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Exact Superconformal and Yangian Symmetry of Scattering Amplitudes

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Abstract

We review recent progress in the understanding of symmetries for scattering amplitudes in $\mathcal{N} = 4$ superconformal Yang–Mills theory. It is summarized how the superficial breaking of superconformal symmetry by collinear anomalies and the renormalization process can be cured at tree and loop level. This is achieved by correcting the representation of the superconformal group on amplitudes. Moreover, we comment on the Yangian symmetry of scattering amplitudes and how it inherits these correction terms from the ordinary Lie algebra symmetry. Invariants under this algebra and their relation to the Graßmannian generating function for scattering amplitudes are discussed. Finally, parallel developments in $\mathcal{N} = 6$ superconformal Chern–Simons theory are summarized. This article is an invited review for a special issue of Journal of Physics A devoted to *Scattering Amplitudes in Gauge Theories*.

Contents

1 Introduction

An efficient unitarity-based construction of the scattering matrix[1] in a quantum field theory relies heavily on the concepts of locality, analyticity and symmetry. Symmetries are tremendously important because they strongly constrain the permissible building blocks from the start and guide reliably towards the desired final result. This is especially true for highly symmetric theories such as $\mathcal{N} = 4$ super Yang–Mills theory (SYM), which is believed to be integrable in the planar limit, cf. [3]. During the last few years, remarkable structures in this theory's scattering amplitudes have been discovered. Most notably, planar amplitudes display a hidden 'dual' superconformal symmetry [4–6],[2] which together with the ordinary superconformal symmetry combines into Yangian symmetry [9]. The latter is a typical feature of integrable models (see [10] for reviews), and was observed earlier in the spectral problem of the theory [11]. Commonly, the spectrum and dynamics of integrable models are strongly constrained or even completely determined by the extended symmetry. Conceivably, this is also the case for $\mathcal{N} = 4$ SYM amplitudes. For

[1] See [1, 2] within this special issue.
[2] See also [7, 8] within this special issue.

exploiting the constraints, a thorough understanding of the symmetries is indispensable. Here, we review the status of superconformal and Yangian symmetry for $\mathcal{N} = 4$ SYM scattering amplitudes. We also comment on parallel developments in three-dimensional $\mathcal{N} = 6$ super Chern–Simons (SCS) theory.

Scattering amplitudes in conformal field theories show infrared divergences. Their regularization by means of a mass scale superficially breaks conventional conformal symmetry. On the other hand, superconformal symmetry in $\mathcal{N} = 4$ SYM is expected to be exact also at the quantum level. Is it possible to reconcile the symmetry with a non-vanishing regulator that is required for a consistent formulation of scattering amplitudes? Can the symmetry breaking be assessed quantitatively, or, even better, can the broken symmetry be restored in a modified way? Interestingly, a careful study reveals that superconformal symmetry is broken already at tree level [12]. Namely, acting with a free generator on a tree-level amplitude produces residual contributions whenever two external legs become collinear. Exact superconformal invariance can be restored by introducing a non-linear correction to the generator that cancels the residual term [13]. Importantly, only the $\mathcal{N} = 4$ SYM scattering matrix as a whole is exactly invariant, its individual entries (the amplitudes) are not. While only contributing to singular momentum configurations at tree level, collinear residues become inevitable at higher orders, where loop momenta are integrated over. At one-loop order, superconformal symmetry can again be restored by further generator corrections, which cures residual contributions both from collinear terms and from infrared regularization [14, 15].

The corrections for the superconformal generators straightforwardly carry over to the Yangian symmetry of scattering amplitudes. The formally very simple Graßmannian function of [16] generates invariants of the free (uncorrected) Yangian [17].[3] In fact, it is believed to generate *all* free Yangian invariants [18]. Scattering amplitudes are linear combinations of these invariants satisfying physicality requirements such as correct collinear limits or cancellation of unphysical poles [19]. Free Yangian symmetry alone is insufficient for fixing the right linear combination. The missing piece is provided by the generator corrections: It appears that they single out the physical linear combination as the unique *exact* invariant [13], thus paving the way for an algebraic determination of loop amplitudes.

Compared to $\mathcal{N} = 4$ SYM, much less is known about scattering amplitudes in its three-dimensional cousin, $\mathcal{N} = 6$ SCS theory [20,21] (or 'ABJM' named after the authors of [21]). Both theories are surprisingly similar, and indeed, counterparts to some of the most important symmetry structures known from $\mathcal{N} = 4$ SYM amplitudes have been found in $\mathcal{N} = 6$ SCS during the last year. In particular, there is compelling evidence for Yangian and dual superconformal symmetry [22–24]. Nevertheless, several fundamental questions regarding symmetries of the S-matrix in $\mathcal{N} = 6$ SCS remain to be answered.

This work is structured as follows: We review how exact superconformal symmetry is restored at tree level in Section 2, and we also comment on the extension to loops. In Section 3, we briefly recapitulate Yangian symmetry in the context of $\mathcal{N} = 4$ SYM scattering amplitudes, and remark on the corrections to Yangian generators. The Graßman-

[3] The 'invariants' generated by the Graßmannian function are not *exact* invariants – they are only invariant under the free (undeformed) symmetry up to residual contributions at collinear momenta.

nian generating function for tree-level invariants and the implications of the generator corrections for these invariants are discussed in Section 4. Section 5 summarizes what is known about the symmetry structures of scattering amplitudes in $\mathcal{N}=6$ SCS theory.

2 Exact Superconformal Symmetry

Maximally supersymmetric Yang–Mills theory is a four-dimensional conformal field theory. It is therefore natural to assume that its S-matrix is exactly invariant under the superconformal algebra $\mathfrak{psu}(2,2|4)$. Invariance of the S-matrix is, however, not straightforward. First of all, the presence of massless particles inevitably leads to infrared divergences in scattering amplitudes at loop level. A regulator for the divergences breaks conformal symmetry, e.g. by moving away from four spacetime dimensions or by introduction of a mass scale. Only after the regulator is removed, we can hope for a restoration of conformal symmetry, but a priori there is no guarantee. In case of success, the procedure will most likely have deformed and thus obscured the action of the symmetry on the (renormalized) S-matrix.[4]

2.1 Tree Level

In fact, the situation is even more subtle than this. Let us consider a color-ordered MHV amplitude at tree level,[5]

$$A_n^{\mathrm{MHV}} = \frac{\delta^4(P)\,\delta^8(Q)}{\prod_{k=1}^n \langle k,k+1\rangle}\,, \qquad \begin{aligned} P^{b\dot a} &= \textstyle\sum_{k=1}^n p_k^{b\dot a}, \quad p_k^{b\dot a} = \lambda_k^b \tilde\lambda_k^{\dot a}, \\ Q^{bA} &= \textstyle\sum_{k=1}^n q_k^{bA}, \quad q_k^{bA} = \lambda_k^b \eta_k^A. \end{aligned} \tag{2.1}$$

We use the spinor helicity formalism to encode particle momenta and flavors: The k-th particle is described by the bosonic spinor $\lambda_k \in \mathbb{C}^2$ with complex conjugate $\tilde\lambda_k = \pm\bar\lambda_k$ (the sign determines the sign of the energy) and the fermionic spinor $\eta_k \in \mathbb{C}^{0|4}$. The two mutually conjugate Lorentz-invariant spinor products are denoted by

$$\langle\lambda,\mu\rangle = \varepsilon_{ac}\lambda^a\mu^c, \qquad [\tilde\lambda,\tilde\mu] = \varepsilon_{\dot a\dot c}\tilde\lambda^{\dot a}\tilde\mu^{\dot c}. \tag{2.2}$$

Now we act on the MHV amplitude with a free superconformal generator

$$\bar{\mathfrak{S}}_{\dot a}^B = \sum_{k=1}^n \eta_k^B \frac{\partial}{\partial\tilde\lambda_k^{\dot a}}\,. \tag{2.3}$$

Superficially, the derivative acts only on P in $\delta^4(P)$ and produces a factor of Q. The fermionic delta function $\delta^8(Q)$ makes the result vanish, i.e., the MHV amplitude is invariant under $\bar{\mathfrak{S}}$.

Interestingly, this is not the full story: There is a subtle contribution when the derivative w.r.t. $\tilde\lambda_k$ hits a pole of the (otherwise) holomorphic denominator in (2.1) [12].

[4]There is an alternative treatment of (dual) conformal symmetries for $\mathcal{N}=4$ SYM on the Coulomb branch. This is discussed in detail in [8] within this special issue. We will not comment on it here.

[5]For a more detailed introduction to the formalism, see [25–27, 1] within this special issue.

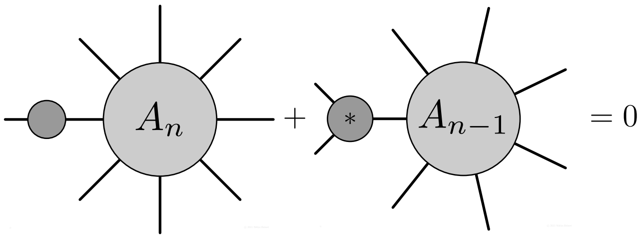

Figure 1: Deformed superconformal invariance relation for tree amplitudes.

