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FROM $4d$ SUPERCONFORMAL INDICES TO $3d$ PARTITION FUNCTIONS

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Abstract. An exact formula for partition functions in $3d$ field theories was recently suggested by Jafferis, and Hama, Hosomichi, and Lee. These functions are expressed in terms of specific q-hypergeometric integrals whose key building block is the double sine function (or the hyperbolic gamma function). Elliptic hypergeometric integrals, discovered by the second author, define $4d$ superconformal indices. Using their reduction to the hyperbolic level, we describe a general scheme of reducing $4d$ superconformal indices to $3d$ partition functions. As an example, we consider explicitly the duality pattern for $3d$ $\mathcal{N} = 2$ SYM and CS theories with $SP(2N)$ gauge group with adjoint matter.

Introduction

Superconformal indices (SCIs) of supersymmetric Yang Mills field theories in four dimensions [1] may be most conveniently expressed in terms of elliptic hypergeometric integrals, discovered in [2]. This fact was observed and utilised, in the context of Seiberg duality, first in [3] and discussed in detail in [4, 5] (see also [6, 7]). SCIs provide perhaps currently the most rigorous mathematical test of $4d$ supersymmetric dualities whereby indices for theories with quite different matter content may be shown to coincide due to non-trivial identities for elliptic hypergeometric integrals. Moreover, they give a powerful tool for searching for new dualities by utilising transformation formulae for elliptic hypergeometric integrals, as was first shown in [4].

SCIs of $3d$ field theories have an essentially more involved form (due to monopole contributions, that do not analogously arise for $4d$ theories) – see [8, 9, 10, 11, 12] and references therein. In [12] an attempt was made to find a connection between $4d$ and $3d$ SCIs, but no simple relation was found. In the present work we concentrate on the partition functions of $3d$ theories [13, 14, 15, 17, 16, 18, 19, 20, 21]. More precisely, we demonstrate that these partition functions as well as duality relations among different theories can be obtained by a reduction of $4d$ SCIs and corresponding duality relations.

The study of $3d$ partition functions using the localization techniue was initiated by Kapustin, Willett, and Yaakov [14]. In the work of Jafferis [17] and Hama, Hosomichi, and Lee [18] a general recipe for building $3d$ partition functions was suggested. It was found that these functions are expressed in terms of q-hypergeometric integrals admitting the $|q| = 1$ regime [22, 23] (which are referred also as the hyperbolic q-hypergeometric integrals) and having equal quasiperiods $\omega_1 = \omega_2$. In [19] this result was generalized to arbitrary values of the quasiperiods ω_1, ω_2.

Technically, the observation that we make here (that there is an explicit connection between SCIs of $4d$ supersymmetric field theories and the partition functions of $3d$ theories, that allows also for a recipe for finding possible duals) is realized by the reduction of elliptic hypergeometric integrals [2] to hyperbolic q-hypergeometric integrals, which was rigorously established by Rains in [24]. A detailed consideration of such limiting cases was given by van de Bult [25] (see also [26]). This suggests perhaps that the SCIs of $4d$ theories are more fundamental objects than the partition functions of $3d$ theories in that the properties of the latter are inherited from elliptic hypergeometric integrals, which, as functions, are more general, nevertheless having a simpler form. While the connection between $4d$ and $3d$ dualities for supersymmetric field theories was explored in [27] and studied further in the context of the three-dimensional analog of Seiberg duality in [28, 29], here the connection between $4d$ SCIs and $3d$ partition functions gives a different perspective with strong predictive power.

In the following, we illustrate how a reduction in $4d$ SCIs lead to formulae equivalent to $3d$ partition functions by considering particular examples. The same reduction may be applied essentially to other $4d$ SCIs and expressions for $3d$ partition functions recovered so that these examples suffice to show the general procedure. Since $4d$ SCIs for dual theories are obtained from transformation formulae for these elliptic hypergeometric integrals, the same reduction applied to these integrals yields corresponding $3d$ partition functions for dual theories, from which matter content and their representations may be read off. While this procedure is applied to a few examples here, obviously if more examples were considered, due to the profusion of transformation formulae available for elliptic hypergeometric integrals, a whole plethora of new dualities for $3d$ theories would potentially be implied.

