Pound A 2010 *Phys. Rev.* D **81** 024023
[20] Mino Y, Sasaki M, and Tanaka T 1997 *Phys. Rev.* D **55** 3457
[21] Quinn T C and Wald R M 1997 *Phys. Rev.* D **56** 3381
[22] Dixon W G 1974 *Phil. Trans. R. Soc.* A **277** 59
[23] Harte A I 2008 *Class. Quantum Grav.* **25** 205008
[24] Wald R M 1984 *General Relativity* (Chicago: University of Chicago Press)
[25] Dixon W G 1970 *Proc. R. Soc.* A **314** 499
[26] Schattner R 1979 *Gen. Rel. Grav.* **10** 377; **10** 395
[27] Harte A I 2007 *Class. Quantum Grav.* **24** 5161
[28] Bini D, Fortini P, Geralico A and Ortolan A 2008 *Class. Quantum Grav.* **25** 125007; **25** 035005
[29] Schattner R and Streubel M 1981 *Ann. de l'Inst. H. Poincaré* **34** 117
[30] Streubel M and Schattner R 1981 *Ann. de l'Inst. H. Poincaré* **34** 145
[31] Synge J L *Relativity: The General Theory* (Amsterdam: North-Holland)
[32] Friedlander F G 1975 *The Wave Equation on a Curved Space-Time* (Cambridge: Cambridge University Press)
[33] Thorne K S 1980 *Rev. Mod. Phys.* **52** 299
[34] Ehlers J and Rudolph E 1977 *Gen. Rel. Grav.* **8** 197
[35] Ehlers J 2008 *Acta Phys. Polon. B Proc. Suppl.* **1**, 123
[36] Choquet-Bruhat Y *General Relativity and the Einstein Equations* (Oxford: Oxford University Press)
[37] Barneby T A 1974 *Phys. Rev.* D **10** 1741
[38] Gralla S E, Harte A I, and Wald R M 2010 *Phys. Rev.* D **81** 104012