The so-called holomorphic anomaly for the spinor product reads ($E(\lambda)$ is the energy associated to the spinor $\lambda, \tilde{\lambda}$)

$$\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \frac{1}{\langle \lambda, \mu \rangle} = 2\pi \operatorname{sign}\bigl(E(\lambda)E(\mu)\bigr) \varepsilon_{\dot{a}\dot{c}} \tilde{\mu}^{\dot{c}} \delta^2\bigl(\langle \lambda, \mu \rangle\bigr). \tag{2.4}$$

In the MHV amplitude this yields distributional contributions supported on kinematical configurations where $\langle k, k+1 \rangle = [k, k+1] = 0$. In other words, this means that invariance of the S-matrix under free superconformal transformations is violated where the momenta of two adjacent particles are collinear. The terms that break invariance can be summarized as follows [13]

$$\bar{\mathfrak{S}}^B_{\dot{a}} A_n = -\sum_{k=1}^{n} \int d^{4|4}\Lambda \, \bar{S}^B_{\dot{a}}(k, k+1, \bar{\Lambda}) \, A_{n-1}(1, \ldots, k-1, \Lambda, k+2, \ldots, n). \tag{2.5}$$

Here $\Lambda := (\lambda, \tilde{\lambda}, \eta)$ and $\bar{\Lambda} := (\lambda, -\tilde{\lambda}, -\eta)$. The integral over Λ sums over all flavors and light-like momenta in a Lorentz-invariant fashion. The kernel of superconformal violation reads

$$\begin{aligned} \bar{S}^B_{\dot{a}}(1,2,3) = &- 2\varepsilon_{\dot{a}\dot{c}} \tilde{\lambda}^{\dot{c}}_3 \int d^{4|4}\Lambda' \, \delta^4(\lambda') \, \eta'^B \int_0^{\pi/2} d\alpha \int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \, e^{i\varphi + i\vartheta} \\ &\cdot \delta^{4|4}(e^{-i\varphi} \bar{\Lambda}_3 \sin\alpha + e^{i\vartheta} \bar{\Lambda}' \cos\alpha - \Lambda_1) \\ &\cdot \delta^{4|4}(e^{-i\vartheta} \bar{\Lambda}_3 \cos\alpha - e^{i\varphi} \bar{\Lambda}' \sin\alpha - \Lambda_2) + 2 \text{ cyclic images}, \end{aligned} \tag{2.6}$$

where $z\Lambda := (z\lambda, \bar{z}\tilde{\lambda}, \bar{z}\eta)$ for $z \in \mathbb{C}$. The delta function $\delta^4(\lambda')$ enforces collinearity of all three momenta, and $\sin^2\alpha, \cos^2\alpha$ represent the momentum fractions for particles $1, 2$, respectively, in terms of particle 3.

Importantly, the free superconformal violation of A_n in (2.5) is expressed through another tree amplitude A_{n-1}. We introduce an operator $\bar{\mathfrak{S}}_+$ which attaches the kernel $\bar{S}$ to an amplitude function as in minus the r.h.s. of (2.5), cf. Figure 1. Then we can write

$$\bar{\mathfrak{S}} A_n + \bar{\mathfrak{S}}_+ A_{n-1} = 0. \tag{2.7}$$

Although individual scattering amplitudes A_n with a fixed number of external legs are not exactly conformally invariant, the full S-matrix (representing the generating functional for all amplitudes) is invariant under the deformed superconformal generator $\bar{\mathfrak{S}} + \bar{\mathfrak{S}}_+$.

Until now we have only discussed MHV amplitudes (2.1). Luckily, all of the above applies to generic N^kMHV tree amplitudes as well. The reason lies in the universality of collinear behavior: A scattering amplitude A_n diverges in the vicinity of collinear momentum configurations [28, 29]. The pole is given by the amplitude A_{n-1} with one fewer leg times a universal splitting function. Superconformal generators yield a distributional term (2.5) at these poles. The kernel $\bar{S}$ is essentially the superconformal variation of the splitting function. Note that the splitting function is more or less equivalent to the three-point function A_3 which cannot exist in Minkowski signature. In split signature, however, one can derive the kernel as the variation of the three-point function, $\bar{S} = -\frac{1}{2}\bar{\mathfrak{S}}A_3$ [15].

Similar considerations hold for the conjugate superconformal generator $\mathfrak{S}$ and the bosonic conformal generator $\mathfrak{K}$. The latter in fact receives a further correction $\mathfrak{K}_{++}$ that maps one leg to three. On the other hand, all super-Poincaré generators $\mathfrak{P}, \mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{R}$ as well as the dilatation generator $\mathfrak{D}$ are manifest symmetries of the tree level S-matrix.

2.2 Further Considerations

All in all this shows that the complete tree-level S-matrix is indeed exactly conformally invariant, but only under the interacting superconformal generator $\bar{\mathfrak{S}} + \bar{\mathfrak{S}}_+$. This observation calls for a few clarifications to be discussed in the following.

Does the above apply to gauge groups other than $\mathrm{SU}(N_\mathrm{c})$*?* The answer is affirmative: The kernel in (2.6) must be complemented with the structure constants for the gauge group. The free superconformal variation in (2.5) generalizes canonically [13].

Does it mean that superconformal symmetry is anomalous at tree level? In quantum field theory an anomaly refers to a violation of symmetry which cannot be repaired, at least not by a local deformation. Here superconformal symmetry becomes exact when the deformation is included. Moreover the deformation has no poles or cuts, it is local. It is not an anomaly, but rather a careful treatment of a non-manifest symmetry.[6]

Does the deformation alter the $\mathfrak{psu}(2,2|4)$ *superconformal algebra?* It is a proper representation, albeit of a somewhat bigger algebra [13]: First of all, the anticommutator of the superconformal generator $\mathfrak{S}$ and its conjugate $\bar{\mathfrak{S}}$ consistently defines the deformed conformal generator $\mathfrak{K}$. The only subtlety is in the anticommutators between two $\mathfrak{S}$'s or two $\bar{\mathfrak{S}}$'s: They ought to vanish for $\mathfrak{psu}(2,2|4)$, but they do not. Instead they represent a gauge variation which transforms a covariant field X according to $X \mapsto [G, X]$. Here the gauge variation parameter G is actually a field itself, namely the zero mode of the scalar field. Such a deformation of the algebra is not harmful because it vanishes for all physically meaningful, i.e. gauge invariant, observables. In fact, it is very common in gauge theories that symmetry algebras are deformed by gauge variations, e.g. the supersymmetry algebra for gauge theories with extended supersymmetry.

Is there a physical reason for the deformation? What does it mean? Notice that the violation of free superconformal symmetry occurs at collinear momentum configurations. This points at the problems encountered in scattering theory for a model without a mass gap (in particular for a CFT), see also [30]. Scattering amplitudes require a notion of asymptotic particles which do not interact further. However, nothing prevents a

[6] We thank H. Nicolai for discussions of this issue.

massless particle from decaying into two or more massless particles at any time. Lorentz invariance implies that these particles necessarily have strictly collinear momenta. In physical terms such asymptotic decays have no implications because a detector would merely measure the total deposited momentum and energy of all particles emitted in a specific direction. In other words, the Fock space for massless asymptotic particles is bigger than necessary. The physical space must be supplemented with an equivalence relation to factor out particle configurations with collinear momenta. It is reasonable to relate the deformed representation with this issue. Presumably, the deformation makes superconformal transformations compatible with the structure of representatives of the equivalence relation used for scattering amplitudes.[7]

Is there a relation to the deformations for superconformal representations on local operators? It is very analogous, and the same structures [31] are observed, cf. [32]; quite likely it is equivalent to some extent. There are, however, important differences. For local operators the representation must deal with UV divergences. These are absent for scattering amplitudes leading to simplifications. For instance, the free super-Poincaré representation is undeformed for amplitudes whereas it requires non-trivial deformations for local operators. On the other hand, scattering amplitudes have IR divergences which are absent for local operators.

Do multi-particle poles introduce further violations of free superconformal symmetry? Yes and no. The holomorphic anomaly produces a codimension-two distribution (2.4). This matches with the codimension $D-2$ of collinear configurations of two massless particles in $D=4$ Minkowski space. Multi-particle poles are always codimension-one, thus the free superconformal generators do not yield distributional terms [13]. This exhausts all singularities at tree level, nevertheless one has to be careful [14, 15]: Multi-particle poles originate from Feynman propagators $1/(p^2 \pm i\epsilon)$. The principal part $1/p^2$ is harmless as explained above, but the on-shell contribution $\pm i\pi\delta(p^2)$ requires further deformations. The holomorphic anomaly also appears when an internal momentum becomes collinear with an external one or even if two internal momenta become collinear. In a graphical representation where (2.5) (Figure 1) is given by Figure 2a,b, the additional terms take the form of Figure 2d,e [14]. This completes the analysis at tree level.

2.3 Loop Level

At tree level it is easy to ignore the distributional terms (2.5) which break invariance under free superconformal transformations. For generic configurations of the external momenta, none of the internal or external momenta are collinear. Consequently, the free superconformal generators annihilate the scattering amplitude. At loop level, the situation is different. Within the loop integrals some internal momenta inevitably become collinear with others. Thus for generic configurations of the external momenta, invariance under free superconformal transformations is broken.

To understand superconformal transformations at loop level it is important to quantify the violation terms. Unfortunately, loop integrals are off-shell, and we cannot immediately address superconformal transformations using the framework outlined above.

[7]We thank D. Skinner for discussions of this issue.

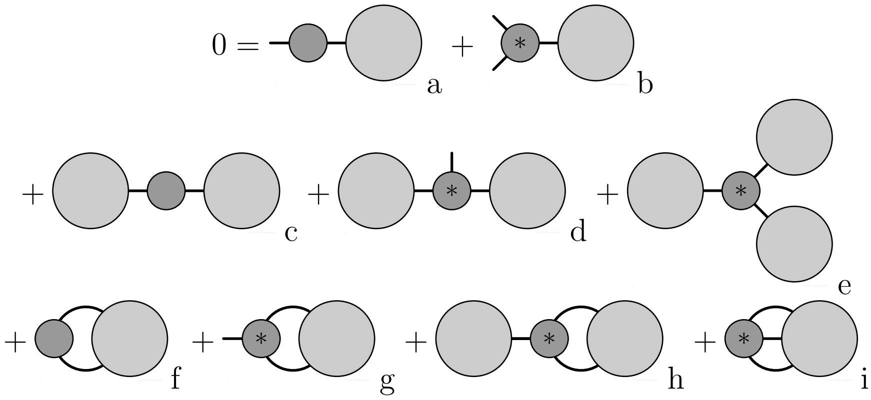

Figure 2: General superconformal invariance relation (qualitatively). Terms a–b correspond to Figure 1, terms c–e are needed for factorized amplitudes, and terms f–i are needed for loops. A big circle represents a connected scattering amplitude. The small circle represents a free conformal generator (empty) or the three-point kernel (starred).