Reduction from $4d$ superconformal indices to $3d$ partition functions

Let us take an $SP(2N)$ gauge group four-dimensional $\mathcal{N}=1$ SQCD obeying the multiple duality phenomenon which was described in [4]. The electric theory has the overall symmetry group

$$G \;=\; SP(2N)\times SU(8)\times U(1)_R.$$

This has one chiral scalar multiplet Q belonging to the fundamental representation f of $SP(2N)$ gauge group, a vector multiplet V in the adjoint representation of the gauge group, and an antisymmetric $SP(2N)$-tensor field X, as described in the table below.

	$SP(2N)$	$SU(8)$	$U(1)$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	$\frac{1}{2}$
X	T_A	1	1	0

For $N=1$, the field X is absent and the group $U(1)$ is completely decoupled.

The electric index is given by the following elliptic hypergeometric integral, namely,

$$I_E=\frac{(p;p)_\infty^N(q;q)_\infty^N}{2^N N!}\Gamma((pq)^s;p,q)^{N-1}\int_{\mathbb{T}^N}\prod_{1\le i<k\le N}\frac{\Gamma((pq)^s z_i^{\pm1}z_k^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_k^{\pm1};p,q)}$$
$$\times\prod_{j=1}^N\frac{\prod_{i=1}^8\Gamma((pq)^r y_i z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{1}$$

where $r=1/4-(N-1)s/4$, $r_X=s$, and s is an arbitrary chemical potential associated with the group $U(1)$. The parameters satisfy the constraints $|t|,|t_j|<1$, where $t=(pq)^s$ and $t_j=(pq)^r y_j, j=1,\ldots,8$, and the balancing condition,

$$t^{2N-2}\prod_{j=1}^{8} t_j \;=\; (pq)^2.$$

Here $\Gamma(z;p,q)$ is the elliptic gamma function, defined as,

$$\Gamma(z;p,q) \;=\; \prod_{i,j=0}^{\infty}\frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \qquad |p|,|q|<1,$$

and $\Gamma(a,b;p,q)\equiv\Gamma(a;p,q)\Gamma(b;p,q)$, $\Gamma(az^{\pm1};p,q)\equiv\Gamma(az;p,q)\Gamma(az^{-1};p,q)$, as per usual conventions. Function (1) is a two-parameter generalization of the elliptic Selberg integral suggested in [30] and a multidimensional extension of the elliptic analogue of the Gauss hypergeometric function of [2, 31].

The multiple dual theories of [4] are constructed by means of $W(E_7)$-symmetry transformations for the integral,

$$I(t_1,\ldots,t_8;t,p,q)=\prod_{1\le j<k\le 8}\Gamma(t_jt_k;p,q,t)\frac{(p;p)_\infty^N(q;q)_\infty^N}{2^N N!}$$
$$\times\int_{\mathbb{T}^N}\prod_{1\le j<k\le N}\frac{\Gamma(tz_j^{\pm1}z_k^{\pm1};p,q)}{\Gamma(z_j^{\pm1}z_k^{\pm1};p,q)}\prod_{j=1}^{N}\frac{\prod_{k=1}^{8}\Gamma(t_kz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi\mathrm{i}z_j}, \tag{2}$$

where the nine variables $t,t_1,\ldots,t_8\in\mathbb{C}$ satisfy the inequalities $|t|,|t_j|<1$. Here,

$$\Gamma(z;p,q,t) \;=\; \prod_{j,k,l=0}^{\infty}(1-zt^jp^kq^l)(1-z^{-1}t^{j+1}p^{k+1}q^{l+1}),$$

is the elliptic gamma function of the second order satisfying the key t-difference equation,

