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Zero-Point Energies in Cosmology

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Nullpunktsenergien in der Kosmologie

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Zero-Point Energies in Cosmology

Theoretical physics has devised a wealth of methods to deal with the various problems associated with a cosmological constant of order $H_0^2 M_{\text{Pl}}^2 \sim 10^{-47}\,\text{GeV}^4$. The smallness of this value puts it solidly outside the reach of Standard Model particle physics. We introduce the basics of quantum field theory in flat and in curved spacetime in order to investigate the Casimir effect, that is, the emergence of a finite difference in the vacuum energy of a given spacetime when the spacetime itself is altered. Specifically, we consider a circular chain of quantum harmonic oscillators as a simple model of a discrete spacetime, and the de Sitter space of the early inflationary era of the universe. In the former case, we discover a Casimir energy that scales as the inverse of the circumference of the chain, while in the latter, we find the well-known Casimir energy $\propto H^4$ of a conformally invariant field. We attempt to extrapolate our findings to simple two-dimensional structures, on the one hand, and to the accelerated universe we live in today, on the other. It turns out that the effects we investigate in the inflationary and the current universe are of mostly academic interest.

Nullpunktsenergien in der Kosmologie

Mit einer Vielzahl verschiedener Methoden versuchen Physiker die Probleme zu lösen, die sich in Verbindung mit einer kosmologischen Konstante stellen, deren geringe Größe von $H_0^2 M_{\text{Pl}}^2 \sim 10^{-47}\,\text{GeV}^4$ sich der Erklärung durch das Standardmodell der Teilchenphysik gänzlich entzieht. Wir führen hier die Grundlagen der Quantenfeldtheorie in flachen und gekrümmten Raumzeiten ein, um den Casimir-Effekt untersuchen zu können. Dieser besteht darin, dass die Veränderung einer gegebenen Raumzeit zu einer endlichen Änderung der Vakuumenergie in dieser Raumzeit führt. Wir betrachten insbesondere eine kreisförmige Kette von quantenmechanischen harmonischen Oszillatoren als simples Modell einer diskreten Raumzeit, sowie den de-Sitter-Raum des frühen, inflationären Universums. Im ersten Fall stellen wir eine Casimir-Energie fest, die sich wie das Inverse des Umfangs der Kette verhält, während wir im zweiten Fall auf die bekannte Casimir-Energie $\propto H^4$ eines konform invarianten Feldes stoßen. Wir verallgemeinern diese Ergebnisse auf einfache zweidimensionale Strukturen einerseits und auf das beschleunigte Universum der heutigen Zeit andererseits. Wir gelangen zu dem Schluss, dass der Einfluss der Casimir-Energie sowohl auf das frühe inflationäre, als auch auf das heutige Universum von hauptsächlich akademischem Interesse ist.

Contents

Chapter 1

Introduction

Cosmological observations indicate that the energy density of baryons in the universe is much less than the critical density. They also suggest, however, that the universe is very nearly flat, which implies that the total energy density should in fact be equal to the critical density. There are commonly two approaches to dealing with this discrepancy.

First, we could introduce new energy components, dark matter and dark energy, to make up the remainder of the energy density. In order to add up to the critical density, these components have to be far more abundant in the cosmos than baryons are. This insight puts an end to the age-old tradition of baryon chauvinism (Linder, 2008), that is, the misconception that what we are made of must be typical of the rest of the universe as well.

Or, second, we could change the underlying theoretical framework by replacing Einstein's general theory of relativity with some alternative theory of gravity. One widespread example of this is the modification of general relativity by replacing the Ricci scalar R in the gravity action by some function $f(R)$. A review of these theories was given by Sotiriou and Faraoni (2010), for example.

The present work shall be an instance of the first approach. While we will not discuss the effects of dark matter on cosmology and astrophysics, nor speculate on its nature, we will give an overview of the dark energy hypothesis and discuss a few popular models of how dark energy arises, including our own work involving the Casimir effect.

The simplest dark energy model one can think of is a cosmological constant: an unchanging, homogeneously distributed energy density that has no interactions with any other components of the universe. All cosmological data, so far, are perfectly compatible with this possibility. There are, however, two main theoretical problems to trying to explain the missing cosmic energy density by a cosmological constant.

First of all, the cosmological constant and the background matter or radiation density evolve at different rates. While the cosmological constant is, by definition, constant in time, matter—dark or otherwise—decreases as the inverse cube of the scale factor. This is simply a manifestation of the intuitively obvious dilution of matter with the expansion of the volume. Radiation, which dominated before matter took over, dilutes even faster: it is redshifted by the expansion of the universe, and therefore its energy density decreases as the inverse fourth power of the scale factor. Hence, for the background energy density—that of matter—and the dark energy density to be of roughly the same order of magnitude today, we would require very carefully engineered conditions in the early universe. We refer to this as the coincidence problem.

The second problem we shall mention here is the extreme smallness of the cosmological constant as measured by recent cosmological experiments. It turns out that the energy density of the cosmological constant is far below any scale that could reasonably be expected to enter into the dynamics of the universe from Standard Model particle physics.

There is, of course, no shortage of models attempting to provide a plausible resolution to both of these issues. Among the most popular approaches are various adjustment mechanisms that dynamically compensate an arbitrary initial cosmological constant, be it via the introduction of additional (scalar) fields or via certain modifications of the underlying gravity theory (see, for instance, the $f(R)$ variant of Bauer *et al.* (2010)). For a more comprehensive review of the various approaches to the cosmological constant problems, consult, for example, the classic paper by Weinberg (1989) or the more recent categorisation by Nobbenhuis (2006).

A particularly interesting modification of our conception of the world manifests in the many efforts to describe the universe in terms of discrete, rather than continuous spacetime. Sakharov (1968) is typically considered to have originated the concept of a microscopic structure of spacetime beyond the Planck scale by introducing the notion of spacetime elasticity. Related ideas were proposed by Padmanabhan (2002, 2004), who suggests that this elasticity of the microscopic degrees of freedom might allow them to compensate a large cosmological constant, so that the observed value is always small.

There is also a multitude of authors who think of spacetime and gravity in terms of thermodynamics, probably starting with Bardeen *et al.* (1973) and Bekenstein (1973), who derive thermodynamic laws governing the behaviour of black holes. This idea was later expanded on by Jacobson (1995), Verlinde (2010), and others, in an attempt to derive gravity itself from the thermodynamics of spacetime. The successes of black hole thermodynamics, such as the fascinating conclusion that the entropy of a black hole should be proportional to its surface area measured in terms of the Planck area $1/M_{\text{Pl}}^2$, have also been taken as an indication that, fundamentally, we live in a (2+1)-dimensional spacetime ('t Hooft, 1993), made up of discrete pieces of Planck-area size, each of which represents one microscopic degree of freedom. This idea, then, gave rise to the development of the holographic principle (Bousso, 2002).

Starting out from black hole physics, Cohen *et al.* (1999) have proposed that in an effective local quantum field theory, i.e., one that is a good approximation to the underlying high-energy theory up to a certain energy scale Λ, no states should exist that allow the energy contained within a region of size L to exceed the mass of a black hole of the same size. They obtain from this requirement the bound $L^3\Lambda^4 \lesssim LM_{\mathrm{Pl}}^2$, which relates the ultraviolet cutoff Λ and the infrared cutoff L. By using the size of the observable universe, the Hubble radius $1/H$, as the infrared cutoff, they obtain for the energy density that saturates this bound $\Lambda^4 \sim L^{-2}M_{\mathrm{Pl}}^2 = H^2M_{\mathrm{Pl}}^2$ which is of the order of the observed dark energy density without any need for fine-tuning.

While it turned out that the result of Cohen *et al.* (1999) does not yield the correct behaviour of the dark energy density with the expansion of the universe, it was nonetheless a useful step in the development of the notion of a holographic dark energy (Li, 2004) via Susskind (1995) and Fischler and Susskind (1998). Li (2004) finds that using the event horizon, that is, the boundary of the volume a fixed observer may eventually be in causal contact with, as the infrared cutoff, one can obtain a dark energy density of the correct magnitude and time dependence.

All of this may serve to illustrate the great amount of creativity physicists have displayed in trying to deal with the cosmological constant. Inspired by the notion of a discrete spacetime, we shall investigate the effect that the zero-modes of microscopic degrees of freedom making up the spacetime might have on the evolution of the cosmos as a whole. We will consider several cases of boundary conditions being imposed on the dynamics within the spacetime, either by the structure of the spacetime itself or by some other influence. These boundary conditions will then alter the zero-mode sum of the fields contained within the spacetime, which gives rise to a finite, non-zero change of the zero-point energy with respect to fields in the unperturbed spacetime. Conceptually, this energy shift is of the same origin as the one observed between two conducting plates in the Casimir effect. The Casimir effect will therefore be one of the main foci of this work

For the remainder of this chapter, we shall briefly present standard FRW cosmology, introduce the usual evolution equations, and give a few examples of observations in support of the existence of a dark energy component. Chapter 2 will provide an overview of the relevant notions of quantum field theory. There, we will show how a cosmological constant emerges in quantum field theory, and cite a few basic concepts of quantum field theory in flat spacetime, before moving on to curved spacetimes. In chapter 3, we will discuss several instances of the Casimir effect and its possible influence on the evolution of the cosmos. We will conclude, and provide an outlook on possible future avenues of investigation in chapter 4.

1.1. Fundamentals of Cosmology

1.1.1. The Friedmann-Robertson-Walker Universe

The framework for our considerations of cosmology is the general theory of relativity. One of the tenets of general relativity is a fundamental connection between the geometry of spacetime, characterised by the metric tensor $g^{\mu\nu}$, and its matter content, described by the energy-momentum tensor $T^{\mu\nu}$. This relation is expressed in the field equations also known as the Einstein equations, (Peacock, 1999)

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} , \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R = R_{\mu}{}^{\mu} = g_{\mu\nu} R^{\mu\nu}$ is the Ricci scalar, and G is Newton's constant. The Ricci tensor is defined in terms of the Riemann curvature tensor, which is the unique tensor that is constructed from the metric and its first and second derivatives, and is linear in the second derivatives (Weinberg, 1972, chapter 6.2); for details, including the Riemann tensor in terms of the affine connection, see appendix A. The left-hand side of equation (1.1) contains the relevant information on the curvature of spacetime, while the right-hand side comprises the matter contained in the universe.

Since, on very large scales—at least hundreds of megaparsecs—, the matter distribution of the universe can be approximated as very nearly homogeneous and isotropic, the metric is usually assumed to take the Friedmann-Robertson-Walker (FRW) or Friedmann-Lemaître-Robertson-Walker (FLRW) form

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu} = \mathrm{d}t^2 - a^2(t) R_0^2 \left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\, \mathrm{d}\Omega^2 \right] , \tag{1.2}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$. The units of r can be chosen such that the curvature parameter k takes on values $k = +1, 0, -1$ for positively curved, flat, or negatively curved spatial sections, respectively. The size of the universe relative to its size today, R_0, is given by the scale factor $a(t) = R(t)/R_0$. The scale factor $a(t)$ is related to the redshift z, and thus to the wavelength change experienced by light emitted at time t, by $1 + z = \lambda_{\mathrm{obs}}/\lambda_{\mathrm{em}} = a(t)^{-1}$. Time intervals can then be related to redshift intervals by $\mathrm{d}t = -\mathrm{d}z/(H(z)(1+z))$.

The energy-momentum tensor of the FRW universe is that of a perfect fluid that is on average at rest (Weinberg, 1972, chapter 14.2):

$$T_{\mu\nu} = (\rho + p)\, U_{\mu} U_{\nu} - p\, g_{\mu\nu} , \tag{1.3}$$

where $U_{\mu} = (1, 0, 0, 0)$ is the four-velocity of the fluid, and ρ and p are its energy density and pressure. With the Ricci tensor for the FRW metric (cf. appendix A), the time-time component of the Einstein equations gives the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) , \tag{1.4}$$

while the space-space components yield

$$\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2} = 4\pi G (\rho - p) .$$

By eliminating $\ddot{a}$ from the last two equations, we obtain a first-order differential equation for the scale factor—the Friedmann equation:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.5}$$

where $H(t) \equiv \dot{a}/a$ is the Hubble function.

For a more detailed derivation of equations (1.4) and (1.5), consult Weinberg (1972, chapter 15.1), from whence this calculation has been adapted.

The Friedmann equation (1.5) illustrates the intimate connection between the energy density of the universe and its global geometry: there is a critical density ρ_c that produces a flat universe, $k = 0$,

$$\rho_c(a) = \frac{3H^2(a)}{8\pi G} = 3H^2 M_{Pl}^2, \tag{1.6}$$

where we have introduced the reduced Planck mass $M_{Pl} = 1/\sqrt{8\pi G} \approx 2.43 \times 10^{18}\,\text{GeV}$. The critical density today is

$$\rho_c(a_0) = 3H_0^2 M_{Pl}^2 \approx 4.1 \times 10^{-47}\,\text{GeV}^4, \tag{1.7}$$

where the Hubble parameter today is $H_0 \approx 71.4\,\text{km s}^{-1}\,\text{Mpc}^{-1} \approx 1.52 \times 10^{-42}\,\text{GeV}$ (Komatsu *et al.*, 2010). It is convenient, then, to normalise all densities to the critical density, introducing the density parameters

$$\Omega_i(a) \equiv \frac{\rho_i(a)}{\rho_c(a)} = \frac{8\pi G \rho_i(a)}{3H^2(a)}, \tag{1.8}$$

where, for clarity, we made the scale-factor dependence of the density parameters explicit; the density parameters today will be denoted by dropping the a-dependence, i.e., $\Omega_i \equiv \Omega_i(a_0)$.

1.1.2. The Cosmological Constant in the FRW Universe

The Einstein equation as given in equation (1.1) is the form of the field equations originally presented by Einstein (1915).[1] Based on the cosmological views and astronomical data at the time, it seemed reasonable to assume that the universe is static. However, the equations (1.4) and (1.5) imply that a solution with constant scale factor ($\dot{a} = 0$ and $\ddot{a} = 0$) is only possible if

$$\rho = -3p = \frac{3k}{8\pi G a^2} \tag{1.9}$$

[1]Actually, Einstein (1915) writes the field equations in the form $R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T_\mu{}^\mu)$, which is, however, equivalent to our equation (1.1), as can be seen by inserting the trace of equation (1.1), $R = 8\pi G T_\mu{}^\mu$.

in equation (1.4), which implies that either the energy density ρ or the pressure p has to be negative. In order to avoid this result, Einstein (1917) introduced the so-called cosmological constant Λ:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} . \tag{1.10}$$

By moving $\Lambda g_{\mu\nu}$ to the right-hand side, we introduce the modified energy-momentum tensor

$$\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu} , \tag{1.11}$$

which is still of the perfect-fluid form (1.3) with energy density ρ and pressure p replaced by $\tilde{p} = p - \Lambda/(8\pi G)$, and $\tilde{\rho} = \rho + \Lambda/(8\pi G)$. The condition for a static universe is then simply

$$\tilde{\rho} = -3\tilde{p} = \frac{3k}{8\pi G a^2} , \tag{1.12}$$

which, for a universe filled with pressure-less matter (dust, $p = 0$), implies

$$\Lambda = \frac{k}{a^2} , \qquad \rho = \frac{\Lambda}{4\pi G} . \tag{1.13}$$

Therefore, for a positive energy density ρ, we require positive Λ, in which case the above equations yield

$$k = +1 , \qquad a = \frac{1}{\sqrt{\Lambda}} . \tag{1.14}$$

The curvature radius of the static Einstein universe is therefore finite and constant—this is why this model is sometimes referred to as the cylinder universe (Friedman, 1922).

When, expanding on work done by Pease (1915), Slipher (1917), Humason (1929), and others astronomers, Hubble (1929) established an approximately linear relation between the distance and the redshift of galaxies,[2] the assumption of a static universe became, of course, untenable. Nevertheless, it remains true that equation (1.10) is the most general form of the field equations with the energy-momentum tensor equalling a tensor constructed from the metric and its first and second derivatives that is linear in the second derivatives. Therefore, the constant Λ, cannot, a priori, be assumed to vanish.

Introducing the cosmological term $\Lambda g_{\mu\nu}$ in equation (1.10) modifies the equations (1.4) and (1.5), which describe the evolution of the scale factor of the FRW universe to

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} , \tag{1.15}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} . \tag{1.16}$$

[2]"Extra-galactic nebulae", in contemporary terminology.

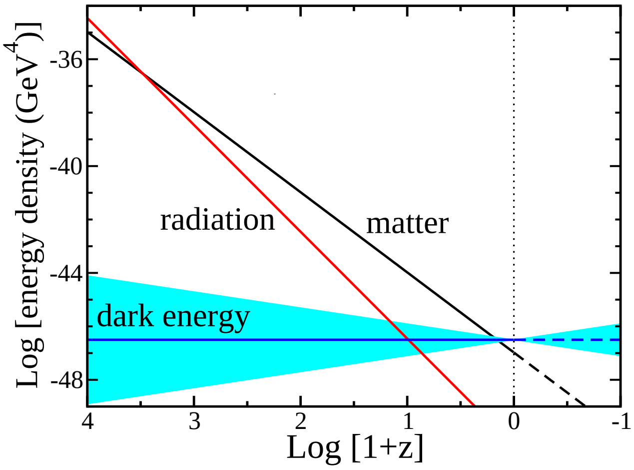


Figure 1.1.: Scaling of matter, radiation, and dark energy with $w = -1.0 \pm 0.2$, represented by the blue band. From Frieman *et al.* (2008).

While the cosmological constant is displayed explicitly in equations (1.15) and (1.16), it is usually expressed as a contribution to the energy density and pressure by writing $\Lambda = 8\pi G\rho_\Lambda = -8\pi G p_\Lambda$.

The evolution of the energy density ρ_i of a given component can be derived from covariant energy-momentum conservation, $T^{\mu\nu}{}_{;\nu} = 0$, which, for an energy-momentum tensor of the perfect-fluid form (1.3), yields the continuity equation

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0. \tag{1.17}$$

It is often described in terms of the equation-of-state parameter $w_i \equiv p_i/\rho_i$,

$$\rho_i \propto \exp\left[\int_0^z \mathrm{d}z'\,\frac{3(1+w_i)}{1+z'}\right] = \exp\left[-\int_0^a \frac{\mathrm{d}a'}{a'}\,3(1+w_i)\right]. \tag{1.18}$$

For constant w_i, this implies

$$\rho_i \propto (1+z)^{3(1+w_i)} \propto a^{-3(1+w_i)}. \tag{1.19}$$

These scaling relations are illustrated for the dominant energy components in figure 1.1.