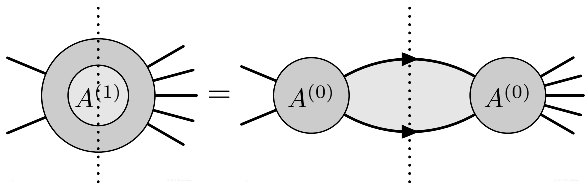

Figure 3: Essential contribution to the one-loop unitarity cut.

This problem is circumvented by considering (generalized) unitarity cuts [19, 14] which are expressed through on-shell amplitudes at lower loop orders. Superconformal transformations of cuts can thus be obtained recursively through the transformations at tree level. At the end we will have to lift the transformation rule from a cut integral to a full loop integral.

Let us thus consider a simple one-loop unitarity cut [15]. The essential contribution is written as an on-shell integral over two tree-level amplitudes, see Figure 3 (we discard various other ways to cut the amplitude). Now we can substitute the relevant superconformal transformation rule for tree amplitudes from Figure 2. Discarding the correction terms already present at tree level, there are three new terms due to the superconformal generators acting on internal legs, see Figure 4. Let us briefly discuss these terms.

The term in Figure 4a essentially represents a superconformal transformation of an

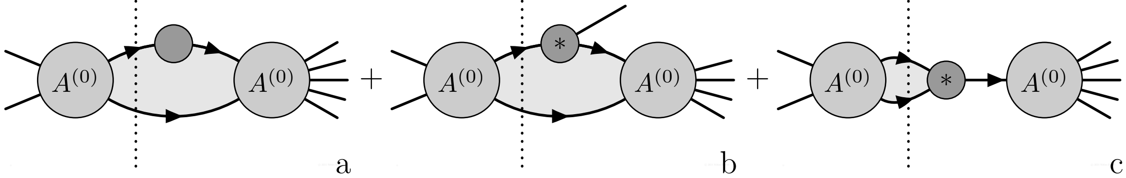

Figure 4: Essential contributions to the one-loop superconformal variation.

internal on-shell propagator. Propagators are superconformal invariants, so this contribution ought to be zero. Due to the appearance of IR divergences in the integrals, however, we have to work with regularized propagators which violate invariance by a small amount. In this case, divergences only appear when one of the two subamplitudes has four legs, and when there is no momentum transfer between the two pairs of legs. In dimensional regularization, for example, the (planar) correction to the generator $\mathfrak{D}$ of conformal rescalings reads

$$\mathfrak{D}(\lambda_{\text{YM}}) = \mathfrak{D}^{(0)} + \mathit{\Gamma}(\lambda_{\text{YM}}, \epsilon) \sum_{k=1}^{n} \mathfrak{D}^{(1)}_{k,k+1}, \qquad \mathfrak{D}^{(1)}_{k,k+1} = -\frac{1}{2\epsilon} \left(\frac{(p_k + p_{k+1})^2}{-\mu^2} \right)^{-\epsilon}. \tag{2.8}$$

For $\mathfrak{D}$ this is the only correction at loop level, and the coefficient in front

$$\mathit{\Gamma}(\lambda_{\text{YM}}, \epsilon) = \mathit{\Gamma}_{\text{cusp}}(\lambda_{\text{YM}}) + \epsilon \mathit{\Gamma}_{\text{coll}}(\lambda_{\text{YM}}) + \mathcal{O}(\epsilon^2) \tag{2.9}$$

includes the cusp dimension $\mathit{\Gamma}_{\text{cusp}}(\lambda_{\text{YM}}) = \lambda_{\text{YM}}/4\pi^2 + \mathcal{O}(\lambda^2_{\text{YM}})$ as well as the collinear dimension $\mathit{\Gamma}_{\text{coll}}(\lambda_{\text{YM}}) = \mathcal{O}(\lambda^2_{\text{YM}})$. The other superconformal generators $\mathfrak{S}, \bar{\mathfrak{S}}, \mathfrak{K}$ receive analogous IR corrections, but they also receive corrections due to the collinear behavior discussed in Section 2.1.

The term in Figure 4b quantifies the effect of superconformal transformations when internal and external legs become collinear. It has the particularly nice property that the momenta running in the triangular loop are all fixed by the on-shell conditions. Hence there is no integral to be performed and the result is rational. Removing the cut can be achieved simply by inserting an appropriate logarithm. These terms take a lengthier form.

The last term in Figure 4c is rather strange. The kernel forces the two momenta across the cut to be collinear. Consequently, the subamplitude on the other side is evaluated at two collinear external momenta, i.e. right on the pole. The problem of defining the result is directly related to the ambiguity in defining the loop correction to the splitting function, see [29]. A justifiable resolution to the problem is to discard this term.

Importantly, all three terms are expressible through combinations of tree amplitudes and a simple kernel. We have thus determined how general one-loop amplitudes transform under the free superconformal symmetries corrected by the collinear deformations discussed in Section 2.1. The deformed transformation law was verified explicitly for the example of one-loop MHV amplitudes in the dimensional reduction scheme in [15]. One can even contemplate deforming the superconformal representation further by these three terms to make amplitudes manifestly invariant.

To continue to higher loops, a promising proposal has been made in [14]. It consists in adding the terms in Figure 2g–i to the general transformation rule. The complete rule apparently respects unitarity in the sense that it appears to formally commute with taking cuts (this might require to further add the contributions in Figure 2c,f). One may therefore expect that a unitarity-based construction of loop amplitudes will respect the rule. A practical problem is that the terms Figure 2f–i suffer from the same problems as Figure 4c: the subamplitude has to be evaluated right on a singularity. Here it does not suffice to discard the result, as it also contains important finite contributions. In [14] the terms are evaluated using the CSW rules formally leading to agreement. We can also identify the three one-loop terms discussed above in the various terms in the higher-loop rule: The cusp anomalous dimension term in Figure 4a, the collinearity term in Figure 4b and the one-loop splitting term in Figure 4c represent cuts of the terms in Figure 2f,g,h, respectively.

In conclusion, higher-loop superconformal transformations can be investigated by taking unitarity cuts. The rule in Figure 2 is promising, but its evaluation in practice is subtle. A two-loop analysis would be highly desirable to settle several open questions.

3 Yangian Symmetry

In the previous Section 2 we have discussed correction terms to the free superconformal symmetry representation on scattering amplitudes in $\mathcal{N} = 4$ SYM theory. Here we review how this Lie algebra symmetry extends to an integrable structure in the planar limit.

One of the most fundamental properties of the AdS/CFT system is that the underlying symmetry of both $\mathcal{N} = 4$ SYM theory as well as type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is given by the superconformal algebra $\mathfrak{psu}(2, 2|4)$. This symmetry is realized on different observables on the two sides of the duality by respective representations of the superconformal generators. In addition to this Lie algebra symmetry, there is a T-duality that leaves the bulk action invariant and thus maps the string theory onto itself [33]. As a consequence, one may study the action of this T-self-duality on the representation of the Lie algebra symmetry for different observables. On the gauge theory side, however, a counterpart to the string T-duality is not known. This fact obscures the explicit investigation of the dual image of the Lie symmetry representation on scattering amplitudes in $\mathcal{N} = 4$ SYM theory, which is given in terms of generators $\mathfrak{J}_\alpha \in \{\mathfrak{P}, \mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{D}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{R}, \bar{\mathfrak{S}}, \mathfrak{S}, \mathfrak{K}\}$.[8] Remarkably though, planar gauge theory amplitudes reveal a second $\mathfrak{psu}(2, 2|4)$ Lie symmetry represented by $\mathfrak{j}_\alpha \in \{\mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{q}}, \mathfrak{d}, \mathfrak{l}, \bar{\mathfrak{l}}, \mathfrak{r}, \bar{\mathfrak{s}}, \mathfrak{s}, \mathfrak{k}\}$ [5]. Via the AdS/CFT duality, this second, so-called dual, superconformal symmetry is interpreted as the image of the ordinary superconformal symmetry under the string theory T-duality, cf. Figure 5. While the first symmetry corresponds to the ordinary superconformal symmetry of scattering amplitudes, its dual image can be understood as the ordinary symmetry of Wilson loops in $\mathcal{N} = 4$ SYM theory.[9]

The above implies that the Lie algebra generators $\mathfrak{J}$ on scattering amplitudes are supplemented by additional operators $\mathfrak{j}$ that furnish dual symmetries. The latter have various different roles: The operators $\{\bar{\mathfrak{q}}, \mathfrak{d}, \mathfrak{l}, \bar{\mathfrak{l}}, \mathfrak{r}, \bar{\mathfrak{s}}\}$ are identical to $\{\bar{\mathfrak{S}}, \mathfrak{D}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{R}, \bar{\mathfrak{Q}}\}$, (T-duality maps this $\mathfrak{su}(2) \times \mathfrak{su}(2|4)$ subalgebra to itself). The operators $\mathfrak{q}$ and $\mathfrak{p}$ are trivial when evaluated on amplitudes. None of the above thus implies new symmetries. Only the operators $\mathfrak{s}$ and $\mathfrak{k}$ are unrelated to the $\mathfrak{J}$'s, implying that the closure of these two superconformal algebras is bigger. To be more precise, it is a Yangian algebra [9]

[8] Here α labels the different generators of $\mathfrak{psu}(2, 2|4)$.

[9] In fact, gauge theory scattering amplitudes and Wilson loops can also be shown to map to each other [34–36] – at least in the case of MHV amplitudes and bosonic Wilson loops (cf. [37] for the supersymmetric extension). See also [38] within this special issue.