$$\Gamma(tz;p,q,t) \;=\; \Gamma(z;p,q)\Gamma(z;p,q,t).$$

The relevant generating transformation, which was established by the second author (for $N=1$) in [31] and by Rains (for arbitrary N) in [32], is given by,

$$I(t_1,\ldots,t_8;t,p,q) \;=\; I(s_1,\ldots,s_8;t,p,q), \tag{3}$$

where the variables are related by

$$s_j=\rho^{-1}t_j,\quad j=1,2,3,4,\qquad s_j=\rho t_j,\quad j=5,6,7,8, \tag{4}$$
$$\rho=\sqrt{\frac{t_1t_2t_3t_4}{pqt^{1-n}}}=\sqrt{\frac{pqt^{1-n}}{t_5t_6t_7t_8}}.$$

Fixing the variables as in [24],

$$t=e^{2\pi\mathrm{i}v\tau},\quad t_i=e^{2\pi\mathrm{i}v\mu_i},\quad i=1,\ldots,8,\quad p=e^{2\pi\mathrm{i}v\omega_1},\ q=e^{2\pi\mathrm{i}v\omega_2},$$

with the restriction,

$$2(N-1)\tau+\sum_{i=1}^{8}\mu_i \;=\; 2(\omega_1+\omega_2), \tag{5}$$

playing the role of the balancing condition, we first consider the limit $v\to 0$.

Using the notation of [26], the limiting integral [24] reduces the electric superconformal index to the form,

$$I_E^{reduced}(\mu_1,\ldots,\mu_8,\tau;\omega_1,\omega_2) = \frac{1}{2^N N!}\gamma^{(2)}(\tau;\omega_1,\omega_2)^{N-1} \tag{6}$$
$$\times \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{1\le i<k\le N} \frac{\gamma^{(2)}(\tau\pm u_i\pm u_k;\omega_1,\omega_2)}{\gamma^{(2)}(\pm u_i\pm u_k;\omega_1,\omega_2)} \prod_{j=1}^N \frac{\prod_{i=1}^8 \gamma^{(2)}(\mu_i\pm u_j;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u_j;\omega_1,\omega_2)} \prod_{j=1}^N \frac{du_j}{\sqrt{\omega_1\omega_2}},$$

where,

$$\gamma^{(2)}(u;\omega_1,\omega_2) = e^{-\pi \mathrm{i} B_{2,2}(u;\omega_1,\omega_2)/2}\frac{(e^{2\pi i u/\omega_1}\widetilde{q};\widetilde{q})_\infty}{(e^{2\pi i u/\omega_1};q)_\infty}, \tag{7}$$

with,

$$q = e^{2\pi i\omega_1/\omega_2}, \qquad \widetilde{q} = e^{-2\pi i\omega_2/\omega_1},$$

and for $B_{2,2}(u;\omega_1,\omega_2)$ denoting the second order Bernoulli polynomial,

$$B_{2,2}(u;\omega_1,\omega_2) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}. \tag{8}$$

Here the conventions, $\gamma^{(2)}(a,b;\omega_1,\omega_2) \equiv \gamma^{(2)}(a;\omega_1,\omega_2)\ \gamma^{(2)}(b;\omega_1,\omega_2)$, $\gamma^{(2)}(a \pm u;\omega_1,\omega_2) \equiv \gamma^{(2)}(a+u;\omega_1,\omega_2)\gamma^{(2)}(a-u;\omega_1,\omega_2)$, are used.

The same limiting result can be obtained after considering the modified elliptic hypergeometric integrals constructed from the modified elliptic gamma function $G(u;\omega_1,\omega_2,\omega_3)$, [31], after taking the limit $\omega_3 \to \infty$, see a detailed consideration of some examples in [23, 26]. In Appendix A of [26] different forms of the function $\gamma^{(2)}(u)$ baring different names are listed. In particular, $1/\gamma^{(2)}(u)$ is known as the double sine function. In [24, 25] the hyperbolic gamma function $\Gamma_h(u)$ is used which is obtained after replacing the quasiperiods ω_1,ω_2 by $\mathrm{i}\omega_1, \mathrm{i}\omega_2$ in $\gamma^{(2)}(u)$.