Pressureless matter (baryonic, and dark non-relativistic matter), for example, has an equation-of-state parameter $w_M = 0$, which implies $\rho_M \propto (1+z)^3 \propto a^{-3}$, while radiation (relativistic matter) with an equation of state $w_R = \frac{1}{3}$ evolves as $\rho_R \propto (1+z)^4 \propto a^{-4}$. For a cosmological constant, we find $\rho_\Lambda = -p_\Lambda =$ constant, i.e., $w_\Lambda = -1$. More generally, dark energy satisfies $\rho_{\mathrm{DE}} \propto \exp\left[\int_0^z \mathrm{d}z'\,\frac{3(1+w_{\mathrm{DE}})}{1+z'}\right]$, where for a time-varying equation of state, w_{DE} is usually parametrised as some function of redshift z, for instance as (Chevallier and Polarski, 2001; Linder, 2003)

$$w_{\mathrm{DE}}(z) = w_0 + w_a(1-a) = w_0 + w_a \frac{z}{1+z}. \tag{1.20}$$

A cosmological constant, in this parametrisation, would correspond to $w_0 = -1, w_a = 0$.

By using the critical density (1.6) to rewrite the Friedmann equation (1.15) in terms of the density parameters (1.8), we find

$$1 = \Omega(a) + \Omega_\Lambda(a) - \frac{k}{a^2 H^2(a)} \tag{1.21}$$

where $\Omega \equiv \rho/\rho_c$ and $\Omega_\Lambda \equiv \Lambda/(8\pi G \rho_c)$. Writing $\Omega_{\mathrm{total}} = \Omega + \Omega_\Lambda$, we conclude

$$\Omega_{\mathrm{total}}(a) - 1 = \frac{k}{a^2 H^2(a)}. \tag{1.22}$$

Therefore, in a flat universe ($k = 0$), the density parameters add up to 1 at all times, whatever the character of the density components may be.

For the case where the above scalings for the density components apply, we can rewrite the Friedmann equation in the form

$$\frac{H^2}{H_0^2} = \Omega_r(1+z)^{-4} + \Omega_m(1+z)^{-3} + \Omega_k(1+z)^{-2} + \Omega_{\mathrm{DE}} \exp\left[\int_0^z \mathrm{d}z' \, \frac{3(1+w_{\mathrm{DE}})}{1+z'}\right], \tag{1.23}$$

where we have introduced the Hubble parameter today H_0 and the curvature density parameter, $\Omega_k \equiv \rho_k/\rho_c$, $\rho_k \equiv -3k/(8\pi G)$. For dark energy in the form of a cosmological constant ($\Omega_{\mathrm{DE}} = \Omega_\Lambda$, $w_{\mathrm{DE}} = -1$), this turns into

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda. \tag{1.24}$$

1.1.3. Cosmic Evolution

Astrophysical and cosmological data (see section 1.2) indicate that the universe today is composed of three main components, baryonic matter making up about 4%, cold dark matter about 22%, and dark energy about 74% of the critical density (Komatsu *et al.*, 2010), supplemented by a small contribution of radiation that is negligible today.

At the same time, Komatsu *et al.* (2010) place a tight bound on the curvature density Ω_k, constraining it, at the 95% confidence level, to $-0.0178 < \Omega_k < 0.0063$ or $-0.0133 < \Omega_k < 0.0084$ using data from WMAP+BAO+SN or WMAP+BAO+H_0, respectively, which is perfectly compatible with a flat universe. This implies, via equation (1.22), that all the remaining density parameters must add up to 1, $\sum_i \Omega_i = 1$.

As can be seen from figure 1.1 and equation (1.24), radiation, matter (dark and baryonic combined), and dark energy dominate the total density of the universe during different epochs. In order to describe the evolution of the universe, we can first combine the continuity equation (1.17) with the time derivative of the Friedmann equation (1.15) to obtain the time development of the Hubble parameter,

$$\dot{H} = -4\pi G(\rho + p). \tag{1.25}$$

We can then use equations (1.15), (1.16), and (1.25) to derive

$$H = \frac{2}{3(1+w)(t-t_0)}, \tag{1.26}$$

$$a(t) \propto (t-t_0)^{\frac{2}{3(1+w)}}, \tag{1.27}$$

where t_0 is a constant. Note, that these solutions only apply for $w \neq -1$. For the radiation-dominated universe ($w = 1/3$) and the matter-dominated ($w = 0$) universe, we obtain

$$\text{Radiation:} \quad a(t) \propto (t-t_0)^{1/2}, \quad \rho \propto a^{-4}, \tag{1.28}$$

$$\text{Matter:} \quad a(t) \propto (t-t_0)^{2/3}, \quad \rho \propto a^{-3}. \tag{1.29}$$

Both of these epochs therefore exhibit decelerated expansion.

As first indicated by observations made in the late 20th century (Perlmutter *et al.*, 1999; Riess *et al.*, 1998), the universe is actually experiencing accelerated expansion today. In order to accommodate this, we need dark energy with an equation of state $w_{\mathrm{DE}} < -1/3$. As can be seen from equation (1.16), when dark energy dominates the universe, the expansion is indeed accelerating, $\ddot{a}(t) > 0$.

The simplest case, dark energy with $w_{\mathrm{DE}} = -1$, is of special interest—it is referred to as a cosmological constant. Equation (1.17) implies that in a universe dominated by a cosmological constant, the energy density is constant. From both equation (1.15) and equation (1.25) it then follows that the Hubble parameter is constant as well, which implies that the scale factor evolves as

$$a \propto e^{Ht}. \tag{1.30}$$

This case was first investigated by de Sitter (1917). In the standard model of cosmology, called the ΛCDM model, the universe is flat and filled with cold dark matter (CDM, about 22%), non-relativistic baryonic matter (about 4%), a cosmological constant (Λ, about 74%), and small amounts of relativistic matter (radiation).

Though perfectly compatible with current experimental data, the ΛCDM model is by no means uncontested. Fig. 1.6, for instance, shows that a wide range of values for w_{DE} is allowed by the data. This includes dark energy with $w_{\mathrm{DE}} < -1$, which is sometimes called phantom or ghost energy. Integrating the dark-energy-dominated Friedmann equation, $(\dot{a}/a)^2 = H_0^2\, \Omega_{\rm DE}\, a^{-3(1+w)}$, from time t to some future time t_s yields

$$a(t) \propto (t_s - t)^{\frac{2}{3(1+w)}}, \tag{1.31}$$

$$H = -\frac{2}{3(1+w)(t_s - t)}. \tag{1.32}$$

This shows that both the scale factor and the Hubble rate reach a singularity ("Big Rip") at some finite time t_s. This corresponds to a growth of the dark energy density to infinity at t_s, which leads to all currently bound structures being ripped apart at some finite time before the final singularity.[3] For example, for $w_{\rm DE} = -3/2$, the Milky Way will get stripped about 60 million years before the Big Rip. The Earth itself will be ripped apart about 30 minutes before the end (Caldwell *et al.*, 2003).

1.2. Observational Evidence for Dark Energy

Ever since its discovery in 1998 (Perlmutter *et al.*, 1999; Riess *et al.*, 1998), observational evidence for the acceleration of the cosmic expansion has been accumulating. While the first indications for cosmic acceleration were based on surveys of supernovae Ia (SN Ia), subsequent observations include measurements from cosmic microwave background (CMB) radiation, large scale structure (LSS), and galaxy clusters. We will focus on measurements involving SN Ia, the CMB, and baryon acoustic oscillations (BAO).

1.2.1. Supernovae Ia

The luminosity distance to an object is defined simply via the inverse-square flux dilution, $\mathcal{F} = L_{\rm obs}/(4\pi d_L^2)$, where $L_{\rm obs}$ is the observed absolute luminosity of the source, and $\mathcal{F}$ is the observed flux at distance d_L. Luminosity—defined as the energy emitted per time interval—scales with the scale factor at the time of emission as $L \propto a(t_{\rm emit})^{-2}$, because time intervals grow as a and energies decrease as $1/a$. When we combine this with the comoving distance obtained from integrating the light-cone condition $\mathrm{d}s^2 = 0$, the luminosity distance turns out to be (Bean, 2010; Tsujikawa, 2010)

$$d_L(z) = \frac{1+z}{H_0\sqrt{\Omega_k}} \sinh\left(\sqrt{\Omega_k}\int_0^z \frac{\mathrm{d}z'}{H(z')/H_0}\right), \tag{1.33}$$

where the function $f_k(\chi) = 1/\left(\sqrt{\Omega_k}\right)\, \sinh\left(\sqrt{\Omega_k}\chi\right)$ behaves as $f_k(\chi) = \sin\chi$ for $k = +1$, as $f_k(\chi) = \chi$ for $k = 0$, and as $f_k(\chi) = \sinh\chi$ for $k = -1$. For the flat case $k = 0$,

[3] Note, that a Big Rip singularity is not, in fact, inevitable with $w_{\rm DE} < -1$: some modified gravity models, e.g., $f(R)$ gravity, permit $w_{\rm DE} < -1$ without a Big Rip (Amendola and Tsujikawa, 2008).

equation (1.33) reduces to $d_L(z) = (1+z)\int_0^z \mathrm{d}z'/H(z')$, which turns equation (1.33) into

$$H(z) = \left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1} . \tag{1.34}$$

This relation allows us to determine the expansion history of the universe by measuring the luminosity distance to faraway objects.

The luminosity distance can also be expressed in terms of the distance modulus $m - M$ with the apparent magnitude m and the absolute magnitude M:

$$m - M = 5\log_{10}\left(\frac{d_L}{10\,\mathrm{pc}}\right) . \tag{1.35}$$

Now, if we somehow knew the absolute magnitude of an object, we could get at its luminosity distance simply by measuring its apparent magnitude m. Fortunately, there are certain classes of objects, called standard candles, for which just this is possible. There exists, for example, a relationship between the period and the luminosity of Cepheid variable stars, first noticed by Leavitt (1908) and later confirmed by Leavitt and Pickering (1912), that can be used to measure distances within neighbouring galaxies (Peacock, 1999). For cosmological purposes, SN Ia turn out to be useful standard candles: their brightness follows a characteristic light curve whose descending slope is a good predictor of the supernova luminosity (Riess *et al.*, 1998).

Expanding the luminosity distance (1.33) about $z = 0$, we find (Tsujikawa, 2010)

$$d_L(z) = \frac{1}{H_0}\left[z + \frac{1}{4}\left(1 - 3w_{\mathrm{DE}}\Omega_{\mathrm{DE}} + \Omega_k\right) z^2 + \mathcal{O}(z^3)\right] , \tag{1.36}$$

For a curvature density parameter $\Omega_k \simeq 0$, as indicated by a combination of data from WMAP, baryon acoustic oscillations (BAO, see section 1.2.2), and the Hubble parameter today H_0 (Komatsu *et al.*, 2010), equation (1.36) shows that in the presence of dark energy ($w_{\mathrm{DE}} < -\frac{1}{3}$ and $\Omega_{\mathrm{DE}} > 0$), the luminosity distance to a given redshift tends to be greater than in a flat universe without dark energy.

The SN Ia data of Riess *et al.* (1998) and Perlmutter *et al.* (1999) do indeed show that for redshifts $0.2 < z < 0.8$ the luminosity distances of observed supernovae Ia tend to be greater than those predicted for a flat universe without dark energy. Assuming a flat universe and $w_{\mathrm{DE}} = -1$, Perlmutter *et al.* (1999) conclude that a non-zero, positive cosmological constant is present at the 99% confidence level. More recent SN Ia surveys (Astier *et al.*, 2006; Davis *et al.*, 2007; Riess *et al.*, 2004, 2007; Wood-Vasey *et al.*, 2007) confirm this result. They also show, however, that supernova data alone is insufficient to constrain the equation of state, once the assumption of a cosmological constant $w_{\mathrm{DE}} = -1$ is dropped. This is illustrated in figure 1.2, which shows the observational constraints

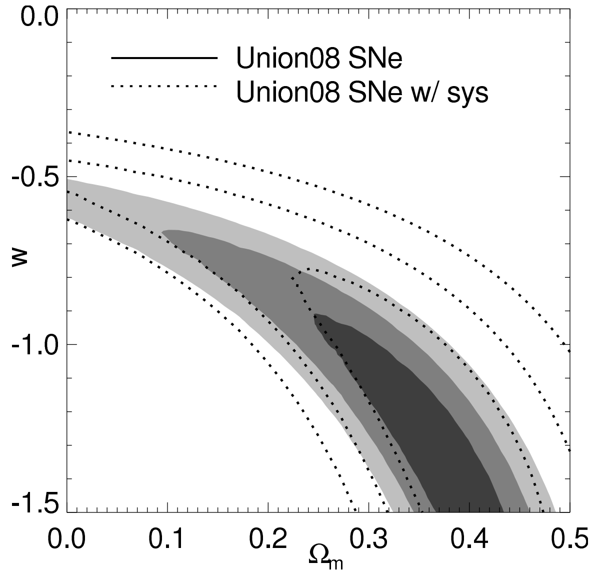


Figure 1.2.: 68.3 %, 95.4 %, and 99.7 % confidence level contours on w_{DE} (here denoted as w) and Ω_m constrained by the Union08 SN Ia data sets. The equation of state w_{DE} is assumed to be constant. It is obvious that SN Ia data do not place a tight bound on a varying w_{DE}. From Kowalski *et al.* (2008).

on the dark energy equation of state (assumed to be constant) and the matter density parameter from the Union08 SN Ia data by Kowalski *et al.* (2008).

For w_{DE} varying with redshift, the constraints on w_0 and w_a (in the parametrisation (1.20)) from a combination of several kinds of observations (including supernovae Ia, WMAP, BAO, H_0, and gravitational lensing time delays ($D_{\Delta t}$)) are shown in figure 1.3. Komatsu *et al.* (2010) obtain from WMAP+BAO+H_0+$D_{\Delta t}$+SN the joint constraint

$$w_0 = -0.93 \pm 0.12, \qquad w_a = -0.38^{+0.66}_{-0.65}, \tag{1.37}$$

which is consistent with a cosmological constant ($w_0 = -1, w_a = 0$).

1.2.2. BAO

The use of standard candles like SN Ia to probe the expansion history of the universe can be complemented by data from the observation of standard rulers. These are objects whose physical size is well-known from fundamental physics. We can then extract information about the cosmic expansion history by measuring the redshift evolution of the object's size.

One prominent standard ruler used in cosmology is the sound horizon at recombination. It is given by the coordinate distance that a sound wave in the primordial photon-baryon-electron plasma can travel within the time from the Big Bang to recombination.

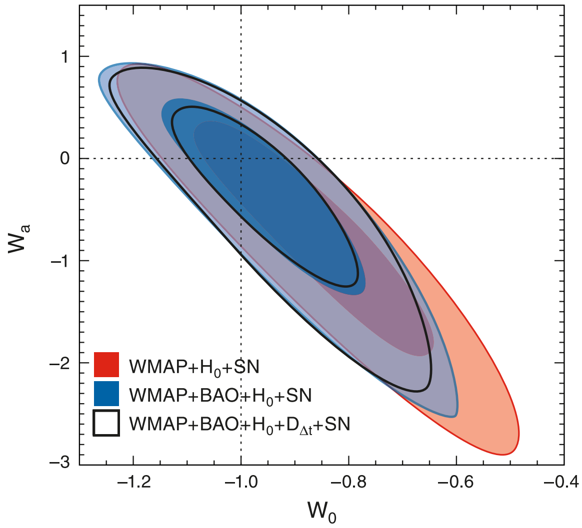


Figure 1.3.: 68% and 95% confidence level contours on w_0 and w_a in the parametrisation $w_{\mathrm{DE}}(a) = w_0 + w_a(1-a)$. From Komatsu *et al.* (2010).

This distance is reflected in the galaxy distribution, because galaxies tend to form in over-dense regions.

As illustrated in figure 1.4, primordial over-densities, seeded by small inhomogeneities in the early universe, are initially adiabatic, which means that the density perturbations in all components (cold dark matter, baryons, photons) are proportional to one-another (Liddle and Lyth, 2000, chapter 4.8.1). While the heavy and pressureless cold dark matter (CDM) stays put, the pressure of the photon-baryon fluid drives it away from the origin. Once the photons and baryons decouple at the time of recombination ($z_{\mathrm{dec}} \simeq 1080$), the baryons stop expanding as well, whereas the photons free-stream away. The CDM over-density at the origin is thus surrounded by a shell of baryon over-density. Owing to the gravitational attraction, CDM and baryons from the homogeneous background then fall into the potential wells formed by the over-densities, thus giving rise to CDM-baryon over-densities, which turn out to be preferred sites for the formation of galaxies.

This effect on the galaxy distribution has been observed both in the Sloan Digital Sky Survey (SDSS) (Eisenstein *et al.*, 2005) and in the 2dF Galaxy Redshift Survey (2dF-GRS) (Percival *et al.*, 2007). The correlation function between pairs of SDSS galaxies is presented in figure 1.5; it shows a significant peak for galaxies separated by a distance of $105\,\mathrm{h}^{-1}\mathrm{Mpc} \simeq 150\,\mathrm{Mpc}$, which corresponds to the sound horizon at recombination. The constraint that BAO place on the dark energy equation of state w_{DE} is shown in figure 1.6, along with bounds from supernovae and the CMB.

For a more detailed account of the use of the sound horizon as a standard ruler, consult Rich (2010, ch. 5.3) and references therein. Furthermore, the following section, 1.2.3, gives a somewhat more mathematical description of how the sound horizon affects CMB anisotropy and how this is influenced by the presence of dark energy.