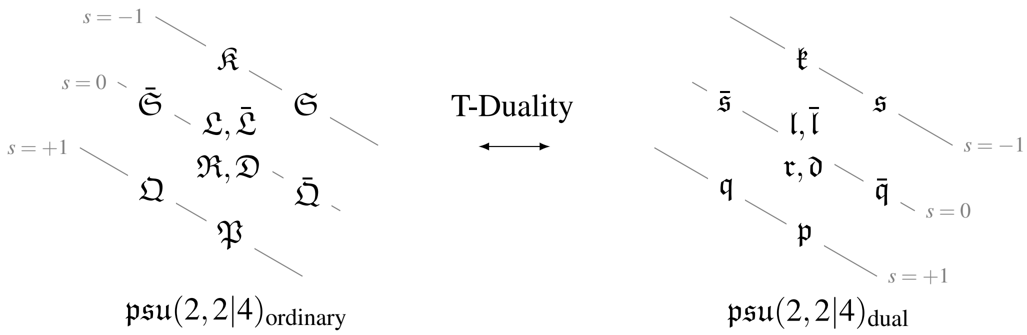

Figure 5: The ordinary representation of the superconformal algebra and its dual symmetry. Lines of constant eigenvalue under the commutator with the sum of dilatation generator and hypercharge of $\mathfrak{pu}(2,2|4)$ are indicated in gray.

generated by an infinite tower of $\mathfrak{psu}(2,2|4)$-like charges. Since in these arguments the role of amplitudes and Wilson loops as well as that of their symmetries is interchangeable, one can express the charges in either picture:

$$\{\mathfrak{J} = \mathfrak{J}^{[0]},\ \widehat{\mathfrak{J}} = \mathfrak{J}^{[1]},\ \mathfrak{J}^{[2]},\ \ldots\} \quad \simeq \quad \{\mathfrak{j} = \mathfrak{j}^{[0]},\ \widehat{\mathfrak{j}} = \mathfrak{j}^{[1]},\ \mathfrak{j}^{[2]},\ \ldots\}. \tag{3.1}$$

Here we have chosen specific names for the first two levels since these are sufficient to recursively generate the whole tower of generators via commutation, e.g. $\mathfrak{J}^{[2]} \simeq [\widehat{\mathfrak{J}}, \widehat{\mathfrak{J}}]$. The algebra underlying each of these infinite sets of generators is the so-called Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ of the superconformal group whose definition will be made precise below. It consists of an infinite number of levels whose structure will be explained in the next Section 3.1.

The T-duality can then be understood to map between the different levels of the Yangian symmetry [39], cf. Figure 6:[10]

$$\mathfrak{j}_\alpha^{[r]} \simeq \pm\mathfrak{J}_{-\alpha}^{[r+s(\alpha)]}, \qquad\qquad [\mathfrak{D} + \mathfrak{B}, \mathfrak{J}_\alpha^{[r]}] = s(\alpha)\,\mathfrak{J}_\alpha^{[r]}. \tag{3.2}$$

Here $\mathfrak{D}$ and $\mathfrak{B}$ are the dilatation generator and hypercharge of $\mathfrak{psu}(2,2|4)$, respectively. Furthermore, for the index $\alpha = [\mathfrak{P}, \mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{D}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{R}, \bar{\mathfrak{S}}, \mathfrak{S}, \mathfrak{K}]$ we define the conjugate index $-\alpha = [\mathfrak{K}, \mathfrak{S}, \bar{\mathfrak{S}}, \mathfrak{D}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{R}, \bar{\mathfrak{Q}}, \mathfrak{Q}, \mathfrak{P}]$ and the shift $s(\alpha) = [1, 1, 0, 0, 0, 0, 0, 0, -1, -1]$.

3.1 Yangian Algebra

The realization of Yangian symmetry on scattering amplitudes and Wilson loops within the AdS/CFT duality gives an astonishing example of this algebra. Its mathematical structure, however, can be formulated on a more abstract level without any notion of dual symmetry:

A Yangian algebra $\mathrm{Y}[\mathfrak{g}]$ associated with a Lie algebra $\mathfrak{g}$ is defined by two sets of generators $\mathfrak{J}_\alpha$ and $\widehat{\mathfrak{J}}_\beta$ obeying the following axioms [40]:

[10] Also here the role of $\mathfrak{J}$ and $\mathfrak{j}$ can be interchanged.

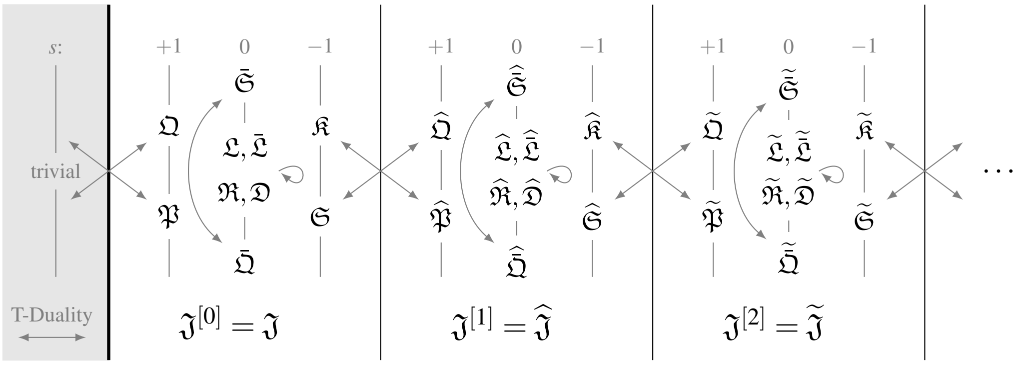

Figure 6: The level-r Yangian generators map under T-duality ($\rightarrow$) to different levels according to their weight $s = \pm 1, 0$ under the sum of the dilatation generator $\mathfrak{D}$ and the hypercharge $\mathfrak{B}$ of $\mathfrak{pu}(2,2|4)$.

$$\begin{aligned}
&\text{i) Ordinary Lie Symmetry:} && [\mathfrak{J}_\alpha, \mathfrak{J}_\beta] = f_{\alpha\beta}{}^{\gamma}\mathfrak{J}_\gamma, \\
&\text{ii) Adjoint Level-One Symmetry:} && [\mathfrak{J}_\alpha, \widehat{\mathfrak{J}}_\beta] = f_{\alpha\beta}{}^{\gamma}\widehat{\mathfrak{J}}_\gamma, \\
&\text{iii) Serre-Relations:} && [\widehat{\mathfrak{J}}_\alpha, [\widehat{\mathfrak{J}}_\beta, \mathfrak{J}_\gamma]] + \text{two cyclic} = f_{\alpha\rho}{}^{\lambda} f_{\beta\sigma}{}^{\mu} f_{\gamma\tau}{}^{\nu} f^{\rho\sigma\tau} \mathfrak{J}_{\{\lambda}\mathfrak{J}_{\mu}\mathfrak{J}_{\nu\}}.
\end{aligned} \tag{3.3}$$

Here $f_{\alpha\beta\gamma}$ denotes the structure constants of the Lie algebra $\mathfrak{g}$ spanned by the generators $\mathfrak{J}_\alpha$, and indices are raised by the Cartan–Killing form. The generalization to superalgebras is given by a straight-forward grading of these relations.

We will assume a specific (tensor product) representation for the level-zero, i.e. the standard Lie symmetry generators $\mathfrak{J}_\alpha$ acting on a tensor product of vector spaces $\mathbb{V}_k$ by (cf. (2.3)) $\mathfrak{J}_\alpha = \sum_k \mathfrak{J}_{\alpha,k}$. Based on such a representation, Drinfel'd introduced the following bilocal definition of additional generators [40]:

$$\widehat{\mathfrak{J}}_\alpha = \sum_{1\leq \ell < k \leq n} f_\alpha{}^{\gamma\beta}\, \mathfrak{J}_{\beta,\ell}\, \mathfrak{J}_{\gamma,k} + \sum_{1\leq k\leq n} u_k\, \mathfrak{J}_{\alpha,k}. \tag{3.4}$$

For many algebras and corresponding representations $\mathfrak{J}_\alpha$ – including all known occurrences within the AdS/CFT correspondence – the definition (3.4) yields generators $\widehat{\mathfrak{J}}_\alpha$ that indeed obey the axioms (3.3) and thus generate a Yangian.[11] Furthermore, in most physical applications including our current one the evaluation parameters $u_k = u$ are all equal.[12]

Let us briefly comment on the relation of the Yangian to standard integrable models without going into details. Typically integrable systems can be based on a Lax operator obeying the Yang–Baxter equation (e.g. spin chains) used to define a monodromy matrix $\mathcal{M}(u)$ as a function of the spectral parameter u. For an $\mathfrak{su}(N)$ symmetric model, the

[11] In order to prove the Serre-relations it suffices to show that the right hand side of iii) in (3.3) vanishes on one vector space of the tensor product. This is due to the fact that the Yangian is a Hopf algebra whose coproduct $\Delta : \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ is compatible with the Serre relations. (cf. [41, 22] for different proofs of the Serre-relations in the context of supersymmetric gauge theories).

[12] The value of u does not make a difference as it merely multiplies the level-zero representation, conventionally one sets $u = 0$.

Yangian levels can then be interpreted as the different orders in the expansion of this monodromy around the point where the Lax operator reduces to the identity (conventionally $u = \infty$):

$$\mathcal{M}(u = \infty) \simeq \mathbb{I} + \sum_{k=0}^{\infty} u^{-1-k} \mathfrak{J}^{[k]}. \tag{3.5}$$

Thus, the monodromy provides a way to determine the Yangian generators or, turning the logic around, the representation of a Yangian algebra allows in principle to reconstruct the monodromy of the integrable model. Such a representation at hand, one may identify an integrable structure by showing that the Yangian commutes with a theory's Hamiltonian (up to boundary terms) or that its observables are invariant under this symmetry.

Color ordered scattering amplitudes in $\mathcal{N} = 4$ SYM theory are *cyclic* functions of the external particle degrees of freedom and cyclicity is typically not compatible with Yangian symmetry. This is due to the form of the Yangian level-one generator (3.4) given by an ordered sum over pairs of particles. In order to investigate this problem, one can evaluate the difference of two generators (3.4) shifted by one site which corresponds to a one-site cyclic permutation of the amplitude's legs [9, 32]:

$$\widehat{\mathfrak{J}}^{\alpha}\left[{\textstyle\sum_{1\le\ell<k\le n}}\right] - \widehat{\mathfrak{J}}^{\alpha}\left[{\textstyle\sum_{2\le\ell<k\le n+1}}\right] = \tfrac{1}{2} f^{\alpha}{}_{\beta\gamma} f_{\delta}{}^{\beta\gamma}\, \mathfrak{J}_1^{\delta} - f^{\alpha}{}_{\beta\gamma}\, \mathfrak{J}_1^{\beta} \mathfrak{J}^{\gamma}. \tag{3.6}$$

In general, this difference does not vanish. In the case of $\mathcal{N} = 4$ SYM theory amplitudes, however, there are two further properties which lead to a well-defined Yangian despite of this cyclicity:

1. The amplitudes transform as *singlets* under the level-zero symmetry $\mathfrak{J}_{\alpha}$ of a Lie algebra $\mathfrak{g}$.