In the limit discussed, transformation formula (3) leads to the following relation [24], namely,

$$I_E^{reduced}(\mu_1,\ldots,\mu_8,\tau;\omega_1,\omega_2) = \frac{1}{2^N N!}\gamma^{(2)}(\tau;\omega_1,\omega_2)^{N-1} \tag{9}$$
$$\times \prod_{j=0}^{N-1} \prod_{1\le i<k\le 4} \gamma^{(2)}(j\tau+\mu_i+\mu_k;\omega_1,\omega_2) \prod_{5\le i<k\le 8} \gamma^{(2)}(j\tau+\mu_i+\mu_k;\omega_1,\omega_2)$$
$$\times \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{1\le i<k\le N} \frac{\gamma^{(2)}(\tau\pm u_i\pm u_k;\omega_1,\omega_2)}{\gamma^{(2)}(\pm u_i\pm u_k;\omega_1,\omega_2)} \prod_{j=1}^N \frac{\prod_{i=1}^8 \gamma^{(2)}(\nu_i\pm u_j;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u_j;\omega_1,\omega_2)} \prod_{j=1}^N \frac{du_j}{\sqrt{\omega_1\omega_2}},$$

where

$$\nu_j = \xi + \mu_j,\ j = 1,2,3,4, \qquad \nu_j = -\xi + \mu_j,\ j = 5,6,7,8,$$
$$2\xi = (\omega_1+\omega_2) - (N-1)\tau - \sum_{i=1}^4 \mu_i = -(\omega_1+\omega_2) + (N-1)\tau + \sum_{i=5}^8 \mu_i.$$

Applying the further limit,

$$\lim_{S\to\infty} I_E^{reduced}(\mu_1,\ldots,\mu_6,\xi_1+\alpha S,\xi_2-\alpha S;\omega_1,\omega_2) \tag{10}$$
$$\times\, e^{-\pi \mathrm{i} N((\xi_2-\alpha S-\omega)^2-(\xi_1+\alpha S-\omega)^2)/\omega_1\omega_2},$$

where $\max\{\arg(\omega_1),\arg(\omega_2)\} - \pi < \arg(\alpha) < \min\{\arg(\omega_1),\arg(\omega_2)\}$ and $\omega = (\omega_1 + \omega_2)/2$, to (9), carried out essentially in [25], leads to an expression coinciding exactly with the partition function for $3d$ $\mathcal{N}=2$ SYM theory with $SP(2N)$ gauge group, six quarks and one chiral field in absolutely antisymmetric representation of a gauge group, namely,

$$Z_E^{3d} = \frac{1}{2^N N!}\gamma^{(2)}(\tau;\omega_1,\omega_2)^{N-1}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\le i<k\le N}\frac{\gamma^{(2)}(\tau\pm u_i\pm u_k;\omega_1,\omega_2)}{\gamma^{(2)}(\pm u_i\pm u_k;\omega_1,\omega_2)}$$
$$\times\prod_{j=1}^N\frac{\prod_{i=1}^6\gamma^{(2)}(\mu_i\pm u_j;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u_j;\omega_1,\omega_2)}\prod_{j=1}^N\frac{du_j}{\sqrt{\omega_1\omega_2}}. \tag{11}$$

Initially the partition function for $3d$ $\mathcal{N}=2$ SYM theories was calculated in [17, 18] albeit in a particular limit that can be obtained from our results in two ways $\omega_1\to\omega_2$ or $\omega_1\omega_2\to 1$. Later in [19] the most general partition function with unrestricted ω_1 and ω_2 was obtained and in [21] the authors established an explicit relation between particular hyperbolic q-hypergeometric integrals and particular partition functions for $3d$ $\mathcal{N}=2$ SYM theories.