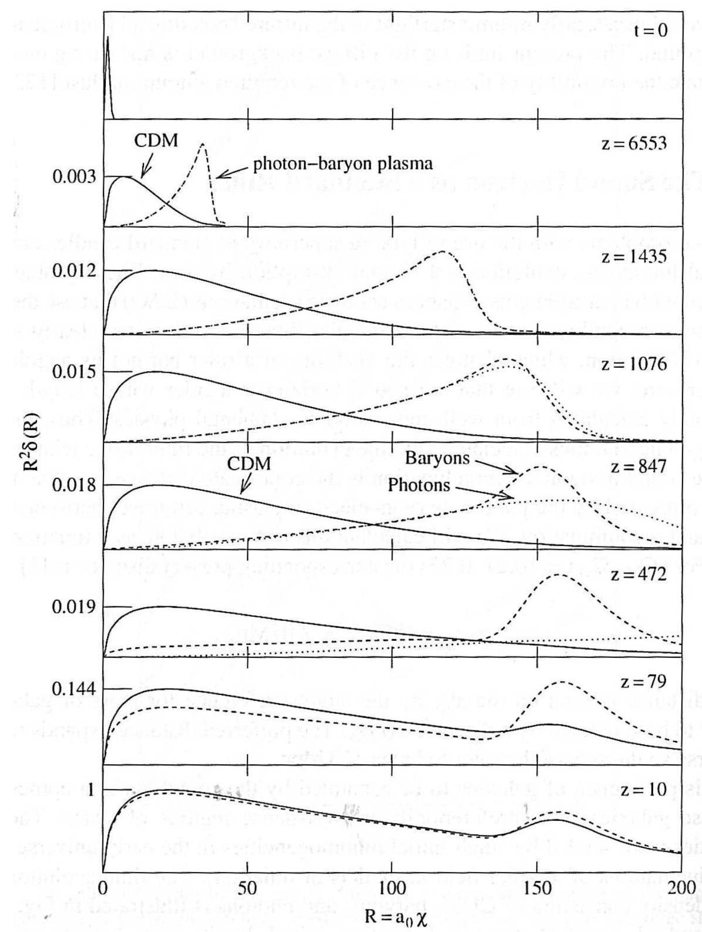


Figure 1.4.: Evolution of initially adiabatic over-density in a universe with CDM, neutrinos (not shown), baryons, and photons. The x-axis is the current distance from the original position of the density perturbation; the y-axis is the density contrast $\delta(R)$ in an arbitrary normalisation. Time increases from top to bottom. See text for details on the density evolution. From Rich (2010).

1.2.3. CMB

Dark energy affects the CMB power spectrum in two ways: (i) It changes the expansion history of the universe after decoupling, when the CMB radiation was released. This leads to a shift in the position of the acoustic peaks. (ii) It induces the integrated Sachs-Wolfe (ISW) effect (Sachs and Wolfe, 1967), which causes a rise in the large-scale (low-multipole) part of the CMB spectrum by "flattening out" gravitational potential wells (e.g., of galaxy clusters) while CMB photons pass through them. Thus the photons lose less energy upon exiting the well than they gained when entering it, leaving them with a net energy boost, or blueshift. Being limited to large scales, or low multipoles, where

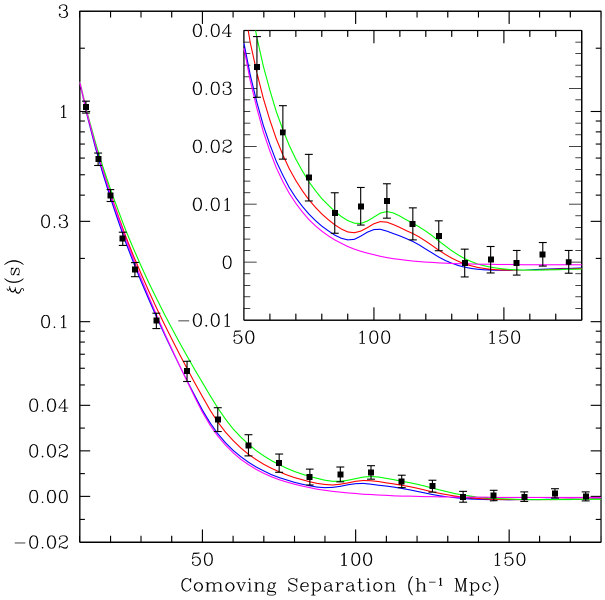


Figure 1.5.: Galaxy-galaxy correlation function as measured by SDSS (Eisenstein *et al.*, 2005). The peak around 105 h^{-1}Mpc corresponds to the sound horizon at recombination.

the CMB power spectrum is fraught with considerable error,[4] the ISW effect is typically less important than effect (i).

The sound horizon at recombination (see section 1.2.2), sets the characteristic size of acoustic oscillations seen in the CMB radiation. It is defined as $r_s(\eta) = \int_0^\eta \mathrm{d}\tilde{\eta}\, c_s(\tilde{\eta})$, where c_s is the speed of sound and the conformal time is defined as $\mathrm{d}\eta = \mathrm{d}t/a$. The speed of sound in the primordial fluid is given by $c_s^2 = 1/\left[3(1+R_s)\right]$, $R_s = 3\rho_b/(4\rho_\gamma)$, where ρ_b and ρ_γ are the baryon and photon energy densities, respectively.

The CMB multipole ℓ_A corresponding to a CMB feature seen under a characteristic angle of θ_A is

$$\ell_a = \frac{\pi}{\theta_A} = \pi \frac{d_A^{(c)}(z_{\mathrm{dec}})}{r_s(z_{\mathrm{dec}})}, \tag{1.38}$$

where $d_A^{(c)}$ is the comoving angular-diameter distance, related to the luminosity distance by $d_A^{(c)} = d_L/(1+z)$ (Weinberg, 1972, chapter 14.4). Inserting the expression (1.33) for the luminosity distance, and the Friedmann equation (1.15) for a background of matter

[4]The accuracy achievable in measurements of low-multipole CMB anisotropy is severely restricted by cosmic variance, which is the $1/\sqrt{N}$ uncertainty in measuring N independent wave modes. The cosmic variance error actually dominates the errors in WMAP data out to $l \sim \mathcal{O}(100)$.

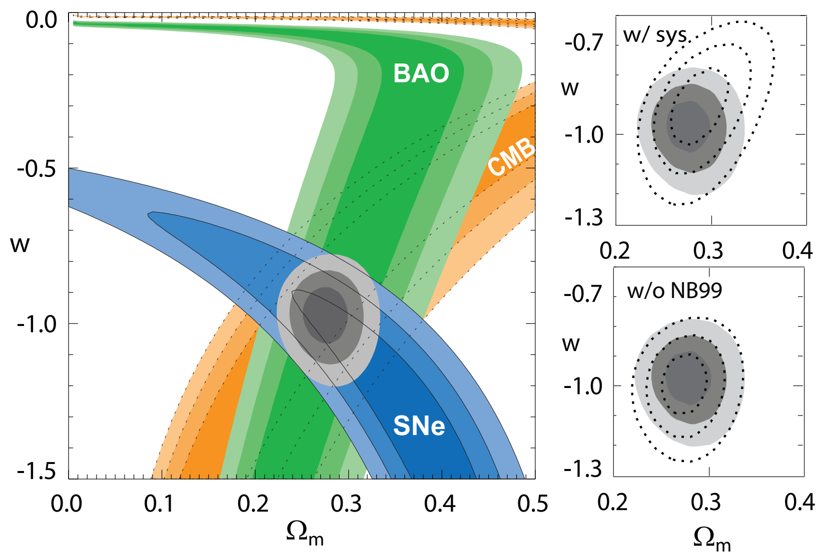


Figure 1.6.: 68.3%, 95.4% and 99.7% confidence level contours on w_{DE} (denoted as w in the figure) and Ω_m for a flat universe. Left: Individual constraints from SN Ia, CMB, and BAO, as well as the combined constraints (filled gray contours, statistical errors only). The upper right panel shows the effect of including systematic errors. The lower right panel illustrates the impact of the Supernova Cosmology Project Nearby 1999 data. From Kowalski *et al.* (2008).

and radiation only, we find

$$\ell_A = \frac{3\pi}{4}\sqrt{\frac{\omega_b}{\omega_\gamma}}\left[\ln\left(\frac{\sqrt{R_s(a_{\mathrm{dec}})+R_s(a_{\mathrm{eq}})}+\sqrt{1+R_s(a_{\mathrm{dec}})}}{1+\sqrt{R_s(a_{\mathrm{eq}})}}\right)\right]^{-1}\mathcal{R}\,, \tag{1.39}$$

where $\omega_b \equiv \Omega_b h^2$ and $\omega_\gamma \equiv \Omega_\gamma h^2$, while h is the Hubble parameter today normalised to $100\,\mathrm{km\,(s\,Mpc)^{-1}}$, and the CMB shift parameter $\mathcal{R}$ is defined as

$$\mathcal{R} \equiv \sqrt{\frac{\Omega_m}{\Omega_K}}\sinh\left(\sqrt{\Omega_K}\int_0^{z_{\mathrm{dec}}}\frac{\mathrm{d}z}{H(z)/H_0}\right). \tag{1.40}$$

The modification of the expansion history by dark energy effects the shift of CMB peaks via this quantity. Using the WMAP 5-year bound $\mathcal{R} = 1.710 \pm 0.019$ (68% CL) (Komatsu *et al.*, 2009), one can constrain the dark energy density to be $0.72 < \Omega_{\mathrm{DE}}^{(0)} < 0.77$ (Tsujikawa, 2010), which is consistent with the SN Ia data.

As $\mathcal{R}$ depends only weakly on the dark energy equation of state w_{DE}, CMB data alone cannot put tight bounds on w_{DE}. In figure 1.6, however, we show that a combination of CMB and SN Ia data is much better at constraining the equation of state. For a flat universe with constant w_{DE}, Kowalski *et al.* (2008) obtain $w_{\mathrm{DE}} = -0.955^{+0.060+0.059}_{-0.066-0.060}$ (with statistical and systematic errors) from combining CMB and SN Ia.

Chapter 2

Quantum Field Theory

Our main focus in this chapter will be on ways of analysing the energy-momentum tensor of scalar fields in various contexts. First of all, in section 2.1, we introduce the circumstances under which a cosmological constant arises in quantum field theory. There we provide, in section 2.1.1, a brief review of the standard formalism of quantum field theory in flat spacetimes. The short section 2.2 then gives an exhibition of the various Green's functions we will need in our investigations. We shall encounter, in section 2.3, where we discuss flat spacetimes with non-trivial topology, the first instances of the spacetime itself affecting the vacuum expectation value of the energy-momentum tensor. Following up on this, section 2.4 reviews quantum field theory in curved spacetime, and introduces the curious phenomenon of cosmological particle creation as an indication of how the very concept of a vacuum loses some of its intuitive meaning when considered in curved spacetime.

In particular, we will have to find ways of handling the divergence of the vacuum energy arising whenever we take products of field operators at the same spacetime point. For this purpose, we will use standard techniques of regularising and then renormalising the infinities appearing in our calculations. Specifically, we will mention the usual normal-ordering procedure in ordinary flat spacetime and the analogous subtraction schemes in flat spacetimes with non-trivial topology.

Finally, when starting to work in curved spacetimes in section 2.5, we shall find that these simple methods fail us and we need more sophisticated ways of handling divergent quantities. Here, we will use dimensional regularisation to temporarily render the energy-momentum tensor finite and isolate several purely geometrical terms that diverge in the limit of four spacetime dimensions. These will simply renormalise the constants Λ and G on the left-hand side of the Einstein equations. Furthermore, we will find two finite, geometrical tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ contributing to the left-hand side. The renormalised Einstein equation (2.140) will then have on its left-hand side these renormalised quantities including the two new tensors, while the source on the right-hand side will be given by the renormalised vacuum expectation value of the energy-momentum tensor.

Our discussion of quantum field theory in this chapter is based on the excellent book by Birrell and Davies (1984), where the curious reader may find elaborations on many other interesting aspects of curved-space quantum field theory, including, for example, the treatment of higher-spin fields.

2.1. Vacuum Contributions and the Cosmological Constant Problem

Lorentz invariance requires that the energy-momentum tensor of the vacuum be proportional to the metric,

$$\langle T_{\mu\nu}\rangle = -\langle\rho\rangle g_{\mu\nu}\,. \tag{2.1}$$

Comparison with equation (1.10) shows that any energy contribution of this form acts just like a cosmological constant, and thus, combined with the original cosmological constant (a parameter in the action) and any other vacuum contributions adds up to an effective cosmological constant

$$\Lambda_{\text{eff}} = \Lambda + 8\pi G\langle\rho\rangle\,. \tag{2.2}$$

Current data (Komatsu *et al.*, 2010) indicate that the vacuum energy density corresponding to the effective cosmological constant,

$$\rho_\Lambda = \frac{\Lambda_{\text{eff}}}{8\pi G}\,, \tag{2.3}$$

is about 74% of the critical density, i.e.,

$$\rho_\Lambda \sim 10^{-47}\,\text{GeV}^4\,. \tag{2.4}$$

2.1.1. Zero-Point Energies in Flat Spacetime

Consider a scalar field $\phi(\boldsymbol{x},t)$ in Minkowski spacetime with Lagrangian density

$$\mathscr{L}(x) = \tfrac{1}{2}\left(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\,, \tag{2.5}$$

where $\eta^{\mu\nu} = \text{diag}(+1,-1,-1,-1)$ is the Minkowski metric. The action for $\phi(\boldsymbol{x},t)$ is given by

$$S = \int \mathrm{d}^4x\, \mathscr{L}(x)\,, \tag{2.6}$$

from which we obtain the Euler-Lagrange equation for ϕ:

$$(\Box + m^2)\phi = 0\,. \tag{2.7}$$

This Klein-Gordon equation has solutions $u_{\boldsymbol{k}}(\boldsymbol{x},t) \propto \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega t}$, where

$$\omega \equiv \left(k^2+m^2\right)^{\frac{1}{2}}, \qquad k \equiv |\boldsymbol{k}| = \left(\sum_{i=1}^{3} k_i^2\right). \tag{2.8}$$

Defining the scalar product

$$\begin{aligned}(\phi_1,\phi_2) &= -\mathrm{i}\int \mathrm{d}^3x\ \left\{\phi_1(x)\partial_0\phi_2^*(x) - \left[\partial_0\phi_1(x)\right]\phi_2^*(x)\right\} \\ &\equiv -\mathrm{i}\int_t \mathrm{d}^3x\ \phi_1(x)\overleftrightarrow{\partial_0}\,\phi_2^*(x),\end{aligned} \tag{2.9}$$

where t denotes a space-like hyperplane of simultaneity at time t; we normalise the solutions $u_{\boldsymbol{k}}$ such that $\left(u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}\right) = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$,

$$u_{\boldsymbol{k}} = \left(2\omega(2\pi)^3\right)^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega t}. \tag{2.10}$$

When restricting the solutions $u_{\boldsymbol{k}}$ to a space-like 3-torus of circumference L with periodic boundary conditions, we choose the normalisation

$$u_{\boldsymbol{k}} = \left(2L^3\omega\right)^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega t}, \tag{2.11}$$

where $k_i = 2\pi j_i/L$, $j_i = 0, \pm 1, \pm 2, \ldots$, $i = 1,2,3$, so $\left(u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}\right) = \delta^{(3)}_{\boldsymbol{k}\boldsymbol{k}'} = \delta_{k_1k_1'}\delta_{k_2k_2'}\delta_{k_3k_3'}$. It is possible to convert from the continuous to the discrete normalisation by replacing

$$\int \mathrm{d}^3k \to (2\pi/L)^3 \sum_{\boldsymbol{k}}. \tag{2.12}$$

The field ϕ can now be expanded in terms of these mode functions:

$$\phi(t,\boldsymbol{x}) = \sum_{\boldsymbol{k}} \left[a_{\boldsymbol{k}} u_{\boldsymbol{k}}(\boldsymbol{x},t) + a_{\boldsymbol{k}}^\dagger u_{\boldsymbol{k}}^*(\boldsymbol{x},t)\right], \tag{2.13}$$

where the coefficients $a_{\boldsymbol{k}}$, $a_{\boldsymbol{k}}^\dagger$ are considered to be creation and annihilation operators satisfying the canonical commutation relations

$$[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}] = 0, \quad [a_{\boldsymbol{k}}^\dagger, a_{\boldsymbol{k}'}^\dagger] = 0, \quad [a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^\dagger] = \delta_{\boldsymbol{k}\boldsymbol{k}'}. \tag{2.14}$$

We can use these operators to define the vacuum state of our theory, the "zero-particle" state in the usual Fock representation of the Hilbert space, as that state which is annihilated by all the $a_{\boldsymbol{k}}$:

$$a_{\boldsymbol{k}}|0\rangle = 0. \tag{2.15}$$

Let us now define the energy-momentum tensor $T_{\mu\nu}$ (sometimes called stress-energy tensor, SET) of ϕ. In general, $T_{\mu\nu}$ is defined as

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \frac{\delta S}{\delta g^{\mu\nu}(x)}, \tag{2.16}$$

where $g(x)$ is the determinant of the metric tensor. In Minkowski space ($g_{\mu\nu} = \eta_{\mu\nu}$) and for a scalar field with action (2.6), equation (2.16) turns into

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} + \tfrac{1}{2}m^2\phi^2\eta_{\mu\nu}. \tag{2.17}$$

Hence we obtain the Hamiltonian and momentum densities

$$\mathcal{H} \equiv T_{00} = \frac{1}{2}\left[(\partial_0\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2 + m^2\phi^2\right], \quad T_{0i} = \partial_0\phi\,\partial_i\phi, \quad i = 1,2,3, \tag{2.18}$$

from which we get, upon inserting the mode expansion (2.13), and using the commutators (2.14), the Hamiltonian and momentum operators

$$H \equiv \int T_{00}\,\mathrm{d}^3x = \sum_{\boldsymbol{k}}\left(a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \tfrac{1}{2}\right)\omega, \qquad P_i \equiv \int \mathrm{d}^3x\, T_{0i} = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} k_i\,. \tag{2.19}$$

Now, while the vacuum carries no momentum, $\langle 0|\boldsymbol{P}|0\rangle = 0$, the corresponding vacuum energy is

$$\begin{aligned}\langle\rho\rangle \equiv \langle 0|H|0\rangle &= \frac{1}{2}\langle 0|0\rangle\sum_{\boldsymbol{k}}\omega = \frac{1}{2}\sum_{\boldsymbol{k}}\omega = \frac{1}{2}\left(\frac{L}{2\pi}\right)^3\int \mathrm{d}^3k\,\omega \\ &= \frac{L^3}{4\pi^2}\int_0^\infty \mathrm{d}k\, k^2\sqrt{k^2+m^2}\,.\end{aligned} \tag{2.20}$$

which exhibits an ultraviolet divergence $\langle\rho\rangle \propto k^4$. If we trust quantum field theory to be correct up to the Planck scale M_{Pl}, we may regulate this divergence by introducing a cutoff $k_{\mathrm{max}} \simeq M_{\mathrm{Pl}}$, which yields

$$\langle\rho\rangle \approx \frac{k_{\mathrm{max}}^4}{16\pi^2} \approx 2\times 10^{74}\,\mathrm{GeV}^4\,. \tag{2.21}$$

This is greater than the observed value ρ_Λ by about 120 orders of magnitude. We urge the reader to remember, that this short calculation does not imply that quantum field theory automatically comes with huge contradictions of observational data. In fact, it just means that we would need to modify the Lagrangian of our theory to include a counterterm that cancels the contribution (2.20) to 120 decimal places. This, however, would be so extreme a case of fine-tuning that physicists feel compelled to investigate alternative ways of dealing with the divergence of the vacuum energy.