2. The Lie algebra $\mathfrak{g}$ has a *vanishing* dual Coxeter number $f^{\alpha}{}_{\beta\gamma} f_{\delta}{}^{\beta\gamma}$.

These two properties guarantee that the right hand side of (3.6) vanishes when evaluated on amplitudes and that in this special case the Yangian level-one generators are compatible with cyclicity. Consequently, the Yangian represents a well-defined symmetry of the color ordered amplitudes.

There is an additional important property of $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$: The algebra may be enhanced to $\mathfrak{pu}(2, 2|4)$ by an external automorphism or so-called hypercharge, typically denoted by the generator $\mathfrak{B}$. This generator measures the overall helicity of scattering amplitudes, i.e. amplitudes are generically not invariant under it. Remarkably, it can be shown that scattering amplitudes of $\mathcal{N} = 4$ SYM theory are invariant under the level-one generator $\widehat{\mathfrak{B}}$ associated to $\mathfrak{B}$ [42]. This bilocal generator together with the ordinary superconformal symmetry yields all previously known symmetries (e.g. the dual symmetry) of scattering amplitudes in $\mathcal{N} = 4$ SYM theory.[13]

[13] The symmetry algebra $\mathfrak{g} = \mathfrak{osp}(6|4)$ of $\mathcal{N} = 6$ superconformal Chern–Simons theory (to be discussed in Section 5 below) is not enhanceable by an external automorphism. This reflects the fact that helicity is absent in the three-dimensional theory. Understanding the algebraic difference between the symmetries of $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 6$ SCS theory might eventually resolve the problems to formulate a T-self-duality for the string dual of the latter gauge theory and to put the discovered dual symmetry in three dimensions on solid grounds. Note in this context that the role of the generator $\mathfrak{B}$ in $\mathcal{N} = 4$ SYM theory shows formal similarities to the trace of the R-symmetry in $\mathcal{N} = 6$ SCS theory.

3.2 Dual versus Yangian Symmetry at Tree-Level

It is instructive to see the explicit relation between dual conformal and Yangian level-one generators [9]. Let us therefore consider the example of the dual conformal boost. On the full superspace spanned by ordinary supersymmetric spinor $(\lambda, \tilde{\lambda}, \eta)$ and dual (x, θ) coordinates, it takes the form

$$\mathfrak{k}^{a\dot{a}} = \sum_{i=1}^{n} \left(x_i^{a\dot{b}} x_i^{\dot{a}b} \frac{\partial}{\partial x_i^{b\dot{b}}} + x_i^{\dot{a}b} \theta_i^{aB} \frac{\partial}{\partial \theta_i^{\beta B}} + x_i^{\dot{a}b} \lambda_i^{a} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{a\dot{b}} \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{b}}} + \tilde{\lambda}_i^{\dot{a}} \theta_{i+1}^{aB} \frac{\partial}{\partial \eta_i^{B}} \right). \quad (3.7)$$

Here the dual coordinates providing the natural variables of Wilson loops in $\mathcal{N} = 4$ SYM theory are defined by

$$x_i^{a\dot{a}} - x_{i+1}^{a\dot{a}} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}, \qquad \theta_i^{aA} - \theta_{i+1}^{aA} = \lambda_i^a \eta_i^A. \quad (3.8)$$

The form of (3.7) on the full superspace results from requiring that the generator commutes with the constraints (3.8). Scattering amplitudes transform covariantly under the action of the dual conformal boost generator, i.e. $\mathfrak{k}^{a\dot{a}} A_n = -\sum_{i=1}^n x_i^{a\dot{a}} A_n$, which motivates the redefinition

$$\tilde{\mathfrak{k}}^{a\dot{a}} = \mathfrak{k}^{a\dot{a}} + \sum_{i=1}^{n} x_i^{a\dot{a}}. \quad (3.9)$$

Acting on amplitudes such that we can neglect terms annihilating A_n, this operator can be rewritten as the level-one Yangian generator $\widehat{\mathfrak{P}}^{a\dot{a}}$, whose form follows from the definition (3.4):

$$\tilde{\mathfrak{k}}^{a\dot{a}}\Big|_{A_n} = \widehat{\mathfrak{P}}^{a\dot{a}} = \sum_{\ell<k} \left[\mathfrak{P}_{\ell,c\dot{c}} (\mathfrak{L}^{c}_{k,a} \delta^{\dot{c}}_{\dot{a}} + \bar{\mathfrak{L}}^{\dot{c}}_{k,\dot{a}} \delta^{c}_{a} - \mathfrak{D}_k \delta^c_a \delta^{\dot{c}}_{\dot{a}}) - \mathfrak{Q}^{C}_{\ell,a} \bar{\mathfrak{Q}}_{k,\dot{a}C} - (\ell \leftrightarrow k) \right]. \quad (3.10)$$

An analogous relation holds for $\mathfrak{s}_a^A$ and $\widehat{\mathfrak{Q}}_a^A$ while all other dual conformal generators can be related to the level-zero symmetry. Note that the bilocality in (3.7) is hidden in the definition of the dual variables (3.8). In the Wilson loop picture, $\mathfrak{k}$ reduces to the ordinary conformal boost in coordinates x and θ and thus to the level-zero dual symmetry.

Invariance of tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM theory under Yangian symmetry can then be seen in two ways: On the one hand, tree-level amplitudes can be written in terms of manifestly dual superconformal invariant expressions making this property obvious with regard to the above relations, cf. Section 4. On the other hand one may in principle explicitly apply the simplest level-one generator $\widehat{\mathfrak{P}}$ as given in (3.10) to the amplitudes and show invariance as done in [32] for the MHV case. The adjoint property ii) of the Yangian (3.3) then guarantees invariance under the full algebra.

3.3 Corrections to Yangian Generators

As discussed in the previous sections, symmetry generators acting on scattering amplitudes in $\mathcal{N} = 4$ SYM theory are affected by singularities. These require corrections to the generators in order to render the symmetry exact. The correction terms have to take into account the holomorphic anomaly starting at tree-level as well as infrared singularities starting at one-loop order. They also affect the level-one Yangian symmetry as will be indicated here:

Two collinear massless particles are not distinguishable in a conformal theory. At tree-level, this manifests itself in the occurrence of collinear singularities of the amplitudes which violate their invariance under the free conformal symmetry. As a consequence, the conformal generators $\mathfrak{S}$, $\bar{\mathfrak{S}}$ and $\mathfrak{K}$ of the ordinary superconformal symmetry acquire correction terms on the subspace of two-particle collinearities as shown above [13]. At tree-level, these are the only correction terms of the level-zero generators. In particular, the tree-level generator of anomalous dimensions $\mathfrak{D}$ does not obtain corrections. The level-one symmetry inherits the correction terms from the conformal level-zero generators via its bilocal definition (3.4). This allows to explicitly determine all level-one corrections at tree level. As an example, the level-one generator $\widehat{\mathfrak{P}}$ (3.10) obtains no tree-level correction since it does not depend on $\mathfrak{S}$, $\bar{\mathfrak{S}}$ or $\mathfrak{K}$.

Then, for instance, the level-one tree-level correction $\widehat{\mathfrak{Q}}_+$ to $\widehat{\mathfrak{Q}}$, can be written as a commutator of the form

$$\delta^{\dot{a}}_{\dot{b}}\, \widehat{\mathfrak{Q}}^{aA}_{+} = [\widehat{\mathfrak{P}}^{a\dot{a}}, \bar{\mathfrak{S}}^{A}_{+,\dot{b}}]. \tag{3.11}$$

Provided the adjoint property ii) of the Yangian can be proved, all other level-one generators – including their corrections – could be obtained by commutation of $\widehat{\mathfrak{P}}$ with the level-zero symmetry. Note that this generically yields bilocal operators that change the number of external particles of the amplitude.

At loop order, conformal symmetry is typically broken by the renormalization scheme, e.g. by dimensional regularization which introduces a mass scale μ and a regularization parameter ϵ. In order to render conformal symmetry exact, these parameters can be included into correction terms to the level-zero symmetry. At one loop order, all four conformal generators $(\mathfrak{S}, \bar{\mathfrak{S}}, \mathfrak{K}, \mathfrak{D})$ obtain such corrections as demonstrated in Section 2 [15]. The loop corrections to the level-one symmetry require local terms reminiscent of those multiplied by the u_k's in (3.4). At one loop order they take the perturbative form[14]

$$\widehat{\mathfrak{J}}^{(1)}_{\alpha} = \sum_{1\le \ell<k\le n} f_{\alpha}{}^{\gamma\beta} \left(\mathfrak{J}^{(1)}_{\beta,\ell}\,\mathfrak{J}^{(0)}_{\gamma,k} + \mathfrak{J}^{(0)}_{\beta,\ell}\,\mathfrak{J}^{(1)}_{\gamma,k}\right) + \sum_{1\le k\le n} \widehat{\mathfrak{J}}^{(1)}_{\alpha,k}. \tag{3.12}$$

Let us again consider the simplest level-one generator $\widehat{\mathfrak{P}}^{(1)}$. Its form can be obtained by acting with $\widehat{\mathfrak{P}}^{(0)}$ onto the one-loop amplitude and requiring invariance $\widehat{\mathfrak{P}}^{(0)}A^{(1)} + \widehat{\mathfrak{P}}^{(1)}A^{(0)} = 0$ [15]:

$$\big(\widehat{\mathfrak{P}}^{(1)}\big)^{a\dot{a}} = \sum_{1\le \ell<k\le n} \big[\mathfrak{D}^{(1)}_{\ell,\ell+1}\mathfrak{P}^{a\dot{a}}_{k} - \mathfrak{P}^{a\dot{a}}_{\ell}\mathfrak{D}^{(1)}_{k-1,k}\big]. \tag{3.13}$$

Here the nontrivial contribution comes from the one-loop correction to the dilatation generator (2.8). In fact, it is well-known that the dual conformal boost $\tilde{\mathfrak{k}}$ alias $\widehat{\mathfrak{P}}$ is anomalous at loop level [35,5,43]. The conjectured all loop form for the former allows to derive a similar expression for its Yangian level-one counterpart in analogy to (2.8) [15]:

$$\widehat{\mathfrak{P}}(\lambda_{\text{YM}})^{a\dot{a}} = (\widehat{\mathfrak{P}}^{(0)})^{a\dot{a}} + \Gamma(\lambda_{\text{YM}}, \epsilon)(\widehat{\mathfrak{P}}^{(1)})^{a\dot{a}}. \tag{3.14}$$

[14] As the corrections do not act on single legs there is no canonical prescription for the summation bounds. The local term $\widehat{\mathfrak{J}}^{(1)}_{\alpha,k}$ thus depends on the prescription and specifies the action at the bounds.