In [25], transformations of the integral in (11) forming the group $W(D_6)$ were deduced as a limit from (9). They lead to the representation,

$$Z_M^{3d} = \frac{1}{2^N N!}\gamma^{(2)}(\tau;\omega_1,\omega_2)^{N-1}\prod_{j=0}^{N-1}\prod_{1\le i<k\le 4}\gamma^{(2)}(j\tau+\mu_i+\mu_k;\omega_1,\omega_2) \tag{12}$$
$$\times\prod_{j=0}^{N-1}\gamma^{(2)}(j\tau+\mu_5+\mu_6,4\omega-\sum_{i=1}^6\mu_i-(2N-j-2)\tau;\omega_1,\omega_2)$$
$$\times\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\le i<k\le N}\frac{\gamma^{(2)}(\tau\pm u_i\pm u_k;\omega_1,\omega_2)}{\gamma^{(2)}(\pm u_i\pm u_k;\omega_1,\omega_2)}\prod_{j=1}^N\frac{\prod_{i=1}^6\gamma^{(2)}(\nu_i\pm u_j;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u_j;\omega_1,\omega_2)}\prod_{j=1}^N\frac{du_j}{\sqrt{\omega_1\omega_2}},$$

where also the reflection identity,

$$\gamma^{(2)}(u,2\omega-u;\omega_1,\omega_2)=1,$$

has been applied, and the transformed variables are given by,

$$\nu_j=\xi+\mu_j,\quad j=1,2,3,4,\qquad \nu_j=-\xi+\mu_j,\ j=5,6,$$
$$2\xi=(\omega_1+\omega_2)-(N-1)\tau-\sum_{i=1}^4\mu_i.$$

Interpreting these integrals in terms of field theory, the global symmetry group for the electric theory is $SU(6)\times U(1)\times U(1)_A\times U(1)_R$, and the field content of the electric theory may be tabulated as,

	$SP(2N)$	$SU(6)$	$U(1)$	$U(1)_A$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	1	$\frac{1}{2}$
X	T_A	1	1	0	0

which can be directly read from the expression for the partition function (11). The denominator terms in the integral kernel represent the vector superfield contribution while in the numerator we see the contribution coming from the chiral superfield X in the antisymmetric representation of the group $SP(2N)$ (given by the terms in the first product and the multiplier in front of the integral), and the contribution of quarks in the fundamental representation is given by the terms in the second product in the kernel. The parameters τ and $\mu_i, i=1,\ldots,6$, are written after absorbing the charges for abelian global groups $\sum_{i=1}^{3} r_i x_i$, where x_i, $i = 1,2,3$, correspond to chemical potentials of the abelian global groups $U(1), U(1)_A$, and $U(1)_R$ and r_i their charges. Here $x_3 = (\omega_1 + \omega_2)/2$ corresponds to the group $U(1)_R$.

For the magnetic theory, the global symmetry group in the UV is $SU(4) \times SU(2) \times U(1) \times U(1)_A \times U(1)_R$, while the matter content may be similarly tabulated as,

	$SP(2N)$	$SU(4)$	$SU(2)$	$U(1)_B$	$U(1)$	$U(1)_A$	$U(1)_R$
q_1	f	f	1	-1	$-\frac{N-1}{4}$	-1	$\frac{1}{2}$
q_2	f	1	f	2	$-\frac{N-1}{4}$	-1	$\frac{1}{2}$
x	T_A	1	1	0	1	0	0
$M_{1,j}$	1	T_A	1	0	$\frac{2j-N+1}{2}$	2	1
$M_{2,j}$	1	1	T_A	0	$\frac{2j-N+1}{2}$	2	1
Y_j	1	1	1	0	$\frac{2j-N+1}{2}$	-6	1

where $j = 0, \ldots, N-1$. By applying the above mentioned transformation formulae we expect the appearance of dim $W(D_6)/S_6 = 32$ different dual theories.