In flat spacetime, the vacuum energy is usually neglected, since, in the absence of gravity, the absolute value of energy is not measurable anyway, which allows us to redefine the zero point of energy—even by an infinite amount. This renormalisation can be accomplished by requiring that in any product of creation and annihilation operators, all annihilation operators stand to the right of the creation operators, $:a_k a_k^\dagger: = a_k^\dagger a_k$, and hence

$$:H: = \sum_k a_k^\dagger a_k \omega \,. \tag{2.22}$$

Thus, this procedure, known as normal ordering, eliminates the infinite vacuum energy, in Minkowski space at least.

2.1.2. Cancelling Vacuum Contributions via Supersymmetry

A partial solution to the problem of the large zero-point energy is provided by supersymmetry (SUSY). In an unbroken supersymmetric theory, every bosonic field is complemented by a fermionic field of equal mass, whose vacuum energy contribution is equal but with opposite sign. For a field with spin j, the expression (2.20) generalises to (Copeland *et al.*, 2006, section IV.B)

$$\langle \rho \rangle = \frac{(-1)^{2j}(2j+1)}{4\pi^2} \int_0^\infty \mathrm{d}k \, k^2 \sqrt{k^2+m^2} \,. \tag{2.23}$$

Thus, in exact supersymmetry, the zero-point energies of the fermionic degrees of freedom are cancelled by the contributions from an equal number of bosonic degrees of freedom.

Obviously, supersymmetry is not an exact symmetry of the universe today. The integral in equation (2.23) can be split into an integral up to the SUSY breaking scale M_{SUSY}, assumed to be $M_{\mathrm{SUSY}} \approx 10^3\,\mathrm{GeV}$, and an integral from M_{SUSY} to infinity,

$$\int_0^\infty = \int_0^{M_{\mathrm{SUSY}}} + \int_{M_{\mathrm{SUSY}}}^\infty \,, \tag{2.24}$$

where the second part vanishes, because SUSY is unbroken at that scale, while the first part gives a contribution of order $\langle \rho \rangle^{\mathrm{SUSY}} \approx (16\pi^2)^{-1} M_{\mathrm{SUSY}}^4 \approx 10^{10}\,\mathrm{GeV}^4$. This is "only" 57 orders of magnitude greater than the observed value of the cosmological constant (2.4). Supersymmetry is therefore sometimes said to solve the cosmological constant problem halfway.

Additionally, SUSY breaking induces another contribution of order M_{SUSY}^4 in the same way as described in section 2.1.3 for the breaking of the electroweak symmetry.

2.1.3. Vacuum Contributions from Phase Transitions

The zero-point energies of the various fields are not the only troublesome contribution to the cosmological constant. Consider, for instance, spontaneous symmetry breaking (SSB) in the electroweak theory. The scalar field potential is

$$V = V_0 - \mu^2 \phi^\dagger \phi + g(\phi^\dagger \phi)^2, \tag{2.25}$$

which, at its minimum, takes the value

$$\langle \rho \rangle = V_{\text{min}} = V_0 - \frac{\mu^4}{4g}. \tag{2.26}$$

In the electroweak theory, we would thus expect a vacuum energy contribution of order

$$\langle \rho \rangle^{EW} \approx (200\,\text{GeV})^4 \approx 10^9\,\text{GeV}^4. \tag{2.27}$$

Again, this contribution is greater than the measured value for ρ_Λ by 56 orders of magnitude. In the same manner, we expect vacuum contributions to arise from the spontaneous breakdown of any number of other symmetries.

We could, of course, require $V_0 \approx \langle \rho \rangle^{EW}$, in order to not violate observations. However, at high temperatures, before symmetry breaking, the coefficient of the $\phi^\dagger\phi$-term in the potential is positive, and the minimum of the potential is at $\phi = 0$. Thus, tuning V_0 such that the cosmological constant is small today, we automatically generate a large vacuum energy $\langle \rho \rangle \approx V_0$ in the early universe (Sahni and Starobinsky, 2000). The energy density $\langle \rho \rangle^{EW}$ induced in this way is, of course, far too low to affect inflation, where typical energy densities of the inflaton field are $V(\phi) \sim 10^{-12} M_{\text{Pl}}^4 \sim 10^{61}\,\text{GeV}^4$ (see section 3.3.4). Similar contributions originating, for instance, from GUT breaking at $M_{\text{GUT}} \sim 10^{-3} M_{\text{Pl}}$, may, however, have some influence on cosmic evolution during that era.

2.2. Green's Functions

In the rest of this work, we will frequently refer to the Green's functions of the wave equation. Here, the vacuum expectation values of the commutator and the anticommutator of a scalar field ϕ,

$$\begin{aligned} \mathrm{i}G(x,x') &= \langle 0|[\phi(x), \phi(x')]|0\rangle, \\ G^{(1)}(x,x') &= \langle 0|\{\phi(x), \phi(x')\}|0\rangle, \end{aligned} \tag{2.28}$$

will be particularly important. They are sometimes known as the Pauli-Jordan function and Hadamard's elementary function, respectively, and can be split into their positive and negative frequency parts as

$$\begin{aligned} \mathrm{i}G(x,x') &= G^+(x,x') - G^-(x,x'), \\ G^{(1)}(x,x') &= G^+(x,x') + G^-(x,x'), \end{aligned} \tag{2.29}$$

where the Wightman functions $G^{\pm}$ are defined as

$$\begin{aligned} G^{+}(x,x') &= \langle 0|\phi(x)\phi(x')|0\rangle \,, \\ G^{-}(x,x') &= \langle 0|\phi(x')\phi(x)|0\rangle \,. \end{aligned} \tag{2.30}$$

As can be seen from the field equation (2.7), these Green's functions all satisfy the equation

$$(\Box_x + m^2)\,\mathcal{G}(x,x') = 0 \,. \tag{2.31}$$

The Feynman propagator is defined as the expectation value of the time-ordered product of fields,

$$\begin{aligned} \mathrm{i}G_{\mathrm{F}}(x,x') &= \langle 0|T(\phi(x)\phi(x'))|0\rangle \\ &= \theta(t-t')G^{+}(x,x') + \theta(t'-t)G^{-}(x,x') \,, \end{aligned} \tag{2.32}$$

with the Heaviside step function $\theta(t)$ defined as

$$\theta(t) = \begin{cases} 1\,, & t>0\,, \\ 0\,, & t<0\,. \end{cases} \tag{2.33}$$

It satisfies $(\Box_x + m^2)\,G_{\mathrm{F}}(x,x') = -\delta^{(n)}(x-x')$.

The retarded and advanced Green's functions are defined as

$$\begin{aligned} G_R(x,x') &= -\theta(t-t')G(x,x') \,, \\ G_A(x,x') &= \theta(t'-t)G(x,x') \,, \end{aligned} \tag{2.34}$$

and their average is $\bar{G}(x,x') = \frac{1}{2}[G_R(x,x') + G_A(x,x')]$. This allows us to relate the Feynman propagator to Hadamard's elementary function by

$$G_{\mathrm{F}}(x,x') = -\bar{G}(x,x') - \tfrac{1}{2}\mathrm{i}G^{(1)}(x,x') \,. \tag{2.35}$$

In the massless case, the Feynman propagator and Hadamard's elementary function reduce to

$$G_F(x,x') = \frac{\mathrm{i}}{8\pi^2\sigma} - \frac{1}{8\pi}\delta(\sigma) \,, \tag{2.36}$$

$$G^{(1)}(x,x') = -\frac{1}{4\pi^2\sigma} \,, \tag{2.37}$$

where $\sigma = \frac{1}{2}(x-x')^2$ is half the square of the separation of x and x'.

By inserting the mode decomposition of ϕ into the definitions of the Green's functions, we find that they can all be represented as

$$\mathcal{G}(x,x') = \int \frac{\mathrm{d}^n k}{(2\pi)^n}\, \frac{\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}') - \mathrm{i}k^0(t-t')]}{(k^0)^2 - |\boldsymbol{k}|^2 - m^2} \,, \tag{2.38}$$

from which we obtain the various Green's functions by specifying the way to perform the contour integration in the complex plane. For example, using Feynman's prescription, we would shift the poles at $k^0 = \pm(|\boldsymbol{k}|^2 + m^2)^{\frac{1}{2}}$ off the real axis by the replacement $m^2 \to m^2 - \mathrm{i}\varepsilon$ to recover $G_{\mathrm{F}}(x, x')$.

2.3. Stress-Tensor Renormalisation in Flat Spacetime with Non-Trivial Topology

2.3.1. Vacuum Energy in Cylindrical Spacetime

Before proceeding to the case we are ultimately interested in — the stress-energy in curved spacetime — let us consider, as a simple, yet instructive example, a flat spacetime with one time-like dimension and one compactified space-like dimension with periodic boundary conditions, i.e., we identify the points x and $x + L$. This $\mathbb{R}^1 \times S^1$ spacetime is visualised in figure 2.1.[1]

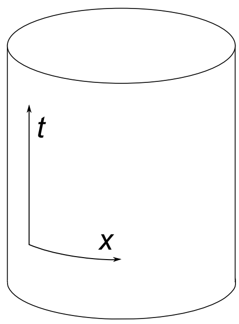


Figure 2.1.: Two-dimensional Minkowski spacetime with periodic boundary conditions in the spatial dimension.

We will restrict our attention to a massless scalar field, whose modes (2.11) in this two-dimensional cylindrical spacetime are

$$u_k = (2L\omega)^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \qquad k = 2\pi n/L,\ n = 0, \pm 1, \pm 2, \ldots \tag{2.39}$$

The energy-momentum tensor (2.17) is given by

$$T_{tt} = T_{xx} = \tfrac{1}{2}\left[(\partial_t \phi)^2 + (\partial_x \phi)^2\right], \qquad T_{tx} = \partial_t \phi\, \partial_x \phi, \tag{2.40}$$

[1] Note that there is no relation to Einstein's cylinder universe mentioned in section 1.1.2. There, we introduced a finite energy density proportional to the cosmological constant in an attempt to make the universe static; this lead to the finite and constant curvature radius of the Einstein universe which justified the name "cylinder spacetime". Here, on the other hand, we are merely considering a spatially finite $(1+1)$-dimensional Minkowski spacetime whose spatial edges we have glued together by imposing periodic boundary conditions.

which we will evaluate in the vacuum state $|0_L\rangle$ associated with the modes (2.39). In the limit of the extent L of the compactified dimension tending to infinity, we recover the usual Minkowski-space modes in 2 dimensions, and the vacuum state tends to that of Minkowski space, $\lim_{L\to\infty}|0_L\rangle \to |0\rangle$.

Inserting in (2.17) the mode expansion (2.13), one easily finds that in general the vacuum expectation value of the energy-momentum tensor of the massless field is

$$\langle 0|T_{\mu\nu}|0\rangle = \frac{1}{2}\sum_k \left(\partial_0 u_k\, \partial_0 u_k^* + \partial_i u_k\, \partial_i u_k^*\right) . \tag{2.41}$$

Evaluating this for the modes (2.39) and the corresponding vacuum $|0_L\rangle$, we obtain for the vacuum energy

$$\langle 0_L|T_{tt}|0_L\rangle = \frac{1}{2L}\sum_{n=-\infty}^{\infty}|k| = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n\,. \tag{2.42}$$

Thus, as for Minkowski space, the vacuum energy of $\mathbb{R}^1\times S^1$ is infinite. This is hardly surprising, as the divergence arises from the ultraviolet behaviour of the field. The compactification, on the other hand, only affects the infrared modes, as only these can actually probe the global structure of the spacetime.

In the Minkowski case, we dealt with the divergence by normal ordering with respect to the creation and annihilation operators of the Fock space associated with the modes (2.10). With respect to a general state $|\Psi\rangle$, this reduces to the prescription

$$\langle\Psi|:T_{\mu\nu}:|\Psi\rangle = \langle\Psi|T_{\mu\nu}|\Psi\rangle - \langle 0|T_{\mu\nu}|0\rangle\,. \tag{2.43}$$

Considering $|0_L\rangle$ as a state in this Fock space, we find

$$\begin{aligned}\langle 0_L|:T_{tt}:|0_L\rangle &= \langle 0_L|T_{tt}|0_L\rangle - \langle 0|T_{tt}|0\rangle \\ &= \langle 0_L|T_{tt}|0_L\rangle - \lim_{L'\to\infty}\langle 0_{L'}|T_{tt}|0_{L'}\rangle\,.\end{aligned} \tag{2.44}$$

Hence, in order to remove the infinity from $\langle 0_L|T_{\mu\nu}|0_L\rangle$, we need to subtract the Minkowski-space equivalent, which contains the same divergence. The subtraction of the two individually divergent terms in (2.42), however, is far from trivial and we will postpone a careful treatment until section 2.5. Here, we shall simply introduce an ultraviolet cutoff $e^{-\alpha|k|}$ into divergent sums like (2.42). This procedure yields

$$\begin{aligned}\langle 0_L|T_{tt}|0_L\rangle &= \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n e^{-2\pi\alpha n/L} = \frac{2\pi}{L^2}\frac{e^{2\pi\alpha/L}}{\left(e^{2\pi\alpha/L}-1\right)^2} \\ &= \frac{1}{2\alpha^2} - \frac{\pi^2}{6L^2} + \frac{\pi^4\alpha^2}{30L^4} + \mathcal{O}\left(\alpha^4\right),\end{aligned} \tag{2.45}$$

where, in the last step, we have expanded about $\alpha = 0$. Similarly, we find for the second term in (2.44), the Minkowski vacuum energy,

$$\lim_{L'\to\infty} \langle 0_{L'}|T_{tt}|0_{L'}\rangle = \frac{1}{2\alpha^2}\,. \tag{2.46}$$

Thus, the subtraction (2.44) leads us to conclude that, with respect to Minkowski space ($\langle 0_L|\!:T_{\mu\nu}:\!|0_L\rangle = 0$), the $\mathbb{R}^1 \times S^1$ universe contains the uniformly distributed energy and pressure densities[2]

$$\langle\rho\rangle = \langle 0_L|\!:T_{tt}:\!|0_L\rangle = -\frac{\pi}{6L^2}, \qquad \langle p\rangle = \langle 0_L|\!:T_{xx}:\!|0_L\rangle = -\frac{\pi}{6L^2}\,. \tag{2.47}$$

It is possible to more elegantly reach the above result by using Green's functions instead of $T_{\mu\nu}$ and an ultraviolet cutoff function. For convenience, we rewrite $T_{\mu\nu}$ in terms of null coordinates $u = t - x$, $v = t + x$ and find for a massless scalar field in two dimensions

$$T_{uu} = (\partial_u\phi)^2\,, \quad T_{vv} = (\partial_v\phi)^2\,, \quad T_{uv} = \tfrac{1}{2}\partial_u\phi\partial_v\phi\,. \tag{2.48}$$

Conversely,

$$T_{tt} = T_{uu} + T_{vv} + 2T_{uv}\,. \tag{2.49}$$

With the Green's function $G^{(1)}$ of equation (2.28), we find

$$\langle 0_L|T_{uu}(u,v)|0_L\rangle = \lim_{v'',v'\to v}\ \lim_{u'',u'\to u}\ \partial_{u''}\partial_{u'}\tfrac{1}{2}G_L^{(1)}(u'',v'';u',v'), \tag{2.50}$$

which is symmetric under exchange of (u'',v'') and (u',v'). Using the modes (2.39), we obtain

$$\begin{aligned} G_L^{(1)}(u'',v'';u',v') &= \langle 0_L|\{\phi(u'',v''),\phi(u',v')\}|0_L\rangle \\ &= \sum_{n=-\infty}^{\infty}\left[u_k(u'',v'')u_k^*(u',v') + \text{c.c.}\right] \\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\mathrm{e}^{(-2\pi n\mathrm{i}/L)\Delta u} + \mathrm{e}^{(-2\pi n\mathrm{i}/L)\Delta v}\right) + \text{c.c.}\,, \end{aligned} \tag{2.51}$$

where $\Delta u = u'' - u'$, $\Delta v = v'' - v'$. In the last step, we have discarded the infrared-divergent $n = 0$ term; it would have vanished anyway, had we performed the differentiation in equation (2.50) before passing to the massless limit.

[2] Notice, that for this energy contribution, the equation of state is $w = p/\rho = +1$. In this particular case, it would therefore be difficult to establish a correspondence between the Casimir energy and a cosmological constant in $(3+1)$ dimensions, for which $w = -1$.