This equation including (2.9) is conjectured to guarantee invariance to all loop orders.

The most urgent question concerning the corrected Yangian generators is whether and how the axioms i), ii) and iii) in (3.3) are compatible with the correction terms to the generators. In Section 2.2 it was already indicated that at tree-level the corrections to the Lie algebra symmetry modify axiom i) by gauge transformations.

4 Invariants & Graßmannian

In integrable models, physical quantities are commonly severely constrained or even fully determined by the enlarged symmetry. Thus one may hope that the Yangian symmetry allows to express all $\mathcal{N} = 4$ SYM scattering amplitudes in terms of a finite set of algebraic, differential, or integral equations. In this section, we review the Yangian invariance properties of scattering amplitudes, and comment on the implications of the deformation. The presentation mostly focuses on the tree-level case.

4.1 Free Invariants: The Graßmannian Formula

In the following, we discuss invariants of the free, undeformed symmetries. These are not exact invariants (see Section 2), but we ignore this fact for the moment, and discuss the deformation and its exact invariants in Section 4.2 below. Tree-level amplitudes $A_{n,k+2}$ are linear combinations of terms $R_{n,k,a}$ which are individually (almost) invariant under both ordinary and dual superconformal symmetry, and hence also under Yangian symmetry [5, 6, 44, 45]

$$A_{n,k+2} = A_n^{\mathrm{MHV}} \sum\nolimits_a R_{n,k,a}\,. \tag{4.1}$$

Here, k specifies the degree $4k$ of polynomials in the fermionic variables or equivalently the helicity $h = n - 2(k+2)$ of the amplitude. The MHV prefactor A_n^{MHV} is Yangian invariant by itself. The tree-level dual superconformal invariants $R_{n,k,a}$ were constructed recursively in [45].[15] A generating function for all these invariants was given in [16],[16] which takes a surprisingly compact form. It can be written as [47]

$$\mathcal{R}_{n,k}(\gamma; \mathcal{W}) = \int_\gamma \frac{d\nu(t)}{M_1 \cdots M_n}\, \delta^{4k|4k}(t \cdot \mathcal{W})\,, \tag{4.2}$$

where t is a complex $k \times n$ matrix, the dot denotes matrix multiplication, and $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_n)^{\mathrm{T}}$ are momentum-twistor variables as introduced in [48]:

$$\mathcal{W}_i^{\mathcal{A}} = (\lambda_i^a, \mu_i^{\dot a}, \chi_i^A)\,, \qquad \mu_i^{\dot a} = x_i^{a\dot a}\lambda_{ia}\,, \qquad \chi_i^A = \theta_i^{aA}\lambda_{ia}\,. \tag{4.3}$$

These are the twistors associated to the dual variables (or *region momenta*) x_i, θ_i defined in (3.8). The symbols M_i in the denominator denote minors of the matrix t made of k successive columns, starting with column i. The integration measure $d\nu(t)$ was given explicitly in [47]. It has degree $k(n-k)$, and turns the function into a multi-dimensional

[15] See also [7] within this special issue.

[16] See also [46, 7] within this special issue.

complex contour integral. Different invariants $R_{n,k,a} = \mathcal{R}_{n,k}(\gamma_a)$ are generated by distinct contours γ_a encircling different residues. Including the measure, the integrand is invariant under local $\mathrm{GL}(k)$ "gauge" transformations acting on the rows of the matrix t.[17] The space being integrated over thus is the *Graßmannian* $\mathrm{Gr}(k,n)$ consisting of all k-planes within $\mathbb{C}^n$, where each plane is spanned by the k rows of t. For practical purposes, a gauge can be fixed by setting k^2 components of t to specific values. A convenient gauge fixes the first k columns of t to the identity matrix

$$t = (1|\cdot)\,, \qquad d\nu(t) = \prod_{a=1}^{k}\prod_{i=k+1}^{n} dt_{ai}\,. \tag{4.4}$$

The benefit of using momentum twistors is that the dual superconformal generators are realized linearly in these variables,

$$\mathrm{j}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i=1}^{n} \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}\,. \tag{4.5}$$

Invariance under these generators is ensured by the delta function in (4.2). It has been shown in [17] that taking these dual generators as the level-zero algebra results in the same Yangian as taking the ordinary superconformal symmetry as level-zero generators, which is a consequence of the T-self-duality mentioned in Section 3. The level-one Yangian generators take the usual form (3.4) of bilocal combinations of (dual) level-zero generators,

$$\widehat{\mathrm{j}}^{\mathcal{A}}{}_{\mathcal{B}} = \Big(\sum_{i<j} - \sum_{j<i}\Big)(-1)^{\mathcal{C}}\, \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}}\,. \tag{4.6}$$

Closely following [17], we will now show that the function (4.2) is indeed invariant under the level-one generators and thus under the whole Yangian algebra. It is sufficient to show invariance under the first sum in (4.6), as invariance under the second sum is completely analogous. The first sum can be expressed as

$$\sum_{i<j}\Big(\mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} - \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}\Big)\,. \tag{4.7}$$

Due to the linearity of the delta function's argument, the twistorial operators $\mathcal{W}_j^{\mathcal{C}}\, \partial/\partial\mathcal{W}_i^{\mathcal{C}}$ can be replaced by operators O_{ij} acting on the integration variables t_{ai}.[18] The action of the level-one generators on $\mathcal{R}_{n,k}$ becomes

$$\widehat{\mathrm{j}}^{\mathcal{A}}{}_{\mathcal{B}}\, \mathcal{R}_{n,k} = \int \frac{d\nu(t)}{M_1 \cdots M_n} \sum_{a=1}^{k} (O_a^{\mathcal{A}} - V_a^{\mathcal{A}})\partial_{a\mathcal{B}} \delta^{4k|4k}(t\cdot\mathcal{W})\,, \tag{4.8}$$

[17] This is the reason for the degree of the naive integration measure $d^{k\cdot n}t$ being reduced to $k(n-k)$; otherwise, the integral would be ill-defined.

[18] In the gauge (4.4), the operators are $O_{i,j\le k} = -\sum_{l=k+1}^{n} t_{jl} \frac{\partial}{\partial t_{il}}$ and $O_{i,j>k} = \sum_{b=1}^{k} t_{bi} \frac{\partial}{\partial t_{bj}}$. While the form of O_{ij} for $j>k$ is derived straightforwardly, one needs to make use of the delta function constraints to arrive at the form for $j \le k$.

where

$$O_a^{\mathcal{A}} = \sum_{i<j} \mathcal{W}_i^{\mathcal{A}} O_{ij} t_{aj} \,, \qquad V_a^{\mathcal{A}} = \sum_{i<j} \mathcal{W}_i t_{ai} \,, \tag{4.9}$$

and $\partial_{a\mathcal{B}} = \partial/\partial\mathcal{W}_a^{\mathcal{B}}$. Making use of the triangular form of $O_a^{\mathcal{A}}$, one can show that the $V_a^{\mathcal{A}}$-term cancels when commuting $O_a^{\mathcal{A}}$ past the minors, $[1/M_1 \cdots M_n, O_a^{\mathcal{A}}] = V_a^{\mathcal{A}}/M_1 \cdots M_n$. Thus

$$\widehat{\mathrm{j}}^{\mathcal{A}}{}_{\mathcal{B}}\,\mathcal{R}_{n,k} = \int d\nu(t) \sum_{a=1}^{k} O_a^{\mathcal{A}} \frac{1}{M_1 \cdots M_n} \partial_{a\mathcal{B}} \delta^{4k|4k}(t \cdot \mathcal{W}) \,. \tag{4.10}$$

Now each term in the integrand is a total derivative of a single-valued function in one of the integration variables, hence the integral along any closed contour vanishes.[19] This shows that $\mathcal{R}_{n,k}$ is indeed Yangian invariant.

The function $\mathcal{R}_{n,k}$ (4.2) in fact produces all Yangian invariant terms $R_{n,k,a}$ that, multiplied by the n-point MHV amplitude, form the planar n-point N^{k}MHV amplitude [49]. It is equivalent [50] to the previously proposed [16] generating function $\mathcal{A}_{n,k}(\mathcal{Z})$, which generates planar tree-level amplitudes including the MHV prefactor. Formally, the relation between the two functions is simply

$$\mathcal{A}_{n,k}(\mathcal{Z}) = A_n^{\mathrm{MHV}} \mathcal{R}_{n,k}(\mathcal{W}) \,, \tag{4.11}$$

where $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ are ordinary spacetime twistors $\mathcal{Z}_i^{\mathcal{A}} = (\partial/\partial\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}, \eta_i^A)$ as opposed to momentum twistors.[20] The MHV amplitudes A^{MHV} are Yangian invariant on their own.