$3d$ $\mathcal{N} = 2$ SYM theory with $SP(2N)$ gauge group, $6f$ and T_A

As another example, here we discuss a $4d$ s-confining multiple duality case considered in terms of indices in [4]. The flavor symmetry group is $F = SU(6) \times U(1)$ and the field content of both theories is presented as follows,

	$SP(2N)$	$SU(6)$	$U(1)$	$U(1)_R$
Q	f	f	$N-1$	$2r = \frac{1}{3}$
A	T_A	1	-3	0
A^k		1	$-3k$	0
QA^mQ		T_A	$2(N-1)-3m$	$\frac{2}{3}$

where $k = 2, \ldots, N$ and $m = 0, \ldots, N-1$.

The electric superconformal index is given by the elliptic Selberg integral suggested by van Diejen and the second author in [30],

$$I_E = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \Gamma(t;p,q)^{N-1} \int_{\mathbb{T}^N} \prod_{1 \le i < k \le N} \frac{\Gamma(t z_i^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_k^{\pm 1}; p, q)} \tag{13}$$
$$\times \prod_{j=1}^{N} \frac{\prod_{m=1}^{6} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{N} \frac{dz_j}{2\pi \mathrm{i} z_j},$$

and the magnetic index is,

$$I_M = \prod_{j=2}^{N} \Gamma(t^j; p, q) \prod_{j=1}^{N} \prod_{1 \le m < s \le 6} \Gamma(t^{j-1} t_m t_s; p, q), \tag{14}$$

where the balancing condition reads $t^{2N-2} \prod_{m=1}^{6} t_m = pq$.

Integral (13) can be reduced to the hyperbolic level in the same way as before (see, e.g. [25]) and yields the partition function of an electric $3d$ $\mathcal{N} = 2$ SYM theory, with $SP(2N)$ gauge group, four quarks and one chiral field in antisymmetric representation of a gauge group, given by the following formula,

$$Z_E^{3d} = \frac{1}{2^N N!}\gamma^{(2)}(\tau;\omega_1,\omega_2)^{N-1}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\leq i<k\leq N}\frac{\gamma^{(2)}(\tau\pm z_i\pm z_k;\omega_1,\omega_2)}{\gamma^{(2)}(\pm z_i\pm z_k;\omega_1,\omega_2)}$$
$$\times\prod_{j=1}^N\frac{\prod_{i=1}^4\gamma^{(2)}(\mu_i\pm z_j;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2z_j;\omega_1,\omega_2)}\prod_{j=1}^N\frac{dz_j}{\sqrt{\omega_1\omega_2}}. \tag{15}$$

Evidently, Z_E^{3d} for this case can be evaluated exactly, yielding,

$$Z_M^{3d} = \frac{\prod_{j=2}^N\gamma^{(2)}((j+1)\tau;\omega_1,\omega_2)}{\prod_{j=0}^{N-1}\gamma^{(2)}((2N-2-j)\tau+\sum_{i=1}^4\mu_i;\omega_1,\omega_2)}\prod_{1\leq i<k\leq 4}\gamma^{(2)}(j\tau+\mu_i+\mu_k;\omega_1,\omega_2), \tag{16}$$

or, using the reflection identity for $\gamma^{(2)}(u;\omega_1,\omega_2)$, equivalently expressed by,

$$Z_M^{3d} = \prod_{j=2}^N\gamma^{(2)}(j\tau;\omega_1,\omega_2)\prod_{j=0}^{N-1}\gamma^{(2)}(\omega_1+\omega_2-(2N-2-j)\tau-\sum_{i=1}^4\mu_i;\omega_1,\omega_2)$$
$$\times\prod_{j=0}^{N-1}\prod_{1\leq i<k\leq 4}\gamma^{(2)}(j\tau+\mu_i+\mu_k;\omega_1,\omega_2). \tag{17}$$

The equality $Z_E^{3d}=Z_M^{3d}$ was rigorously established for the first time by a different method in [23], which result was used as a motivation for a systematic consideration of the reduction procedure in [24].