Performing the summation and using the Taylor series for the natural logarithm, which is $\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$, the Green's function turns into

$$G_L^{(1)}(u'', v''; u', v') = -\frac{1}{4\pi} \ln\left[16 \sin^2(\pi \Delta u/L) \sin^2(\pi \Delta v/L)\right], \tag{2.52}$$

and hence we obtain for the vacuum energy

$$\langle 0_L | T_{uu}(u, v) | 0_L \rangle = -\lim_{\Delta u \to 0} \frac{\pi}{4L^2} \operatorname{cosec}^2\left(\frac{\pi}{\Delta u/L}\right). \tag{2.53}$$

Using the Taylor expansion

$$\frac{\pi}{4L^2} \operatorname{cosec}^2\left(\frac{\pi}{\Delta u/L}\right) = \frac{1}{4\pi \Delta u^2} + \frac{\pi}{12L^2} + \frac{\pi^3 \Delta u^2}{60L^4} + \mathcal{O}(\Delta u^4), \tag{2.54}$$

we find that the vacuum energy diverges as $(\Delta u)^{-2}$ in the limit $\Delta u \to 0$. In order to remove this divergence, we subtract from equation (2.54) the Minkowski-space limit $\langle 0 | T_{uu} | 0 \rangle = \lim_{L \to \infty} \langle 0_L | T_{uu} | 0_L \rangle = -1/(4\pi \Delta u^2)$. The limit $\Delta u \to 0$ then just leaves the term $-\pi/(12L^2)$.

As $G_L^{(1)}$ is symmetric under interchange of u and v, we have $\langle 0_L | T_{vv} | 0_L \rangle = \langle 0_L | T_{uu} | 0_L \rangle$. Additionally, as $G_L^{(1)}$ can be written as the sum of a u- and a v-independent function (cf. equation (2.51)), we can use $\langle 0_L | T_{uv} | 0_L \rangle = \langle 0_L | T_{vu} | 0_L \rangle = 0$ to simplify equation (2.49) and obtain for the energy density

$$\langle \rho \rangle = \langle 0_L | : T_{tt} : | 0_L \rangle = 2 \langle 0_L | T_{uu} | 0_L \rangle = -\frac{\pi}{6L^2}. \tag{2.55}$$

This is the same result we had already found in equation (2.47) by analysing the energy-momentum tensor explicitly and introducing an ultraviolet cutoff.

2.3.2. The Casimir Effect

We saw in section 2.3.1 that the divergent $T_{\mu\nu}$ in Minkowski spacetime changes by a finite, non-zero amount when a non-trivial topology is introduced. There, we changed the topology of the flat spacetime by introducing periodic boundary conditions, which changed the zero-modes of a massless scalar field propagating in the spacetime, thus effecting the change in the zero-mode sum. Here, we will take a look at a similar effect produced by introducing conducting surfaces, which also modify the topology of the field configuration.

Consider first an infinite plane in unbounded four-dimensional Minkowski spacetime. We require a massless scalar field to vanish on the conducting surface placed at $x_3 = 0$ (Dirichlet boundary condition). The modes are then no longer of the form (2.10), but instead take the form $\sin(|k_3|x_3)e^{ik_1x_1+ik_2x_2-i\omega t}$, because the field reflects off the boundary. The Green's function $G^{(1)}$ of equation (2.37) has to be replaced by

$$G^{(1)}_{\mathrm{B}}(x,x')=\frac{1}{2\pi^2}\left(\frac{1}{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2-(t-t')^2}\right.\\ \left.-\frac{1}{(x_1-x_1')^2+(x_2-x_2')^2+(x_3+x_3')^2-(t-t')^2}\right), \tag{2.56}$$

which is obtained by using the method of images familiar from electrostatics.

The Green's function (2.56) vanishes for $x_3=0$ or $x_3'=0$. Its first term is identical to the Green's function (2.37) for unbounded Minkowski space, and diverges quadratically as $x\to x'$. In order to compare the vacuum energy of the space containing the boundary surface to the unbounded case, we simply discard the first term. In analogy to equation (2.50), we obtain for the vacuum energy density

$$\begin{aligned}\langle 0|T_{tt}|0\rangle &= \lim_{\substack{x'\to x\\ x''\to x}} \tfrac{1}{4}\left(\partial_{t''}\partial_{t'}+\partial_{x_1''}\partial_{x_1'}+\partial_{x_2''}\partial_{x_2'}+\partial_{x_3''}\partial_{x_3'}\right)[G^{(1)}_B(x'',x'')-G^{(1)}(x'',x'')]\\ &= -\frac{1}{16\pi^2x_3^4}\,. \end{aligned}\tag{2.57}$$

Similarly, $\langle 0|T_{ii}|0\rangle=(16\pi^2x_3^4)^{-1}$; all other components vanish.

Obviously, the vacuum stress due to the boundary diverges near the surface ($x_3\to 0$), while the effect of the distortion vanishes at infinity ($x_3\to\infty$). Even though the vacuum is completely unaffected far from the boundary, the total vacuum energy per area of the boundary surface is infinite. Apparently, subtracting the infinite Minkowski space vacuum energy does not altogether rid us of divergences. In fact, it can be seen from equation (2.56) already, that after removing the Minkowski space contribution represented by the first term, the remaining term diverges on the boundary $x_3=0$ in the coincidence limit $x'\to x$.

It seems clear, then, that $G^{(1)}_{\mathrm{B}}-G^{(1)}$ will in general diverge near an arbitrary conducting surface. The vacuum stress $\langle T_{\mu\nu}\rangle$ may still be finite, however. For symmetry reasons, $\langle T_{\mu\nu}\rangle$ near a plane boundary at $x_3=0$ has to be constructed from $\eta_{\mu\nu}$ and $\hat{x}_3^\mu\hat{x}_3^\nu$, where $\hat{x}_3^\mu$ is the unit vector orthogonal to the boundary. Furthermore, $\langle T_{\mu\nu}\rangle$ can only be a function of x_3, $\langle T^{\mu\nu}\rangle=f(x_3)\eta^{\mu\nu}+g(x_3)\hat{x}_3^\mu\hat{x}_3^\nu$. Covariant conservation $\langle T^{\mu\nu}\rangle_{,\mu}=0$ (remember that we are working in flat spacetime, for the moment) implies that f and g can only differ by a constant, and hence

$$\langle T_{\mu\nu}\rangle=g(x_3)(\eta^{\mu\nu}+\hat{x}_3^\mu\hat{x}_3^\nu)+\alpha\eta^{\mu\nu}, \tag{2.58}$$

whose trace is $3g(x_3)+4\alpha$. If we require this trace to vanish, we obtain

$$g(x_3)=-4\alpha/3=\text{constant}\,. \tag{2.59}$$

Now, as we wish the vacuum distortion to vanish at infinity, once we have subtracted the Minkowski space contribution, we may conclude that $g(x_3) = 0 = \alpha$, and hence the renormalised vacuum stress for a field with traceless energy-momentum tensor vanishes.

From the Lagrangian density for a scalar field $\phi(x)$ with mass m,

$$\mathscr{L}(x) = \tfrac{1}{2}[-g(x)]^{1/2}\left\{g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} - \left[m^2 + \xi R(x)\right]\phi^2(x)\right\}, \tag{2.60}$$

and the definition (2.16), we obtain for the energy-momentum tensor of a conformally coupled $\left(\xi = \frac{1}{6}\right)$, massless scalar field in the limit $g_{\mu\nu} \to \eta_{\mu\nu}$ the expression

$$T_{\mu\nu}\left(\xi = \tfrac{1}{6}\right) = \tfrac{2}{3}\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{6}\eta_{\mu\nu}\eta^{\rho\sigma}\phi_{,\sigma}\phi_{,\rho} - \tfrac{1}{3}\phi\phi_{;\mu\nu} + \tfrac{1}{12}\eta_{\mu\nu}\phi\Box\phi. \tag{2.61}$$

The trace of equation (2.61) vanishes, and thus we can employ the above argument to conclude that for a conformally coupled scalar field the vacuum stress vanishes near a plane boundary.[3]

Two comments are necessary at this point. First, the term $\xi R\phi^2$ in the Lagrangian (2.60) represents a coupling of the scalar field to gravity; ξ is a numerical factor and the Ricci scalar $R(x)$ provides the only possible local, scalar coupling of this sort with the correct dimensions. Two particularly interesting cases are the minimal coupling $\xi = 0$ and the conformal coupling

$$\xi = \frac{1}{4}[(n-2)/n - 1] \equiv \xi(n). \tag{2.62}$$

Consider, for instance, a conformal transformation

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \tag{2.63}$$

which is just a local shrinking or stretching of the manifold. In the conformally coupled case, the field equation of the massless field ($m = 0$) is invariant under transformations of this kind if the field is assumed to transform as $\phi(x) \to \bar{\phi}(x) \equiv \Omega^{(2-n)/2}\phi(x)$, where the power $(2-n)/2$ is called the conformal weight of the field. That is to say, if ϕ satisfies the equation of motion $\left[\Box + \frac{1}{4}(n-2)R(x)/(n-1)\right]\phi(x) = 0$, the transformed field $\bar{\phi}$ satisfies the transformed equation $\left[\bar{\Box} + \frac{1}{4}(n-2)\bar{R}(x)/(n-1)\right]\bar{\phi}(x) = 0$.

Second, the reader may wonder why the Lagrangian (2.60) contains the term $\xi R(x)\phi^2$ in the first place. In flat spacetime, $R = 0$ and the coupling term should vanish. We need to remember, however, that the energy-momentum tensor is obtained by varying the action with respect to the metric. This procedure introduces additional terms which then

[3] Nonetheless, for a curved boundary, a divergent surface energy reappears, and as one approaches the surface, the vacuum energy-momentum tensor of a conformally coupled field is $\langle T_{\mu\nu}\rangle \propto \varepsilon^{-3}\chi_{\mu\nu} + \mathcal{O}(\varepsilon^{-2})$ (Deutsch and Candelas, 1979), where ε is the distance from the surface, and $\chi_{\mu\nu}$ is the second fundamental form of the boundary, which describes its extrinsic curvature.

yield the general expression (2.152) for the energy momentum tensor in curved spacetime. Since these new terms do not all vanish in the limit $g_{\mu\nu} \to \eta_{\mu\nu}$, we retain in the flat-space energy-momentum tensor a contribution from the coupling to spacetime curvature. This contribution constitutes the difference between our earlier expression (2.17) for the energy-momentum tensor of a scalar field in flat spacetime and equation (2.61).

Generalising our investigation of boundary effect on the vacuum stress to two boundaries at $x_3 = 0$ and $x_3 = a$, we can write the Green's function $G^{(1)}_{\mathrm{B}}$ as an infinite image sum

$$G^{(1)}_{\mathrm{B}}(x,x') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3'-an)^2-(t-t')^2} - \frac{1}{(x_1-x_1')^2+(x_2-x_2')^2+(x_3+x_3'-an)^2-(t-t')^2} \right). \tag{2.64}$$

We can remove the infinite Minkowski space contribution by dropping the $n = 0$ term. Using equation (2.61), we find

$$\langle 0|T_{\mu\nu}|0\rangle_B = \frac{-\pi^2}{1440a^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{2.65}$$

for the vacuum stress-energy of a conformally invariant scalar field.

In the case of the electromagnetic field, where the vacuum energy-momentum tensor is $\langle 0|T_{\mu\nu}|0\rangle_B = \frac{-\pi^2}{720a^4}\,\mathrm{diag}(\,1,-1,-1,\,3)$ (owing to the fact that the electromagnetic field has twice as many degrees of freedom as a scalar field), which gives rise to an attractive force between two electrically neutral conducting surfaces. The force per unit area is

$$F = -\frac{\partial \langle T_{00}\rangle}{\partial a} = \frac{-\pi^2}{240a^4}. \tag{2.66}$$

A similar force, $F = -\pi^2/480a^4$, would arise in the case of the scalar field.

The physical origin of these non-zero vacuum stresses is that the boundary conditions on the bounding surfaces restrict the field modes in the x_3-direction to form a discrete set. This is completely analogous to the situation of section 2.3.1, where a similar constraint was achieved by imposing periodic boundary conditions on the spacetime and thus on any fields propagating in it.

The existence of a force of this kind was first indicated by Casimir and Polder (1948), who obtained it from a calculation of the Van der Waals force between a neutral atom and a perfectly conducting plate by including the effects of relativistic retardation. Casimir (1948) then showed that the problem of the force between two conducting plates can be investigated in terms of the change of the electromagnetic zero-point energy.[4] It seems appropriate, therefore, to refer to the vacuum energies arising from non-trivial topology, be it due to conducting surfaces placed in Minkowski space or the compactification of a spatial dimension, as Casimir energies.

2.4. Vacuum Energy in Curved Spacetime

Before proceeding to calculate the vacuum energy for certain curved spacetimes, we will give a brief overview of curved-spacetime quantum field theory, with special emphasis on the fact that the concept of particles loses its intuitive meaning when the spacetime curvature is non-zero.

2.4.1. Scalar Field Quantisation in Curved Spacetime

We will consider scalar fields in a spacetime that is a smooth (C^∞) n-dimensional, globally hyperbolic, pseudo-Riemannian manifold. Quantisation then proceeds much like in Minkowski space (see section 2.1.1). Starting out with the Lagrangian density (2.60), and varying the action $S=\int \mathrm{d}^n x\, \mathscr{L}(x)$ with respect to ϕ, we obtain the field equation

$$\left[\Box_x + m^2 + \xi R(x)\right] \phi(x) = 0, \tag{2.67}$$

which contains the gravity coupling term we elaborated on in our discussion following equation (2.61).

Equation (2.67) admits a complete set of mode solutions $u_i(x)$, which are orthonormal with respect to the scalar product

$$(\phi_1, \phi_2) = -\mathrm{i} \int_\Sigma \mathrm{d}\Sigma^\mu\, \phi_1(x) \overleftrightarrow{\partial_\mu} \phi_2^*(x) \left[-g_\Sigma(x)\right]^{\frac{1}{2}}, \tag{2.68}$$

where $\mathrm{d}\Sigma^\mu = n^\mu\, \mathrm{d}\Sigma$, with a future-directed unit vector n^μ orthogonal to the space-like hypersurface Σ and $\mathrm{d}\Sigma$ the volume element in Σ; $g_\Sigma(x)$ is the metric on the space-like hypersurfaces. As can be easily shown by the use of Gauss's law, the scalar product is independent of the choice of the hypersurface Σ.

The mode solutions of equation (2.67) satisfy the relations

$$(u_i, u_j) = \delta_{ij}, \qquad (u_i^*, u_j^*) = -\delta_{ij}, \qquad (u_i, u_j^*) = 0 \tag{2.69}$$

and enable us to write the field as

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right]. \tag{2.70}$$

[4] This Casimir force was first measured by Sparnaay (1958).

We can then quantise the theory by implementing the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = 0, \qquad [a_i^\dagger, a_j^\dagger] = 0. \tag{2.71}$$

In principle, we could now proceed to construct a vacuum state and Fock space just like in the flat-space case (2.1.1). For the flat-space construction, however, it was crucial to find a natural set of mode solutions (2.10), associated with the invariance of Minkowski spacetime under the action of the Poincaré group which provides us with a natural coordinate system (t, x, y, z). In other words, the vector $\partial/\partial t$ is a Killing vector of Minkowski spacetime, and the modes (2.10) are eigenfunctions of this Killing vector with eigenvalues $-i\omega$ for positive frequency ω.[5]

Now, in a general curved spacetime, there are no Killing vectors with respect to which we could define positive-frequency modes. Without this natural choice of mode decomposition of the field ϕ, we are free to use a second complete orthonormal set of modes $\bar{u}_j(x)$ to expand ϕ in:

$$\phi(x) = \sum_i \left[\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)\right] . \tag{2.72}$$

This defines a new vacuum state $|\bar{0}\rangle$ which is annihilated by the $\bar{a}_j$,

$$\bar{a}_j|\bar{0}\rangle = 0, \quad \forall j\,, \tag{2.73}$$

and thus a new Fock space of many-particle states. We can expand the two complete sets of modes in terms of one another:

$$\bar{u}_j = \sum_i \left[\alpha_{ji} u_i + \beta_{ji} u_i^*\right], \qquad u_i = \sum_j \left[\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right] . \tag{2.74}$$

These relations are called Bogoliubov transformations. The Bogoliubov coefficients α_{ij}, and β_{ij} can be found by using the normalisation (2.69) of the modes u_i,

$$\alpha_{ij} = (\bar{u}_i, u_j), \qquad \beta_{ij} = -(\bar{u}_i, u_j^*). \tag{2.75}$$

The operator sets a_i and $\bar{a}_i$ can be expressed in terms of one another by equating the two expansions (2.70) and (2.72) and using again the orthogonality of the modes, (2.69),

$$a_i = \sum_j \left(\alpha_{ji}\bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger\right), \qquad \bar{a}_j = \sum_i \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger\right) . \tag{2.76}$$

[5]Killing vectors are associated with symmetries of the metric $g_{\mu\nu}$. The metric is said to be form-invariant with respect to a coordinate transformation f: $x \to x'$, if $g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x)$. Infinitesimally, the transformation f can be written as $x'^\mu = x^\mu + \varepsilon X^\mu$, where $|\varepsilon| \ll 1$. The requirement for form-invariance of the metric then amounts to $0 = X_{\sigma;\rho} + X_{\rho;\sigma}$. Any four-vector field $X_\sigma(x)$ satisfying this relation is called a Killing vector of the metric $g_{\mu\nu}(x)$. An n-dimensional metric admitting the maximum number of Killing vectors, $n(n+1)/2$, is called maximally symmetric. (Weinberg, 1972, chapter 13.1)

Clearly, the Fock spaces based on the two sets of modes are inequivalent if any of the $\beta_{ij} \neq 0$. In that case, the operators a_i will not annihilate the vacuum $|\bar{0}\rangle$, and the expectation value of the number operator $N_i = a_i^\dagger a_i$ for the u_i-mode particles in the $\bar{u}_i$ vacuum $|\bar{0}\rangle$ is

$$\langle \bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2, \tag{2.77}$$

that is, the $\bar{u}_i$-vacuum contains $\sum_j |\beta_{ji}|^2$ particles in the mode u_i.

If the u_i are positive-frequency modes with respect to some time-like Killing vector field X, the linear combinations $\bar{u}_i$ in (2.74) will be positive-frequency modes only if they contain no admixture of the negative-frequency u_i^*, i.e., if all β_{ij} vanish. Only then do both sets of modes share a vacuum state.