It has been argued that the Graßmannian integral $\mathcal{R}_{n,k}$ (4.2) in fact generates *all* invariants of the free Yangian symmetry [18]. Assuming that all invariants of $\mathfrak{psu}(2,2|4)$ in the representation (4.5) are of the form $\delta^{4k|4k}(t \cdot W)$, the most general $\mathfrak{psu}(2,2|4)$ invariant is exactly given by $\mathcal{R}_{n,k}$, except for the integration measure being generalized by an arbitrary function $f(t)$ of the integration variables. Requiring invariance under the level-one generators (4.6), constraints on the function $f(t)$ are derived in [18]. Under certain assumptions, the only remaining solution is a constant.

4.2 Exact Invariants

So far, we have discussed invariants of the free, undeformed Yangian symmetry. Physical scattering amplitudes are linear combinations of these free invariants. On their own, the free Yangian invariants have no local interpretation. They have unphysical 'spurious' singularities, and a wrong behavior in collinear limits. While the free Yangian symmetry determines amplitudes to a large extent, it puts no constraints on the coefficients of the physical linear combination. On the other hand, if $\mathcal{N} = 4$ SYM is an integrable theory, one would expect all dynamical quantities to be completely determined by the extended symmetry. The deformations introduced in Section 2 exactly appear to provide

[19] This argument relies on the integration measure being a gauge-invariant generalization of the standard measure (4.4) [47].

[20] $\partial/\partial\lambda$ indicates a Fourier transform w.r.t. λ. Formally this required that λ and $\tilde{\lambda}$ are unrelated as in split spacetime signature $(2,2)$.

the missing piece. Namely, under mild assumptions, the coefficients of the physical linear combination appear to be uniquely fixed by requiring the correct behavior in collinear limits (or, alternatively, the cancellation of all spurious poles) [19].[21] As the interaction terms in the deformed superconformal and Yangian generators impose precisely the correct collinear behavior, it is plausible that only the physical linear combinations form invariants of the full (deformed) classical Yangian.

Of course, the correct coefficients for all tree-level amplitudes are known explicitly [45]. Nevertheless, the extent to which the symmetries determine the amplitudes is an important question. In particular, a unique invariant at tree level is essential for a complete algebraic determination of loop-level amplitudes. Namely, tree-level invariants form the space of homogeneous solutions to the invariance equations at loop level. Thus, they can be freely added to loop-level invariants.[22] Hence, if there would be multiple tree-level invariants, loop-level amplitudes could not be determined uniquely.

4.3 Loop Level

At loop level, infrared divergences obscure the symmetry properties of scattering amplitudes. For instance, free Yangian symmetry is broken due to (dimensional) regularization. However, loop amplitudes in $\mathcal{N} = 4$ SYM are to a large extent determined by their singularities. The higher their codimension, the less are these singularities affected by infrared divergences. In particular, the "leading singularities" with maximal codimension localize all loop integrals; they can be expressed entirely in terms of tree-level amplitudes and do not require regularization, which makes them especially accessible. In fact, it is conjectured that the function $\mathcal{A}_{n,k}$ (4.11) besides all tree-level amplitudes generates all leading singularities to all orders in planar perturbation theory, which shows that these are invariant under the free Yangian [16].[23] Recently, this invariance has been generalized to the complete planar all-loop integrand in a manifest way by means of new recursion relations [51]. For the integrated (infrared divergent) amplitudes, exact Yangian symmetry can be restored at one-loop order by appropriate corrections [15], see Section 2.3.

5 Symmetries of ABJM Amplitudes

Recently, a superconformal gauge theory in three dimensions ($\mathcal{N} = 6$ SCS, ABJM) was found [20, 21] which bears remarkable similarities to four-dimensional $\mathcal{N} = 4$ SYM. In particular, its planar spectrum of local operators at weak coupling is described by an integrable spin chain (see [52] for a review). Compared to $\mathcal{N} = 4$ SYM, however, much less is known about scattering amplitudes in its three-dimensional cousin. Nevertheless, counterparts to some of the most important symmetry structures known from $\mathcal{N} = 4$

[21] The authors of [19] show that requiring correct collinear limits is sufficient for determining NMHV amplitudes.

[22] Adding the physical tree-level amplitude can be compensated by rescaling the coupling constant and the overall coefficient, both of which cannot be determined algebraically in any case.

[23] Up to contributions from collinear momenta, of course.

SYM amplitudes have been found for the three-dimensional theory during the last year. First, the four- and six-point tree-level amplitudes of $\mathcal{N} = 6$ SCS were shown to be invariant under a Yangian symmetry algebra [22]. Subsequently, a Graßmannian formula for all tree-level amplitudes [53] as well as a dual superconformal symmetry [23] were proposed. On-shell recursion relations à la BCFW [54] for all $\mathcal{N} = 6$ SCS tree-level amplitudes were presented in [24], and were used to inductively demonstrate their dual superconformal alias Yangian invariance.

Also in this theory, amplitudes can be formulated in terms of a superfield

$$\Phi = \phi^4 + \eta^A \psi_A + \tfrac{1}{2}\varepsilon_{ABC}\eta^A\eta^B\phi^C + \tfrac{1}{6}\varepsilon_{ABC}\eta^A\eta^B\eta^C\psi_4 \,, \tag{5.1}$$

which, together with its conjugate $\bar{\Phi}$, captures all on-shell dynamical degrees of freedom (eight scalars $\phi^A, \bar{\phi}_A$ and eight fermions $\psi_A, \bar{\psi}^A$). The superconformal algebra $\mathfrak{osp}(6|4)$ in three dimensions is realized in terms of the fermionic $\mathfrak{u}(3)$ spinor η^A and the real two-component spacetime spinor λ^a, which parametrizes a three-dimensional momentum as $p^{ab} = \lambda^a\lambda^b$ [55]. On scattering amplitudes $A(\Phi_1, \ldots, \Phi_n)$, the superconformal generators $\mathfrak{J}_\alpha \in \mathfrak{osp}(6|4)$ act locally, while the Yangian level-one generators $\widehat{\mathfrak{J}}_\alpha$ take the usual bilocal form (3.4):[24]

$$\mathfrak{J}_\alpha = \sum_{1\le k\le n} \mathfrak{J}_{\alpha,k} \,, \qquad \widehat{\mathfrak{J}}_\alpha = f_\alpha{}^{\gamma\beta} \sum_{1\le j<k\le n} \mathfrak{J}_{\beta,j}\,\mathfrak{J}_{\gamma,k} \,. \tag{5.2}$$

Here, the generator $\mathfrak{J}_k$ acts only on the coordinates of the k'th leg Φ_k. Interestingly, the R-symmetry is broken by the superfield (5.1) to a manifest $\mathfrak{u}(3)$ and a non-manifest remainder:

$$\mathfrak{R}^{AB} = \eta^A\eta^B \,, \qquad \mathfrak{R}^A{}_B = \eta^A \frac{\partial}{\partial\eta^B} \,, \qquad \mathfrak{R}_{AB} = \frac{\partial}{\partial\eta^A}\frac{\partial}{\partial\eta^B} \,. \tag{5.3}$$

In particular, the $\mathfrak{u}(3)$ contains a non-vanishing trace $\mathfrak{R}^C{}_C = \eta^C\,\partial/\partial\eta^C - 3/2$, which enforces scattering amplitudes to be of homogeneous degree $A_n \sim (\eta)^{3n/2}$ in the fermionic variables. This implies that there are no "MHV-like" amplitudes with a minimal degree in the fermionic variables. The four- and six-point tree-level amplitudes have been computed[25] and shown to be invariant under the level-one momentum generator $\widehat{\mathfrak{P}}$ [22]. Invariance under all other Yangian generators follows by commutation with level-zero generators. For consistency of the Yangian, the Serre relations (3.3) have to be satisfied. While difficult to show in general, a rather direct proof [22] utilizes the fact that the level-zero generators form a singleton representation (as in four dimensions) that can be formulated in terms of spinor-helicity variables.

Scattering amplitudes for higher numbers of legs are hard to compute, even at tree level. However, a generating function for all $\mathcal{N} = 6$ SCS tree-level amplitudes similar to (4.2) has been proposed in [53]. In spinor-helicity variables $\Lambda = (\lambda, \eta)$, it takes the form

$$\mathcal{A}_{2k}(\gamma;\Lambda) = \int_\gamma \frac{d\nu(t)}{M_1\cdots M_k}\,\delta^{k(k+1)/2}(t\cdot t^{\mathrm{T}})\,\delta^{2k|3k}(t\cdot\Lambda)\,. \tag{5.4}$$

[24]This definition is compatible with the cyclicity of scattering amplitudes because the dual Coxeter number of $\mathfrak{osp}(6|4)$ vanishes, see also (3.6) above.

[25]Four-point amplitudes of the mass-deformed theory had been studied before in [56].

Here, t is a $(k \times 2k)$ matrix, and the minors M_j are defined as before. In the four-dimensional case (4.2), the domain of integration was the Graßmannian $\mathrm{Gr}(k,n)$, the space of all k-planes in $\mathbb{C}^n$. Here, the additional delta function enforces the scalar product to vanish on the k-plane spanned by the rows of t, which restricts the domain of integration to the *orthogonal Graßmannian* $\mathrm{OGr}(k,2k)$, see [24]. Again, all terms contributing to the $2k$-point tree-level superamplitude are conjectured to be generated by $\mathcal{A}_{2k}$ evaluated on different integration contours γ. This has been verified for the four-point [53] and the six-point amplitude [24]. Moreover, the integral (5.4) is Yangian invariant [53], which, assuming Yangian symmetry for scattering amplitudes, is a strong hint for its correctness.