The dual theories obtained from the equality of the partition functions both have the global symmetry group $SU(4)\times U(1)\times U(1)_A\times U(1)_R$. The spectrum of the electric theory may be tabulated as follows,

	$SP(2N)$	$SU(4)$	$U(1)$	$U(1)_A$	$U(1)_R$
Q	f	f	0	1	$\frac{1}{3}$
X	T_A	1	1	0	0

while that for the magnetic theory may be similarly tabulated as,

	$SU(4)$	$U(1)$	$U(1)_A$	$U(1)_R$
$M_j=X^jQ^2$	T_A	j	2	$\frac{2}{3}$
$N_k=X^k$	1	k	0	0
Y_j	1	$-(2N-2-j)$	-4	$\frac{2}{3}$

where $k=2,\ldots,N$ and $j=0,\ldots,N-1$.

Further dualities

Further dualities are implied by subsequent reduction of the partition functions implemented by taking similar limits as in (10) and, in contrast to the four-dimensional case where similar reduction of SCIs corresponds to theories with fewer flavors, here such reduction corresponds to theories that, whilst also having fewer flavors than originally, have increased CS level. The technical details of the reduction of corresponding integrals are skipped here and only the final results for implied particular dualities, without detailed description of all dual pairs, are indicated.

Taking a similar limit as in (10) in (15) with (17) leads to further identities for partition functions implying further dualities. One such identity describes a $3d$ $\mathcal{N}=2$ CS theory with gauge group $SP(2N)_{k/2}$ and $SU(4-k)\times U(1)\times U(1)_A\times U(1)_R$ global symmetry group with the spectrum involving, apart from the vector multiplet, $4-k$ quarks $Q_i, i=1,\ldots,4-k$, and one chiral superfield X in the antisymmetric representation of the gauge group. This is a confining theory where the spectrum of the dual theory can be directly read from the expressions of the corresponding partition functions, following from the properties of the integrals presented in Sect. 5.6.3 of [25], summarised by singlets of the $SU(4-k)$ flavor group $Y_j = X^{j+1}, j=1,\ldots,N-1$, and baryons $M_j = X^jQ^2, j=1,\ldots,N$, described by the chiral superfield in the antisymmetric representation of $SU(4-k)$ (for $k=3,4$ we do not have such fields).

A particular example from the above duality happens when $N=1$ and $k=3$ which gives a $\mathcal{N}=2$ CS theory with $SP(2)_{3/2}$ gauge group and one quark. In the dual theory we have only contribution coming from additional topological sector as suggested in [20] (where actually one has also some matter field on the magnetic side since the authors consider different electric theory), reflected by the magnetic partition function involving only an exponent with some phase.

Continuing the reduction of (15) with (17) in a similar fashion, another limit can be identitifed with the confining phase description of $3d$ $\mathcal{N}=2$ SYM theory with $U(N)$ gauge group, $SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)\times U(1)_A\times U(1)_R$, $N_f=1,2$, global symmetry group with three flavors $Q_j, \widetilde{Q}_j$, $j=1,\ldots,N_f$, and one chiral superfield X in the adjoint representation of the gauge group. (The exact evaluation of the corresponding partition functions follows from Theorems 5.6.7 and 5.6.8 in [25].) The dual theory contains singlets of $SU(N_f)\times SU(N_f)$, $Y_j = X^{j+1}$, $j=1,\ldots,N-1$, W_j, $j=0,\ldots,N-1$, and mesons $M_j = X^jQ\widetilde{Q}$, $j=0,\ldots,N-1$. (From the same theorems, dualities for CS theories with $U(N)_{k/2}$ gauge group and one chiral field in the adjoint representation and some number of flavors are implied.)

A whole set of dualities are indicated by appropriate reduction of the partition functions of $\mathcal{N}=2$ SYM theory with $SP(2N)$ gauge group, six flavors and one antisymmetric representation matter field, (11) with (12). These reductions correspond to the relation between integrals described in Fig. 5.8 of [25]. Corresponding models represent both SYM and CS theories with different numbers of flavors and different CS level (specifically, the lines going to the left in Fig. 5.8 of [25] correspond in field theory language to integrating out matter fields, reducing the number of flavors by 1 and increasing the CS-level by 1/2 each time, and the lines going to the right correspond to a passage to $U(N)$ SYM or CS field theories with adjoint matter (see [33] for similar dualities involving two adjoint matter fields).