The crucial insight of this section is that in a curved spacetime, there is no natural choice of the quantum vacuum state. The conventional Minkowski space vacuum state is distinguished by being the agreed vacuum of all inertial observers, because both the vacuum and measurements performed by inertial observers are invariant under Poincaré transformations. In the absence of such symmetry, the definition of the vacuum becomes ambiguous, and, without a well-defined vacuum state, so does the construction of the Fock space. Thus, in curved spacetime, the particle concept itself loses much of its physical meaning.

2.4.2. Cosmological Particle Creation

In some cases, the spacetime we are considering allows us to define regions in the remote past or future, referred to as the *in*- and the *out*-region, respectively, that can be treated as asymptotically Minkowskian. In these asymptotic regions, there exist natural choices for the vacuum state: those states that are free of particles as measured by inertial particle detectors in the respective regions.

Now, let a quantum field ϕ be in the state that an inertial particle detector in the *in*-region would consider to be the vacuum state. Working in the Heisenberg picture, the field will forever remain in that state, even though, outside the *in*-region, a comoving particle detector may actually detect particles. In general, then, the state of the field will not coincide with the conventional Minkowski vacuum in the *out*-region, i.e., inertial particle detectors in the *out*-region, which would naturally measure the *out*-vacuum to be free of particles, would in fact detect ϕ-quanta. Therefore, it could reasonably be said that particles have been created by the changing gravitational field.

Consider, for illustration, a two-dimensional Robertson-Walker spacetime with line element $ds^2 = dt^2 - a^2(t)dx^2$. Introducing conformal time $d\eta = dt/a$, we can rewrite this as

$$ds^2 = C(\eta)(d\eta^2 - dx^2), \tag{2.78}$$

where we use the conformal scale factor $C(\eta) = a^2(\eta)$. This line element is manifestly conformal to the Minkowski line element, that is, it can be obtained from the Minkowski space line element $ds^2 = dt^2 - dx^2$ via a conformal transformation (2.63).

Take the conformal scale factor to be

$$C(\eta) = A + B \tanh \rho\eta, \quad A, B, \rho \text{ constants,} \tag{2.79}$$

which is indeed asymptotically Minkowskian,

$$C(\eta) \to A \pm B\,, \quad \eta \to \pm\infty\,, \tag{2.80}$$

and consider a massive, minimally coupled ($\xi = 0$) scalar field in this spacetime. Since this spacetime is invariant under spatial translations ($C(\eta)$ does not depend on position), the mode functions are still separable, and we can write

$$u_k(\eta, x) = (2\pi)^{-\frac{1}{2}} e^{ikx} \chi_k(\eta)\,. \tag{2.81}$$

Inserting this into the field equation (2.67), we obtain an ordinary differential equation for $\chi(\eta)$,

$$\frac{d^2}{d\eta^2}\chi_k(\eta) + \left(k^2 + C(\eta)m^2\right)\chi_k(\eta) = 0\,. \tag{2.82}$$

Solving this equation in terms of hypergeometric functions, and defining

$$\begin{aligned} \omega_{\text{in}} &= \left[k^2 + m^2(A - B)\right]^{\frac{1}{2}}, \quad \omega_{\text{out}} = \left[k^2 + m^2(A + B)\right]^{\frac{1}{2}}, \\ \omega_{\pm} &= \tfrac{1}{2}\left(\omega_{\text{out}} \pm \omega_{\text{in}}\right), \end{aligned} \tag{2.83}$$

one finds that the mode solutions behaving like positive-frequency Minkowski space modes in the asymptotic past ($\eta, t \to -\infty$) are

$$u_k^{\text{in}}(\eta, x) \xrightarrow[\eta\to-\infty]{} (4\pi\omega_{\text{in}})^{-\frac{1}{2}} e^{ikx - i\omega_{\text{in}}\eta}\,, \tag{2.84}$$

while the modes behaving like positive-frequency Minkowski space modes in the asymptotic future ($\eta, t \to +\infty$) are

$$u_k^{\text{out}}(\eta, x) \xrightarrow[\eta\to+\infty]{} (4\pi\omega_{\text{out}})^{-\frac{1}{2}} e^{ikx - i\omega_{\text{out}}\eta}\,. \tag{2.85}$$

Using the linear transformation properties of hypergeometric functions, as given by Abramowitz and Stegun (1965, eqs. (15.3.6), (15.3.3)), we can expand the u_k^{in} in terms of the u_k^{out} as

$$u_k^{\text{in}}(\eta, x) = \alpha_k u_k^{\text{out}}(\eta, x)_k + \beta_k u_{-k}^{\text{out}*}(\eta, x) \tag{2.86}$$

with coefficients α_k and β_k which are related to the Bogoliubov coefficients in equation (2.74) by $\alpha_{kk'} = \alpha_k \delta_{kk'}$, $\beta_{kk'} = \alpha_k \delta_{-kk'}$. For these coefficients, we obtain

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_+/\rho)}{\sinh^2(\pi\omega_{\text{in}}/\rho)\sinh^2(\pi\omega_{\text{in}}/\rho)}, \tag{2.87}$$

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_-/\rho)}{\sinh^2(\pi\omega_{\text{in}}/\rho)\sinh^2(\pi\omega_{\text{in}}/\rho)}. \tag{2.88}$$

Thus, for the quantum field prepared in the *in*-vacuum state defined in terms of the *in*-modes u_k^{in}, which would be confirmed by inertial particle detectors in the asymptotically Minkowskian past to be devoid of particles, inertial detectors in the asymptotically Minkowskian future, whose physical vacuum was constructed using the *out*-modes u_k^{out}, would register the presence of particles. To be specific, the number of quanta expected in the mode k is given by the $|\beta_k|^2$ of equation (2.88) (compare equation (2.77)).

It is interesting to note that in the massless limit, $\omega_{\text{in}} = \omega_{\text{out}}$, and thus $\omega_- \to 0$, so that $|\beta_k|^2$ vanishes and no particle creation occurs. This situation, a massless, conformally invariant field propagating in a conformally flat spacetime, is referred to as conformally trivial. Since the expansion of a conformally flat cosmos is just a time-dependent conformal transformation of the entire spacetime, a conformally invariant field is not affected by the change in scale factor. However, once we give a mass to the field, we break the conformal invariance by introducing a length scale into the theory. Only then does the expansion couple to the field to cause particle creation.[6]

2.4.3. Adiabatic Vacuum and the Adiabatic Expansion of Green's Functions

In spacetimes such as the one considered in section 2.4.2, where we have seen that the cosmic expansion causes the creation of particles, any measurement of the particle number is inherently uncertain: particle creation at a rate r per time interval Δt allows for a precise measurement of the particle number only if $|r|\Delta t \ll 1$. Owing to the Heisenberg energy-time uncertainty, there is an additional uncertainty of order $(m\Delta t)^{-1}$ on the number of particles of mass m. Nevertheless, we know from our own experience of living in an expanding universe that there must be some approximation to the curved-space theory that allows for a meaningful definition of the particle number.

Indeed, parametrising the expansion rate by ρ, like in equation (2.79), and consulting equation (2.77), we find that particle production declines exponentially as $\rho \to 0$,

$$|\beta_k|^2 \to e^{-2\pi\omega_{\text{in}}/\rho} \to 0, \tag{2.89}$$

[6] A non-cosmological instance of a changing gravitational field causing particle creation is the collapse of a Black Hole, which gives rise to the emission of thermal radiation corresponding to a Black Hole temperature proportional to the surface gravity (Hawking, 1974, 1975).

and we conclude that the changing gravitational field only excites field modes with $\omega \lesssim \rho$. For ω much greater than the expansion rate, that is to say, for high-k or high-mass modes, particle creation is exponentially suppressed.

Thus, in an asymptotically static Robertson-Walker spacetime, if we prepare a quantum field in either the *in*- or the *out*-vacuum state, a comoving detector outside the static regions will likely fail to detect any high-energy particles. In contrast to that, the detector is liable to register quanta in the low-energy modes; for these, neither the *in*- nor the *out*-vacuum is a good approximation to the physical vacuum outside the asymptotic regions.

In the absence of asymptotically static spacetime regions, one might seek those field modes that are "closest" to the Minkowski-space limit in that they are least affected by the expansion of the universe—cosmological particle creation into these modes would be minimal. Although we will not go through the calculation to actually determine these modes (a detailed account can be found in Birrell and Davies (1984, section 3.5)), let us briefly summarise the procedure.

The above requirement that the expansion be slow in order for the definition of particle number to be reasonable, can be stated more precisely by introducing the adiabatic parameter T and temporarily replacing η by $\eta_1 = \eta/T$ (letting $T = 1$ at the end of the calculation). The expansion rate is then characterised by

$$\frac{\mathrm{d}}{\mathrm{d}\eta} C(\eta/T) = \frac{1}{T} \frac{\mathrm{d}}{\mathrm{d}\eta_1} C(\eta_1) \,, \tag{2.90}$$

and the adiabatic limit of slow cosmic expansion, where no particle creation occurs, corresponds to the limit $T \to \infty$. In this limit, $C(\eta_1)$ and all its derivatives vary infinitely slowly. In order to determine a quantity's slow-expansion behaviour, we can expand it in inverse powers of T, where the term of order T^{-n} is referred to as the nth adiabatic order. Note that, according to equation (2.90), the adiabatic order corresponds to the number of η-derivatives. On dimensional grounds, a term of adiabatic order A in the expansion of a quantity with dimensions m^d will contain $A - d$ powers of m^{-1} or k^{-1}.

We can now find mode solutions $u_k^{(A)}$ to the equations of motion that are exact to adiabatic order A, and then define the exact field modes in terms of the adiabatic approximation,

$$u_k = \alpha_k^{(A)}(\eta) u_k^{(A)} + \beta_k^{(A)}(\eta) u_k^{(A)*} \,. \tag{2.91}$$

As the $u_k^{(A)}$ are exact solutions of the field equations to order A, the coefficients $\alpha_k^{(A)}$, $\beta_k^{(A)}$ must be constant up to that order. For example, we can choose the coefficients such that u_k are adiabatic positive frequency modes to adiabatic order A by setting

$$\alpha_k^{(A)}(\eta_0) = 1 + \mathcal{O}(T^{-(A+1)}), \quad \beta_k^{(A)}(\eta_0) = 0 + \mathcal{O}(T^{-(A+1)}) \tag{2.92}$$

at some fixed time η_0. As we are free to perform this matching at any time η_0, the modes u_k are not uniquely defined by this procedure. Nevertheless, they are exact, and α and β are given by equations (2.92) for all time.

If we have matched the field modes outside of the static regions that may or may not exist in the spacetime under consideration, they will not, in general, match the standard, positive-frequency modes in these regions, but will instead be a mixture of positive- and negative-frequency modes. The vacuum defined in terms of equation (2.91) will therefore not correspond to the vacuum in the *in*- or *out*-regions, so that inertial particle detectors in these regions register field quanta in this distorted vacuum. Regardless, the distorted vacuum will only deviate from the physical one by terms of adiabatic order $A+1$, and so the number spectrum of the field quanta detected in the static regions will drop off as $k^{-(A+1)}$.

This definition of a vacuum state that matches the traditional emptiness of the Minkowski vacuum up to adiabatic order A, turns out to give the best approximation to physical particles available for spacetimes that lack any static regions to provide a reference vacuum.

In Minkowski spacetime, divergences are usually cured by various momentum space techniques. Owing to the lack of spatial homogeneity in a general curved spacetime, however, no global momentum space exists. In order to still use the familiar methods involving quantities like the Feynman propagator $G_{\mathrm{F}}(x,x')$, we have to define a local momentum space about every point by using Riemann normal coordinates y^α with origin at the point under consideration. The complete derivation of the Feynman propagator in curved spacetime is rather elaborate, so we shall merely sketched it here and give few intermediate steps; the subtleties involved in obtaining the result (2.101) may be more fully appreciated by consulting Bunch and Parker (1979).

To find a representation of the Feynman Green's function, we expand equation (2.67) in terms of normal coordinates and transform to local momentum space about x', i.e., at $y=0$. We define the quantity $\mathcal{G}_F(x,x')$ by

$$G_F(x,x') \equiv [-g(x)]^{-\frac{1}{4}}\mathcal{G}_F(x,x')[-g(x')]^{-\frac{1}{4}} = [-g(x)]^{-\frac{1}{4}}\mathcal{G}_F(x,x'), \tag{2.93}$$

where in the second step $[-g(x')]^{-\frac{1}{4}}=1$, because in the expansion in normal coordinates about x', the lowest order term corresponds to the Minkowski metric and all higher orders vanish at $y=0$. The representation of $\mathcal{G}_F(x,x')$ in the local momentum space about $y=0$ is obtained from the Fourier transform

$$\mathcal{G}_F(x,x') = \frac{1}{(2\pi)^n}\int \mathrm{d}^n k\, \mathrm{e}^{-iky}\mathcal{G}_F(k), \tag{2.94}$$

where $ky = \eta^{\alpha\beta}k_\alpha y_\beta$.

Iteratively solving the Klein-Gordon equation (2.67) yields the momentum-space Feynman propagator $\mathcal{G}_F(k)$ to any adiabatic order. The expression to fourth order is

$$\mathcal{G}_F(k) = \frac{1}{k^2-m^2} - \left(\frac{1}{6}-\xi\right) R \frac{1}{(k^2-m^2)^2} + \frac{1}{2}\mathrm{i}\left(\frac{1}{6}-\xi\right) R_{;\alpha}\partial^\alpha \frac{1}{(k^2-m^2)^2} - \frac{1}{3}a_{\alpha\beta}\partial^\alpha\partial^\beta \frac{1}{(k^2-m^2)^2} + \left[\left(\frac{1}{6}-\xi\right)^2 R^2 + \frac{2}{3}a^\lambda{}_\lambda\right]\frac{1}{(k^2-m^2)^3}, \tag{2.95}$$

where

$$a_{\alpha\beta} = \tfrac{1}{2}(\xi-\tfrac{1}{6})R_{;\alpha\beta} + \tfrac{1}{120}R_{;\alpha\beta} - \tfrac{1}{40}R_{\alpha\beta;\lambda}{}^\lambda - \tfrac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} + \tfrac{1}{60}R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda} + \tfrac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta}. \tag{2.96}$$

Converting back to coordinate space, we obtain

$$\mathcal{G}_F(x,x') = \int \frac{\mathrm{d}^n k}{(2\pi)^n}\, \mathrm{e}^{-iky}\left[a_0(x,x') + a_1(x,x')\left(-\frac{\partial}{\partial m^2}\right) + a_2(x,x')\left(\frac{\partial}{\partial m^2}\right)^2\right]\frac{1}{k^2-m^2}. \tag{2.97}$$

with coefficients

$$\begin{aligned} a_0(x,x') &= 1, \\ a_1(x,x') &= \left(\tfrac{1}{6}-\xi\right)R - \tfrac{1}{2}\left(\tfrac{1}{6}-\xi\right)R_{;\alpha}y^\alpha - \tfrac{1}{3}a_{\alpha\beta}y^\alpha y^\beta, \\ a_2(x,x') &= \tfrac{1}{2}\left(\tfrac{1}{6}-\xi\right)^2 R^2 + \tfrac{1}{3}a^\lambda{}_\lambda. \end{aligned} \tag{2.98}$$

We insert the integral representation $(k^2-m^2+\mathrm{i}\varepsilon)^{-1} = -\mathrm{i}\int_0^\infty \mathrm{d}s\ \mathrm{e}^{\mathrm{i}s(k^2-m^2+\mathrm{i}\varepsilon)}$ into equation (2.97), dropping the $\mathrm{i}\varepsilon$. Then, completing the square in the exponential, we get an extra factor $\exp(\sigma/2\mathrm{i}s)$, where $\sigma(x,x') = \frac{1}{2}y_\alpha y^\alpha$ is half the square of the geodesic separation of x' and x . Performing the Gaussian integral over k, we obtain

$$\mathcal{G}_F(x,x') = -\frac{\mathrm{i}}{(4\pi)^{n/2}}\int_0^\infty \mathrm{i}\,\mathrm{d}s\ (\mathrm{i}s)^{-n/2}\exp\left[-\mathrm{i}m^2 s + \frac{\sigma}{2\mathrm{i}s}\right]F(x,x';\mathrm{i}s), \tag{2.99}$$

where $F(x,x';\mathrm{i}s)$ up to adiabatic order 4 is given by

$$F(x,x';\mathrm{i}s) = a_0(x,x') + a_1(x,x')\mathrm{i}s + a_2(x,x')(\mathrm{i}s)^2. \tag{2.100}$$

Converting back, via equation (2.94), we finally obtain the DeWitt-Schwinger representation for the Feynman propagator in a curved spacetime

$$G_{\mathrm{F}}^{\mathrm{DS}}(x,x') = -\mathrm{i}\Delta^{\frac{1}{2}}(x,x')(4\pi)^{-\frac{n}{2}}\int_0^\infty \mathrm{i}\,\mathrm{d}s\ (\mathrm{i}s)^{-\frac{n}{2}}\exp[-\mathrm{i}m^2 s + (\sigma/2\mathrm{i}s)]F(x,x';\mathrm{i}s) \tag{2.101}$$

with the Van Vleck determinant $\Delta(x,x') = -\det\left[\partial_\mu \partial_\nu \sigma(x,x')\right][g(x)g(x')]^{-\frac{1}{2}}$, which, in normal coordinates about x', reduces to $[-g(x)]^{1/2}$.

It is also possible to find an expression to all adiabatic orders,

$$F(x,x';\mathrm{i}s) \approx \sum_{j=0}^{\infty} a_j(x,x')(\mathrm{i}s)^j \,, \tag{2.102}$$

where $a_0(x,x') = 1$, and the other a_j are given by a recursion relation (Christensen, 1976). The adiabatic expansion of the Feynman propagator is then

$$G_{\mathrm{F}}^{\mathrm{DS}}(x,x') \approx \frac{-\mathrm{i}\pi \Delta^{\frac{1}{2}}(x,x')}{(4\pi\mathrm{i})^{n/2}} \sum_{j=0}^{\infty} a_j(x,x') \left(-\frac{\partial}{\partial m^2}\right)^j \left[\left(\frac{2m^2}{-\sigma}\right)^{(n-2)/4} H^{(2)}_{(n-2)/2}\left((2m^2\sigma)^{\frac{1}{2}}\right)\right] . \tag{2.103}$$

We have given the derivation of these complicated quantities, because in section 2.5.2 we will find that the energy-momentum tensor may be derived from an effective action which can be expressed in terms of the Feynman propagator (2.101). We shall, in section 2.5.3, renormalise this action in order to remove the divergences of the energy-momentum tensor. It is this procedure for which knowledge of the Feynman propagator will be instrumental.