The discovery of Yangian symmetry made the authors of [23] formulate a dual superconformal symmetry for $\mathcal{N}=6$ SCS amplitudes, as found earlier for $\mathcal{N}=4$ SYM. By going to dual variables x_j^{ab} with

$$\lambda_j^a \lambda_j^b = x_j^{ab} - x_{j+1}^{ab} \tag{5.5}$$

exactly as in four dimensions, the proposed dual conformal symmetry (no super yet) acts on the dual variables x^{ab} in the same way the ordinary conformal symmetry acts on spacetime. As all amplitudes only depend on differences of dual x_j variables, they are trivially invariant under dual translations $\mathfrak{p}_{ab} = \sum_j \partial/\partial x_j^{ab}$. Provided the scattering amplitudes scale as

$$A_{2k} \xrightarrow{\mathrm{I}^{\mathrm{dual}}} \sqrt{\prod\nolimits_{j=1}^{2k} x_j^2}\, A_{2k}\,, \tag{5.6}$$

under dual inversions $\mathrm{I}^{\mathrm{dual}}$, they also transform covariantly under dual special conformal transformations $\mathfrak{k}$,

$$\mathfrak{k}_{ab}\, A_{2k} = \mathrm{I}^{\mathrm{dual}} \mathfrak{p}_{ab} \mathrm{I}^{\mathrm{dual}}\, A_{2k} = -\frac{1}{2}\left(\sum_{j=1}^{2k} x_{j,ab}\right) A_{2k}\,. \tag{5.7}$$

The dual conformal symmetry algebra is completed by Lorentz generators $\mathfrak{l} = \mathfrak{L}$ and the dilatation generator $\mathfrak{d} = \mathfrak{D}$, which are equal to the corresponding generators of the ordinary conformal symmetry.

Trying to extend the dual conformal to dual *super*conformal symmetry, one encounters an important difference to the four-dimensional case. Namely, besides the fermionic variables θ_j^{aA} as known from $\mathcal{N}=4$ SYM, another set of dual variables y_j^{AB} is required for formulating the full dual symmetry. Here,

$$\lambda_j^a \eta_j^A = \theta_j^{aA} - \theta_{j+1}^{aA}\,, \qquad \eta_j^A \eta_j^B = y_j^{AB} - y_{j+1}^{AB}\,. \tag{5.8}$$

Specifically, it is impossible to consistently express the action of some of the dual generators on the original variables (λ, η) without also using the additional variables y.[26]

[26] More precisely, the dual generators cannot be formulated on the "full space" of *independent* variables $(\lambda, \eta, x, \theta)$ while preserving the hypersurface constraints (5.5,5.8) without also using the additional variables y. This formulation is needed though for finding the action of the dual generators on the original variables (λ, η), and for studying their relation to the ordinary symmetry generators.

The presence of a dual superconformal symmetry hints at a scattering amplitude / Wilson loop duality like in $\mathcal{N}=4$ SYM. Light-like Wilson loops were studied and successfully compared to the tree-level[27] four-point scattering amplitude in [57].[28]

As in four dimensions, some of the dual generators $\mathfrak{j}$ are trivial, others are identical to their ordinary-symmetry counterparts, and some are equal to level-one Yangian generators $\widehat{\mathfrak{J}}$ when acting on invariants of the ordinary conformal symmetry [23]:

$$\begin{aligned} (\mathfrak{p}_{ab}, \mathfrak{q}_{ab}, \mathfrak{r}_{AB}) &= \text{trivial}\,, \\ (\mathfrak{l}^a{}_b, \mathfrak{d}, \mathfrak{r}^A{}_B, \mathfrak{q}^A_a, \mathfrak{s}^a_A) &= (\mathfrak{L}^a{}_b, \mathfrak{D}, \mathfrak{R}^A{}_B, \mathfrak{S}^A_a, \mathfrak{Q}^a_A)\,, \\ (\mathfrak{k}^{ab}, \mathfrak{s}^{aA}, \mathfrak{r}^{AB}) &\simeq (\widehat{\mathfrak{P}}^{ab}, \widehat{\mathfrak{Q}}^{aA}, \widehat{\mathfrak{R}}^{AB})\,. \end{aligned} \tag{5.9}$$

Invariance under the full dual superconformal symmetry thus follows from invariance under the ordinary symmetry and under the dual generator $\mathfrak{k} \simeq \widehat{\mathfrak{P}}$, for instance. Furthermore, the dual and the ordinary symmetry together generate the whole Yangian algebra $\mathrm{Y}[\mathfrak{osp}(6|4)]$.

In recent years, a key tool for the investigation of scattering amplitudes in four dimensions have been the on-shell 'BCFW' recursion relations [54].[29] A few months ago, similar relations were found for $\mathcal{N}=6$ SCS scattering amplitudes in three dimensions [24]. Unlike their four-dimensional counterpart, the three-dimensional recursion relations require shifting two external momenta non-linearly in the auxiliary complex variable z. Namely,

$$\lambda_j \to +\tfrac{1}{2}(z+1/z)\lambda_j + \tfrac{i}{2}(z-1/z)\lambda_k\,, \tag{5.10}$$

$$\lambda_k \to -\tfrac{i}{2}(z-1/z)\lambda_j + \tfrac{1}{2}(z+1/z)\lambda_k\,. \tag{5.11}$$

Using the recursion relations, the scaling (5.6) under dual inversions was proved inductively, thus establishing dual superconformal alias Yangian invariance for all tree-level amplitudes. Furthermore, the amplitudes obtained by recursion were successfully matched against the Graßmannian formula (5.4) for up to eight external particles.

The dual superconformal symmetry is particularly surprising because to date no supersymmetric T-self-duality of the AdS/CFT dual sigma model [59] has been found. In the case of $\mathcal{N}=4$ SYM, dualizing the coordinates along the $\mathfrak{P}^{ab}$ and $\mathfrak{Q}^{aB}$ directions of the supercoset $\mathrm{PSU}(2,2|4)/\mathrm{Sp}(1,1)\times\mathrm{Sp}(2)$ maps the sigma model onto itself, while turning ordinary into dual symmetry generators. In contrast, it appears impossible to supersymmetrically extend a bosonic T-duality involving only the translational directions of AdS_4 within the supercoset $\mathrm{OSp}(6|4)/\mathrm{U}(3)\times\mathrm{SO}(3,1)$ [60, 61]. The sigma model on this coset is obtained by a kappa-gauge fixing that is not compatible with all string configurations [62], and it was suspected [63] that the gauge fixing could obstruct a T-self-duality and/or dual symmetry that might be present in the string theory. But even using the full superspace formulation of [62], the extension of a pure AdS_4 T-duality to a full self-duality appears impossible [61]. A resolution could be to also T-dualize some of the coordinates from the $\mathbb{CP}^3$ part of the bosonic background. The structure of both

[27] The one-loop contributions vanish in both cases.

[28] Very recently, also n-point correlation functions were related to polygonal Wilson loops [58].

[29] See also [26] within this special issue.

the R-symmetry realization (5.3) and the dual symmetry (5.9) suggests to dualize the coordinates along the $\mathfrak{R}^{AB}$ directions, which generate three Abelian isometries of $\mathbb{CP}^3$. This has been attempted in [63], but leads to a singular transformation that could not be regularized thus far [64] (see also [65]).

Just as in four dimensions, one would expect the superconformal symmetry generators for scattering amplitudes in $\mathcal{N} = 6$ SCS to receive corrections that have distributional support on collinear momentum configurations. However, no source for anomalous contributions from the free generators has been found thus far.

6 Summary & Outlook

Conformal symmetry implies powerful constraints on a physical theory. Nevertheless, it is not always easy to implement it in a mathematically concise way and the difficulties in finding an adequate representation may be misinterpreted as a breaking of this symmetry. In the first part of this review we have investigated two instances of this problem arising in $\mathcal{N} = 4$ SYM theory:

- The holomorphic anomaly leads to a violation of the free superconformal symmetry that can be overcome by corrections to the symmetry generators. This modification only affects collinear momentum configurations and may thus be associated with the ambiguous description of asymptotic states for massless particles (starting at tree-level).

- A renormalization scheme that introduces a mass scale superficially breaks conformal symmetry. Also this shortcoming can be cured by adapting the symmetry representation to the corresponding scheme (starting at one-loop order).

The resulting corrections to the representation of the superconformal algebra relate amplitudes for different numbers of particles and thereby induce recursive relations among them. The representation obeys the commutator relations modulo gauge transformations that vanish when evaluated on scattering amplitudes.

In the second part of the review we have indicated how dual superconformal symmetry results in a Yangian algebra realized on scattering amplitudes. This Yangian algebra forms a typical mathematical structure underlying integrable models. Its representation inherits the deformations of the Lie algebra symmetry mentioned above. While the free representation is able to distinguish certain symmetry invariant building blocks for the amplitudes, the deformation of the integrable structure is crucial for fixing their exact linear combination. Importantly, the building blocks can be generated by a Graßmannian function and we have commented on its relation to the Yangian symmetry.

Finally, similar observations made in $\mathcal{N} = 6$ SCS theory were summarized. While their investigation is still in its infancy, there are strong indications for Yangian symmetry, dual superconformal symmetry as well as a Graßmannian function paralleling the discoveries in $\mathcal{N} = 4$ SYM theory.

Several interesting problems arise in this context. Firstly, it would be important to determine the conformally exact representation of $\mathfrak{psu}(2,2|4)$ at higher loop orders and to investigate the imposed constraints on scattering amplitudes. This could reveal the full power of the underlying formalism potentially facilitating the computation of so far undetermined amplitudes. It would furthermore be crucial to verify the Yangian algebra relations for the deformed representation at tree and loop level. Only this would a posteriori justify the name *Yangian* for the discovered mathematical structure. It would then be highly desirable to study the action of the deformed Yangian symmetry on the Graßmannian function in order to determine the impact of the correction terms. This might yield a prescription for how to obtain the full superamplitude from a generating function. Moreover it would be very interesting to construct Yangian invariants from scratch, i.e. to study the invariants of the corrected algebra as well as their uniqueness starting from the given representation. The way in which the correction terms relate to the Graßmannian formulas described above might be extremely enlightening. Very recently a new bilocal generator $\widehat{\mathfrak{B}}$ corresponding to the hypercharge of the superconformal algebra has been shown to annihilate the amplitudes [42]. It would be important to find out how this generator can be embedded into the above context. Finally many of the above problems carry over to the scattering problem of $\mathcal{N} = 6$ SCS theory. Here the most pushing question is the relation of the discovered algebraic symmetries to a potential T-duality of the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ superstring theory.

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