For example, one particular reduction of (11) with (12) leads to partition functions for $\mathcal{N}=2$ CS theory with $SP(2N)_{1/2}$ gauge group, with global symmetry group $SU(5)\times U(1)\times U(1)_A\times U(1)_R$ (the spectrum consists of 5 quarks $Q_i, i=1,\ldots,5$, and one chiral antisymmetric field X) which is self-dual and obeys multiple dualities arising from a $W(A_5)$ symmetry of the corresponding partition function. Application of the transformation formula leads to the partition function for $\mathcal{N}=2$ CS theory with global symmetry group $SU(4)\times U(1)_f\times U(1)\times U(1)_A\times U(1)_R$, different to the original one, with four quarks Q_i, $i=1,\ldots,4$, in the fundamental representation of $SU(4)$ and one separate quark Q_5 (in a sense, this corresponds to the split of the original $SU(5)$ group to $SU(4)\times U(1)$), and singlet fields of the gauge group $M_j = X^jQ^2$, $j=0,\ldots,N-1$, lying in the antisymmetric representation of the flavor group $SU(4)$.

A series of dualities are implied by the reduction of partition functions of $\mathcal{N}=2$ SYM or CS theories with $SP(2N)$ gauge group, N_f flavors and different CS level using the results of Sect. 5.5 of [25]. These are the generalizations of the Giveon-Kutasov [34] type of dualities for $SP(2N)$ theories, see, e.g., [21] for a duality for $SP(2N)_{k/2}$ CS theory with N_f flavors.

Conclusion

This paper demonstrates that there is a deep relation between superconformal indices for four-dimensional field theories and partition functions of three-dimensional supersymmetric field theories stemming from the reduction of elliptic hypergeometric integrals to hyperbolic q-hypergeometric integrals. It may be interesting to understand from a field theory perspective how the various limits considered here could possibly be realised. (It is perhaps worth mentioning that certain much simpler limits in elliptic hypergeometric integrals considered in [4, 5] had a field theory interpretation as corresponding to the flavour changing test for Seiberg duality. Furthermore, at least in free field theory, $4d$ SCIs themselves may be obtained as limits in $4d$ partition functions, discussed for $\mathcal{N}=1$ superconformal symmetry in [3].) Every $4d$ duality out of the large list described in [4, 5], and also the more standard ones considered in [3], after the appropriate reduction, may be expected to yield a three-dimensional analogue of Seiberg duality similar to [27, 28, 29]. To understand all the dualities appearing in $3d$ field theories one should investigate the degeneration of elliptic hypergeometric integrals to q-hypergeometric ones using the procedure developed by Rains [24]. The case described in [20] should correspond to the reduction of $4d$ $\mathcal{N}=1$ SYM theory with $SP(2)$ gauge group, $2N_f$ quarks and one chiral superfield in the adjoint representation, for example.

From this point of view the Z-extremization for $3d$ theories [17] may not be so unexpected, since the a-maximization of [35] is related to certain automorphic properties of elliptic hypergeometric integrals describing $4d$ SCIs [36]. We hazard a guess that the reduction to hyperbolic q-hypergeometric integrals preserves some of the needed automorphic properties leading exactly to Z-extremization.

A recent intriguing conjecture made in [16] concerns a connection of the partition function for $3d$ $\mathcal{N}=4$ SYM theory with $SU(2)$ gauge group with a kernel of the $2d$ Liouville field theory connecting conformal blocks in different channels [37] (see also [38]). This observation deserves further detailed investigation since from our perspective these kernels can be obtained by an appropriate reduction of the elliptic hypergeometric integrals pushing the $4d/3d$ correspondences of the present work down to a new $4d/2d$ correspondence.

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