2.5. Renormalised Energy-Momentum Tensor in Curved Spacetime

In section 2.1.1, we found that in Minkowski spacetime, even though the expectation value of the Hamiltonian H diverges, the absence of gravity allows us to simply discard an infinite vacuum energy contribution by normal-ordering the field operators. Subsequently, section 2.3 showed us that in a flat universe with non-trivial topology, we can cut off the ultraviolet divergence in $\langle T_{\mu\nu}\rangle$ by using an ultraviolet regulator $\mathrm{e}^{-\alpha|k|}$ and then subtracting the regulated Minkowski space value, before finally letting $\alpha \to 0$.

Unfortunately, neither of these approaches work in a general spacetime. The energy-momentum tensor is the source of spacetime curvature, so if we want to investigate the interplay between curvature and energy, naïvely rescaling the zero point of energy won't do.

Likewise, the simple subtraction scheme of section 2.3 can be seen to fail by considering a spatially flat Robertson-Walker universe with scale factor

$$a(t) = (1 - A^2 t^2)^{\frac{1}{2}}, \quad A \text{ constant.} \tag{2.104}$$

For a massless, minimally coupled ($\xi = 0$) scalar field, mode solutions to the wave equation $\Box\phi = 0$ are found to be (Bunch and Davies, 1978)

$$u_k = \left(16\pi^3\right)^{-\frac{3}{2}} C^{-\frac{1}{2}}(\eta)\left(k^2 + A^2\right)^{-\frac{1}{2}} \exp\left[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\left(k^2 + A^2\right)^{\frac{1}{2}}\eta\right], \tag{2.105}$$

where the scale factor in terms of the conformal time η is $C(\eta) = a^2(\eta) = \cos^2(A\eta)$. The energy-momentum tensor (2.16) for this setup is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}, \tag{2.106}$$

and by using the modes (2.105) to construct the vacuum $|0\rangle$, we find for the vacuum energy density

$$\langle 0|T_0{}^0|0\rangle = \frac{1}{32\pi^3 C^2}\int \mathrm{d}^3k \left[\left(k^2 + A^2\right)^{\frac{1}{2}} + \left(k^2 + \tfrac{1}{4}D^2\right)\left(k^2 + A^2\right)^{-\frac{1}{2}}\right], \tag{2.107}$$

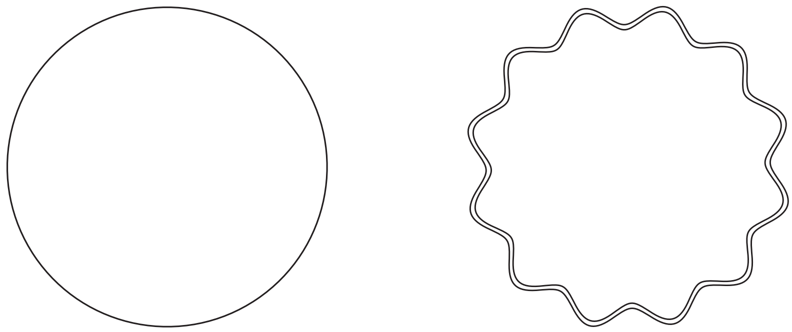
where $D(\eta) = C^{-1}\partial C/\partial\eta$. After regulating the quartic divergence by including a cutoff factor $\exp\left[-\alpha\left(k^2 + A^2\right)^{\frac{1}{2}}\right]$, we can solve the integral in terms of MacDonald functions. Expansion in powers of α yields

$$\rho a^4 = \frac{1}{32\pi^2}\left[\frac{48}{\alpha^4} + \frac{D^2 - 8A^2}{\alpha^2} + A^2\left(\tfrac{1}{2}D^2 - A^2\right)\ln\alpha\right] + \mathcal{O}\left(\alpha^0\right), \tag{2.108}$$

where ρ is the energy density and the left-hand side thus represents the total energy of radiation in a volume a^3, which redshifts with the expansion of the universe.

The Minkowski-space limit of equation (2.108) can be obtained by setting $a = 1$ and $D = A = 0$, which just leaves the first term on the right-hand side. Therefore, trying to regulate the curved space divergence of the energy by subtracting the Minkowski space contribution still leaves the $\mathcal{O}(\alpha^{-2})$ and $\mathcal{O}(\ln\alpha)$ terms, and thus ρa^4 still diverges in the limit $\alpha \to 0$. Hence, we may conclude that in curved spacetimes proper handling of the infinities in $\langle 0|T_{00}|0\rangle$ will require more sophisticated regularisation schemes than our previous treatments of $\langle T_{\mu\nu}\rangle$ in flat spacetimes.

We could now try to find a sufficient number of conditions, for example, the preservation of general covariance, that would allow us to restrict the ways we can go about subtracting the infinities from $\langle T_{\mu\nu}\rangle$, in the hopes of being able to define the procedure uniquely. Alternatively, we could compute $\langle T_{\mu\nu}\rangle$ within the framework of a dynamical theory involving gravity. In this work, we shall consider a semiclassical theory which regards gravity as a classical background field, while treating matter, including gravitons up to one-loop level, as quantum fields. We shall, again, follow the treatment of Birrell and Davies (1984).



(a) (b)

Figure 2.2.: Closed loop of (a) a matter field and (b) gravitons. Loops of this kind represent an infinite contribution to the energy of the vacuum of a theory and need to be removed by some renormalisation procedure.

2.5.1. Semiclassical Theory of Gravity

In a theory that quantises small perturbations to the gravitational field, the Planck length would play the role of a coupling constant, whence one would naïvely expect quantum gravitational effects to be negligible as long as the relevant length and time scales of the process under consideration remain well above the Planck scale. However, as gravity famously couples equally strongly to all forms of matter, including to gravitons themselves, any interaction of other matter fields with gravity should occur equally for gravitons, and thus quantum gravity effects may be hard to ignore, after all.

In order to deal with these effects, we can try separating the classical background metric $g^{\mathrm{cl}}_{\mu\nu}$ and the perturbations in the form of gravitational waves $\bar{g}_{\mu\nu}$ by writing the metric as

$$g_{\mu\nu} = g^{\mathrm{cl}}_{\mu\nu} + \bar{g}_{\mu\nu}\,. \tag{2.109}$$

The waves can then be treated as just another contribution to the right-hand side of the Einstein equations. In the same spirit, gravitons, which represent linearised perturbations of the background metric, will be considered a matter component rather than part of the geometry.

Now, consider this linearised theory to one-loop order. Closed loops like the ones in fig. 2.2 represent the infinite vacuum energy we have already encountered in previous sections. Similar loops also appear in other theories like quantum electrodynamics (QED), of course. In QED, we would deal with these vacuum graphs by absorbing the corresponding infinities into quantities like couplings, particle masses, and wavefunctions. Crucially, the coupling constant of QED, $e^2/\hbar c$, is dimensionless, from which we may conclude that the number of quantities that need to be renormalised in order to cure all divergences of the theory, is finite. In our linearised, semiclassical theory of gravity, however, the coupling constant is the Newton constant G, which has dimensions $(\mathrm{mass})^{-2}$.

This implies that at each higher loop order, new divergences will appear, and an infinite number of new quantities would need to be introduced in order to absorb all infinities. This is why gravity is referred to as a non-renormalisable theory.

If we were to truncate our semiclassical theory at some particular loop order, however, only a finite number of physical quantities would need to be renormalised. Trivially, this truncated theory, with only a finite number of divergences left, could be considered renormalisable.

Unfortunately, many of the interesting aspects of the behaviour of matter fields include higher-loop effects. But then, if we include higher matter loops, we also have to include graviton loops to the same order if we still want our loop expansion to be of consistent order in $\hbar$. Fortunately, each new graviton loop comes with a factor of G, while each matter loop introduces a factor of its relevant coupling, e.g., a factor e^2. Thus, if the relevant length or time scale of the process under consideration is l, the effect of the additional graviton loops will be negligible with respect to matter loops of the same order, as long as $l^{-2}G \ll e^2$. This allows us to restrict our treatment of quantum gravity to the one-loop level for a large range of scales.

2.5.2. The Effective Action

In the semiclassical theory considered here, the gravitational field equations will not be the classical equations (1.10) with the energy-momentum tensor $T_{\mu\nu}$ as the source of the gravitational field. Instead, we will take the source to be the quantum expectation value $\langle T_{\mu\nu}\rangle$,

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda_B g_{\mu\nu} = -8\pi G_B \langle T_{\mu\nu}\rangle \,, \tag{2.110}$$

in which Λ_B and G_B are the bare cosmological constant and Newton's constant, respectively. These parameters shall soon be used to absorb the divergences appearing on the right-hand side of equation (2.110).

The classical Einstein equations (1.10) can be derived from a variational principle by using the action $S = S_g + S_m$, where S_g is the gravitational action

$$S_g = \int \mathrm{d}^n x \, [-g(x)]^{1/2} (16\pi G_B)^{-1} (R - 2\Lambda_B) \,, \tag{2.111}$$

and S_m is the classical matter action. The Einstein equations are obtained from the condition

$$\frac{2}{[-g(x)]^{1/2}} \frac{\partial S}{\partial g^{\mu\nu}} = 0 \,, \tag{2.112}$$

where the variation of S_g yields the left-hand side of equation (1.10), and the variation of S_m produces $T_{\mu\nu}$.

In order to obtain the field equations (2.110), we need to find an effective action W to provide us with the correct source term, i.e.,

$$\frac{2}{[-g(x)]^{1/2}}\frac{\partial W}{\partial g^{\mu\nu}} = \langle T_{\mu\nu}\rangle \,. \tag{2.113}$$

As detailed in section 6.1 of Birrell and Davies (1984), the path-integral formalism allows us to determine

$$W = \tfrac{1}{2}\mathrm{i}\int \mathrm{d}^n x\,[-g(x)]^{1/2} \lim_{x'\to x}\int_{m^2}^{\infty} \mathrm{d}m^2\, G_{\mathrm{F}}^{\mathrm{DS}}(x,x'), \tag{2.114}$$

where $G_{\mathrm{F}}^{\mathrm{DS}}(x,x')$ is the Feynman propagator in the DeWitt-Schwinger representation we laboriously obtained in section 2.4.3. Interchanging the order of integration and taking the limit $x'\to x$ yields

$$W = \tfrac{1}{2}\mathrm{i}\int_{m^2}^{\infty} \mathrm{d}m^2 \int \mathrm{d}^n x\,[-g(x)]^{1/2} G_{\mathrm{F}}^{\mathrm{DS}}(x,x'), \tag{2.115}$$

where the $\mathrm{d}^n x$-integral is the expression for the one-loop Feynman diagram seen in figure 2.2(a). W is therefore called the one-loop effective action.[7]

From equation (2.114), we define an effective Lagrangian density $\mathscr{L}_{\mathrm{eff}}$ and effective Lagrangian L_{eff} by

$$W = \int \mathrm{d}^n x\, \mathscr{L}_{\mathrm{eff}} \equiv \int \mathrm{d}^n x\, [-g(x)]^{1/2} L_{\mathrm{eff}}(x) \tag{2.116}$$

with

$$L_{\mathrm{eff}}(x) = [-g(x)]^{-\frac{1}{2}}\mathscr{L}_{\mathrm{eff}}(x) = \tfrac{1}{2}\mathrm{i}\lim_{x'\to x}\int_{m^2}^{\infty}\mathrm{d}m^2\, G_{\mathrm{F}}^{\mathrm{DS}}(x,x'). \tag{2.117}$$

Let us now take a closer look at the expression (2.101) for $G_{\mathrm{F}}^{\mathrm{DS}}(x,x')$. Convergence of the s integral at the upper end of the integration is ensured by the small imaginary part $-\mathrm{i}\varepsilon$ that is implicitly added to m^2 in the exponential, in keeping with the Feynman prescription for executing the contour integration. At the lower end of the s integration, on the other hand, the integral diverges in the coincidence limit $x'\to x$, because the damping factor $\mathrm{e}^{\sigma/2s}$ vanishes in that limit. It is thus sufficient to insert for $F(x,x';\mathrm{i}s)$ the expansion (2.102) about $s=0$, the first three terms of which are given in equations (2.98). For $n=4$, the potentially divergent terms in the effective Lagrangian (2.117) then turn out to be

$$L_{\mathrm{div}} = -\lim_{x'\to x}\frac{\Delta^{\frac{1}{2}}(x,x')}{32\pi^2}\int_0^{\infty}\frac{\mathrm{d}s}{s^3}\,\mathrm{e}^{-\mathrm{i}(m^2 s-\sigma/2s)}\left[a_0(x,x')+a_1(x,x')\mathrm{i}s+a_2(x,x')(\mathrm{i}s)^2\right],$$

[7]Note, that fermion effective actions additionally require a trace over spinor indices.

(2.118)

with the coefficients a_i defined in equations (2.98). The a_i are entirely geometrical, i.e., constructed from $R_{\alpha\beta\gamma\delta}$ and its contractions, since we are here dealing with an ultraviolet divergence caused by the short-wavelength modes of the field. These modes only probe the local geometry and are thus insensitive to global features of the spacetime, as well as to the specific quantum state. With L_{div} being entirely geometrical, we will consider it to be part of the gravitational Lagrangian, contributing to the left-hand side of the field equations, rather than $\langle T_{\mu\nu} \rangle$.

2.5.3. Renormalisation in the Effective Action

Now that we have found the effective action L_{eff} that, upon variation with respect to the metric, will produce the semiclassical Einstein equations, we are interested in determining the precise form of its divergent parts L_{div}.

Using equation (2.102), we find the following asymptotic adiabatic expansion for L_{eff},

$$L_{\mathrm{eff}} \approx \lim_{x' \to x} \frac{\Delta^{\frac{1}{2}}(x,x')}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x,x') \int_0^{\infty} \mathrm{i}\,\mathrm{d}s\,(\mathrm{i}s)^{j-1-n/2} \mathrm{e}^{-\mathrm{i}(m^2 s - \sigma/2s)}, \tag{2.119}$$

where the first $n/2+1$ terms diverge in the coincidence limit $\sigma \to 0$. If n can be analytically continued throughout the complex plane, we can take the limit $x' \to x$ to find

$$\begin{aligned} L_{\mathrm{eff}} &\approx \frac{1}{2}(4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) \int_0^{\infty} \mathrm{i}\,\mathrm{d}s\,(\mathrm{i}s)^{j-1-n/2} \mathrm{e}^{-\mathrm{i}m^2 s} && (2.120) \\ &= \frac{1}{2}(4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) (m^2)^{n/2-j} \Gamma(j-n/2), && (2.121) \end{aligned}$$

where $a_j(x) \equiv a_j(x,x)$. In order for L_{eff} to still have units $(\mathrm{mass})^4$ even for $n \neq 4$, we introduce a mass scale μ, and write

$$L_{\mathrm{eff}} \approx \frac{1}{2}(4\pi)^{-n/2} \left(\frac{m}{\mu}\right)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma(j-n/2), \tag{2.122}$$

where the first three terms diverge as $n \to 4$ owing to the poles of Gamma function $\Gamma(z)$ for all integer $z \leq 0$. It is convenient to expand these divergent quantities as

$$\begin{aligned} \Gamma\left(-\frac{n}{2}\right) &= \frac{4}{n(n-2)} \left(\frac{2}{4-n} - \gamma\right) + \mathcal{O}(n-4), \\ \Gamma\left(1-\frac{n}{2}\right) &= \frac{2}{2-n} \left(\frac{2}{4-n} - \gamma\right) + \mathcal{O}(n-4), \\ \Gamma\left(2-\frac{n}{2}\right) &= \frac{2}{4-n} - \gamma + \mathcal{O}(n-4), \end{aligned} \tag{2.123}$$

and to write

$$\left(\frac{m}{\mu}\right)^{n-4} = 1 + \frac{1}{2}(n-4)\ln\left(\frac{m^2}{\mu^2}\right) + \mathcal{O}\left((n-4)^2\right). \tag{2.124}$$

We can then write the divergent part of the Lagrangian as

$$L_{\text{div}} = -(4\pi)^{-n/2}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}\left(\frac{4m^4 a_0}{n(n-2)} - \frac{2m^2 a_1}{n-2} + a_2\right), \tag{2.125}$$

where a_0, a_1, and a_2 are given by the coincidence limit of equation (2.98),

$$\begin{aligned} a_0(x) &= 0, \\ a_1(x) &= \left(\tfrac{1}{6} - \xi\right) R, \\ a_2(x) &= \tfrac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \tfrac{1}{180} R_{\alpha\beta} R^{\alpha\beta} - \tfrac{1}{6}\left(\tfrac{1}{5} - \xi\right)\Box R + \tfrac{1}{2}\left(\tfrac{1}{6} - \xi\right)^2 R^2. \end{aligned} \tag{2.126}$$

The astute reader may have noticed that we included in equation (2.125) more than the divergent terms of equation (2.122); we shall comment on this shortly. For the moment, let the reader be assured that finite terms included in equation (2.125) will cause no harm whatsoever, but merely introduce additional finite renormalisations.

We may now absorb the purely geometrical L_{div} into the gravitational Lagrangian to replace the latter in the integrand of equation (2.111) with

$$-\left(A + \frac{\Lambda_B}{8\pi G_B}\right) + \left(B + \frac{1}{16\pi G_B}\right) R - \frac{a_2(x)}{(4\pi)^{n/2}}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}, \tag{2.127}$$

where

$$\begin{aligned} A &= \frac{4m^4}{(4\pi)^{n/2} n(n-2)}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}, \\ B &= \frac{2m^2(\frac{1}{6} - \xi)}{(4\pi)^{n/2}(n-2)}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}. \end{aligned} \tag{2.128}$$

Equation (2.127) illustrates that as part of the renormalisation process, we replace the bare parameters Λ_B and G_B (compare the classical gravitational action (2.111)) by their renormalised values

$$\begin{aligned} \Lambda &\equiv \Lambda_B + 8\pi G_B A, \\ G &\equiv G_B/(1 + 16\pi G_B B). \end{aligned} \tag{2.129}$$

Since, by physical observation, we only ever measure the renormalised values Λ and G, there is no need to know the bare parameters, nor to worry about the formal divergence of the factors A and B in the limit $n \to 4$.

The final term in equation (2.127), upon variation, introduces new terms to the left-hand side of the field equations, which then looks like

$$R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha\, {}^{(1)}H_{\mu\nu} + \beta\, {}^{(2)}H_{\mu\nu} + \gamma H_{\mu\nu} \tag{2.130}$$

where

$$\begin{aligned} {}^{(1)}H_{\mu\nu} &\equiv \frac{1}{[-g(x)]^{1/2}} \frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^n x\, [-g(x)]^{1/2} R^2 \\ &= 2R_{;\mu\nu} - 2 g_{\mu\nu} \Box R - \tfrac{1}{2} g_{\mu\nu} R^2 + 2 R R_{\mu\nu}, \end{aligned} \tag{2.131}$$

$$\begin{aligned} {}^{(2)}H_{\mu\nu} &\equiv \frac{1}{[-g(x)]^{1/2}} \frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^n x\, [-g(x)]^{1/2} R^{\alpha\beta} R_{\alpha\beta} \\ &= 2 R_{\mu;\nu\alpha}^{\alpha} - \Box R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \Box R + 2 R_{\mu}^{\alpha} R_{\alpha\nu} - \tfrac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta}, \end{aligned} \tag{2.132}$$

$$\begin{aligned} H_{\mu\nu} &\equiv \frac{1}{[-g(x)]^{1/2}} \frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^n x\, [-g(x)]^{1/2} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \\ &= -\tfrac{1}{2} g_{\mu\nu} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 2 R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - 4 \Box R_{\mu\nu} + 2 R_{;\mu\nu} \\ &\quad - 4 R_{\mu\alpha} R^{\alpha}_{\nu} + 4 R^{\alpha\beta} R_{\alpha\mu\beta\nu}. \end{aligned} \tag{2.133}$$

In four dimensions, the generalised Gauss-Bonnet theorem states that

$$\int \mathrm{d}^4 x\, [-g(x)]^{1/2} \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + R^2 - 4 R_{\alpha\beta} R^{\alpha\beta} \right) \tag{2.134}$$

is a topological invariant, implying that its metric variation vanishes. It follows that the three new tensors introduced to the left-hand side of the Einstein equations are not independent, but are related by

$$H_{\mu\nu} = -{}^{(1)}H_{\mu\nu} + 4\, {}^{(2)}H_{\mu\nu}. \tag{2.135}$$

The coefficients α, β, and γ in equation (2.130) all inherit the $1/(n-4)$-term from the factor of a_2 in equation (2.127), and thus diverge as $n \to 4$. It is therefore necessary to introduce into the original gravitational Lagrangian order-4-terms with bare coefficients a_B, b_B, and c_B; these new terms can absorb the divergences of the terms involving α, β, and γ, in a manner similar to equation (2.129), to yield renormalised coefficients a, b, c. Owing to the Gauss-Bonnet theorem in four dimensions, only two of these are independent and we can choose $c = 0$. Furthermore, in order to maintain compatibility with experiment, we have to assume, that a and b are very small, such that our theory remains in good agreement with observations.

Now that we have found an expression for L_{div}, we obtain the renormalised effective Lagrangian by subtraction from the effective Lagrangian:

$$L_{\mathrm{ren}} \equiv L_{\mathrm{eff}} - L_{\mathrm{div}}. \tag{2.136}$$

Only the terms with $j \geq 3$ remain of the asymptotic expansion (2.120) in four dimensions, and we can write the coincidence limit ($x' = x$) as

$$L_{\text{ren}} \approx \frac{1}{32\pi^2} \int_0^\infty \mathrm{i}\,\mathrm{d}s \sum_{j=3}^{\infty} a_j(x)(\mathrm{i}s)^{j-3} \mathrm{e}^{-\mathrm{i}m^2 s} . \tag{2.137}$$

This may be integrated three times by parts to yield

$$\begin{aligned} &-\frac{1}{64\pi^2} \int_0^\infty \mathrm{d}(\mathrm{i}s)\, \ln(\mathrm{i}s) \frac{\partial^3}{\partial(\mathrm{i}s)^3} \left[F(x,x';\mathrm{i}s) \mathrm{e}^{-\mathrm{i}sm^2} \right] \\ &+\frac{1}{64\pi^2} \int_0^\infty \mathrm{d}(\mathrm{i}s)\, \ln(\mathrm{i}s) \frac{\partial^3}{\partial(\mathrm{i}s)^3} \left\{ \left[a_0 + a_1(\mathrm{i}s) + a_2(\mathrm{i}s)^2 \right] \mathrm{e}^{-\mathrm{i}sm^2} \right\} , \end{aligned} \tag{2.138}$$

where the second term is finite and of the same form as L_{div}. Thus, it merely renormalises Λ, G, a, b, and c by finite amounts.

At this point, a few remarks on the ambiguities of our renormalisation scheme seem in order. In equation (2.125), we could have equally well included only the divergent $1/(n-4)$-term in L_{div} and renormalised Λ, G, a, b, and c by only absorbing that divergence (Bunch, 1979). Instead, we chose to include the finite term $\frac{1}{2}\left[\gamma + \ln\left(m^2/\mu^2\right)\right]$. This term simply introduces additional finite terms proportional to a_0, a_1, and a_2, and therefore also just renormalises the constants by a finite amount. We can thus write the renormalised Lagrangian as

$$L_{\text{ren}} = -\frac{1}{64\pi^2} \int_0^\infty \mathrm{d}(\mathrm{i}s)\, \ln(\mathrm{i}s) \frac{\partial^3}{\partial(\mathrm{i}s)^3} \left[F(x,x';\mathrm{i}s) \mathrm{e}^{-\mathrm{i}sm^2} \right] , \tag{2.139}$$

keeping in mind that any finite multiple of a_0, a_1, and a_2 may be added to this.

Similarly, rescaling μ changes L_{div} by a finite amount. When working with the renormalised quantities, we would fix μ at some convenient value, and then measure Λ, G, a, and b. Subsequent changes of the scale μ then affect the measured quantities via the usual renormalisation group running.

Regarding equation (2.139), we have to note that, since we used in our derivation of L_{ren} an asymptotic expansion of $F(x,x';\mathrm{i}s)$, equation (2.139) is not the complete Lagrangian associated with the physical, renormalised $\langle T_{\mu\nu} \rangle$, which we will calculate later on. This section has, however, served to illustrate the use of renormalisation techniques for absorbing divergent parts of the effective matter action W into the coupling constants in the gravitational action S_g. We have hence arrived at the semiclassical Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + a\,{}^{(1)}\!H_{\mu\nu} + b\,{}^{(2)}\!H_{\mu\nu} = -8\pi G \frac{\langle \text{out}, 0 | T_{\mu\nu} | 0, \text{in} \rangle_{\text{ren}}}{\langle \text{out}, 0 | 0, \text{in} \rangle} . \tag{2.140}$$

The first three terms on the left-hand side of equation (2.140) correspond to the left-hand side of equation (2.110), where we have now renormalised the cosmological constant Λ. These are the terms usually encountered in the field equations of general relativity; they are of second order in derivatives of the metric. The two remaining left-hand-side terms of the Lagrangian (2.140) are higher-order corrections. They are of fourth order in derivatives of the metric and originate from the variation with respect to the metric of the terms R^2, $R^{\alpha\beta}R_{\alpha\beta}$, and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ contained in the last term of equation (2.127).[8] As a consequence of the renormalisation procedure described in this section, the expression on the right-hand side of equation (2.140) is now a finite quantity. Note that, like the cosmological constant Λ, the coupling constants G, a, and b represent renormalised quantities, which have to be determined from experiment.

Even though we would ordinarily consider $\langle \text{in}, 0|T_{\mu\nu}|0, \text{in}\rangle$ or $\langle \text{in}, 0|T_{\mu\nu}|0, \text{in}\rangle$ the source of the Einstein equations, we will not have to redo the entire renormalisation procedure for every conceivable choice of vacuum combinations. In fact, it can be shown (DeWitt, 1975), that all these forms of $\langle T_{\mu\nu}\rangle$ differ only by finite amounts; they all contain the same divergences. This is a result of the divergences' originating in the short-distance (ultraviolet) behaviour of the propagator, which does not depend on the global structure of the spacetime under consideration, nor on the quantum state.

Incidentally, other regularisation techniques than the dimensional regularisation employed here, for example, ζ-function or point-splitting regularisation, would have arrived at the same result (2.139) (Birrell and Davies, 1984, ch. 6.2).

2.5.4. Conformal Anomaly

In field theories whose classical action S is invariant under conformal transformations $g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$, the trace $T_\rho{}^\rho$ of the energy-momentum tensor vanishes. To see this, consider the behaviour of the action under an infinitesimal conformal transformation $g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x)$,

$$\begin{aligned} S[\bar{g}_{\mu\nu}] &= S[g_{\mu\nu}] + \int \mathrm{d}^n x \, \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta \bar{g}_{\mu\nu}(x)} \delta \bar{g}_{\mu\nu}(x) \\ &= S[g_{\mu\nu}] - \int \mathrm{d}^n x \, [-\bar{g}(x)]^{\frac{1}{2}} \, T_\rho{}^\rho \left[\bar{g}_{\mu\nu}(x)\right] \Omega^{-1}(x) \delta \Omega(x), \end{aligned} \tag{2.141}$$

where we have used $\delta \bar{g}_{\mu\nu}(x) = -2\bar{g}_{\mu\nu}(x)\Omega^{-1}(x)\delta\Omega(x)$ and the definition of the energy-momentum tensor (2.16). Thus, for actions invariant under conformal transformations $\left(S[\bar{g}_{\mu\nu}] = S[g_{\mu\nu}]\right)$, the integral in the second term has to vanish, which, for arbitrary variations $\delta\Omega(x)$, implies $T_\rho{}^\rho \left[g_{\mu\nu}(x)\right] = 0$.

[8] Remember that in equation (2.135), we eliminated the contribution from $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ by means of the Gauss-Bonnet theorem in four dimensions. Furthermore, the term $\Box R$ that also appears in the a_2-factor in the last term of equation (2.127) does not contribute to the field equations, because it is a total divergence.

As discussed before, a conformal transformation represents a rescaling of all lengths. Conformal invariance is therefore broken by the introduction of masses for the fields under consideration. Hence, we shall consider, here, the massless limit of the above regularisation and renormalisation procedure.

Let us return to the expansion (2.122). The terms with $j > 2$ diverge in four dimensions in the limit $m \to 0$, while the first two of the ultraviolet divergent terms $j = 0, 1, 2$ vanish immediately, because they include positive powers of m for $n \approx 4$. The only potentially ultraviolet-divergent term that remains is the ($j = 2$)-term,

$$\tfrac{1}{2}(4\pi)^{-n/2}(m/\mu)^{n-4}a_2(x)\Gamma(2-n/2). \tag{2.142}$$

The divergent part of the effective action arising from this is

$$W_{\rm div} = \frac{1}{2}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\Gamma(2-n/2)\int \mathrm{d}^n x\,[-g(x)]^{1/2}\,[\alpha F(x)+\beta G(x)]+\mathcal{O}(n-4), \tag{2.143}$$

with

$$F = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{3}R^2, \tag{2.144}$$

$$G = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2, \tag{2.145}$$

$$\alpha = \tfrac{1}{120}, \qquad \beta = \tfrac{1}{360}. \tag{2.146}$$

Here, F is the square of the Weyl tensor $C_{\alpha\beta\gamma\delta}$, which is invariant under conformal transformations, and G is the Gauss-Bonnet invariant, whose integral $\int \mathrm{d}^4x\,[-g(x)]^{1/2}G$ is a topological invariant in four dimensions. Therefore, $W_{\rm div}$ is indeed conformally invariant for $n = 4$. However, we are using dimensional regularisation and have thus analytically continued the dimension n away from 4. $W_{\rm div}$ is thus not conformally invariant until we relax the regularisation and pass to $n = 4$. It turns out that a remnant of this breakdown of conformal invariance away from $n = 4$ survives even after we set $n = 4$ at the end of the calculation. This gives rise to the anomalous trace $\langle T_\mu{}^\mu\rangle_{\rm div} = (1/16\pi^2)[\alpha(F - \tfrac{2}{3}\Box R) + \beta G]$ (Birrell and Davies, 1984, chapter 6.3).

Since the total effective action W is conformally invariant in the massless, conformally coupled case, the total energy-momentum tensor $\langle T_\mu{}^\nu\rangle = \langle T_\mu{}^\nu\rangle_{\rm ren} + \langle T_\mu{}^\nu\rangle_{\rm div}$ is traceless, which means that the trace of the renormalised part, $\langle T_\mu{}^\mu\rangle_{\rm ren}$, has to be the negative of $\langle T_\mu{}^\mu\rangle_{\rm div}$,

$$\begin{aligned}\langle T_\mu{}^\mu\rangle_{\rm ren} &= -a_2/16\pi^2\\ &= -\tfrac{1}{16\pi^2}[\alpha(F - \tfrac{2}{3}\Box R) + \beta G]\\ &= -\tfrac{1}{2880\pi^2}\left[R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta} - \Box R\right].\end{aligned} \tag{2.147}$$

We would have obtained the same result for the trace anomaly using the ζ-function method.

The trace anomaly is of special interest in the conformally trivial situation of a conformally coupled, massless field propagating in a conformally flat background, because in that case, the entire energy-momentum tensor is determined by the anomalous trace, once the quantum state is chosen. This can be seen by following the argument of Bunch and Davies (1977). $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$ has to be a covariantly conserved, local tensor. It turns out that the only conserved, local tensors of adiabatic order four (that is, of fourth order in derivatives of the metric) are ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$, defined in equations (2.133), and

$$\begin{aligned}{}^{(3)}H_{\mu\nu} &= \tfrac{1}{12}R^2 g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu} \\ &= R_\mu{}^\nu R_{\rho\nu} - \tfrac{2}{3}RR_{\mu\nu} - \tfrac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \tfrac{1}{4}R^2 g_{\mu\nu}.\end{aligned} \tag{2.148}$$

In conformally flat spacetimes, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ are related by ${}^{(1)}H_{\mu\nu} = \frac{1}{3}{}^{(2)}H_{\mu\nu}$. In addition to these three, there may be another local, conserved tensor ${}^{(4)}H_{\mu\nu}$ that is non-geometrical, i.e., that cannot be expressed in terms of $R_{\mu\nu}$ and R.

The energy-momentum tensor then has to be some linear combination of these tensors

$$\langle T_\mu{}^\nu\rangle_{\mathrm{ren}} = A\,{}^{(1)}H_\mu{}^\nu + B\,{}^{(3)}H_\mu{}^\nu + {}^{(4)}H_\mu{}^\nu, \tag{2.149}$$

the trace of which is $\langle T_\mu{}^\mu\rangle_{\mathrm{ren}} = -6A\,\Box R - B(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2) + {}^{(4)}H_\mu{}^\mu$. Comparing this expression to the first line of equations (2.147), we find

$$A = -\alpha/144\pi^2, \qquad B = -\beta/8\pi^2, \qquad {}^{(4)}H_\mu{}^\mu = 0. \tag{2.150}$$

Therefore, for a conformally invariant field in a conformally flat spacetime, $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$ is entirely determined by the anomalous trace (2.147).

2.5.5. Energy-Momentum Tensor in De Sitter Space

We saw in section 2.5.3 how we would, in principle, go about using dimensional regularisation to remove the divergences from L_{eff} in order to obtain a finite $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$, which we would like to use as the source of the gravitational field equations. Now, however, we shall have to contend that renormalising the action functional is, in general, impractical. This is because in order to functionally differentiate W_{ren} with respect to $g_{\mu\nu}$ to form $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$, we require knowledge of W_{ren} for all geometries $g_{\mu\nu}$, which is impossibly difficult. The conformally trivial case, treated in section 2.5.4, is an important exception; there, knowledge of the trace anomaly alone determines the entire energy-momentum tensor. In all other cases, we have to renormalise $\langle T_{\mu\nu}\rangle$ directly.

One of the cases where dimensional regularisation is useful in handling the divergent portions of $\langle T_{\mu\nu}\rangle$, is de Sitter space, to be discussed in section 3.3.1. Since, in a maximally symmetric spacetime like de Sitter spacetime, the unique (up to a possible constant) maximally form-invariant tensor of rank two is the metric tensor (Weinberg, 1972, chapter 13.4), the energy-momentum tensor for a vacuum state that is chosen to be invariant under the de Sitter group must be proportional to the metric,

$$\langle T_{\mu\nu}\rangle = T g_{\mu\nu}/n\,, \tag{2.151}$$

where $T = \langle T_\mu{}^\mu\rangle$ is the trace of the stress tensor.

Using the Lagrangian (2.60) and the definition (2.16), we find for the energy-momentum tensor of a massive scalar field

$$\begin{aligned} T_{\mu\nu} = {} & (1-2\xi)\phi_{;\mu}\phi_{;\nu} + (2\xi - \tfrac{1}{2})g_{\mu\nu}g^{\rho\sigma}\phi_{;\rho}\phi_{;\sigma} - 2\xi\phi_{;\mu\nu}\phi \\ & + \frac{2}{n}\xi g_{\mu\nu}\phi\Box\phi - \xi\left[R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \frac{2(n-1)}{n}\xi R g_{\mu\nu}\right]\phi^2 \\ & + 2\left[\frac{1}{4} - (1-\frac{1}{n})\xi\right] m^2 g_{\mu\nu}\phi^2\,. \end{aligned} \tag{2.152}$$

The trace of this tensor is

$$T_\mu{}^\mu = m^2\phi^2 + (n-1)[\xi - \xi(n)]\Box\phi^2\,, \tag{2.153}$$

which, with the definition (2.28) of the Green's function $G^{(1)}$, yields

$$T = \langle T_\mu{}^\mu\rangle = \tfrac{1}{2}m^2 G^{(1)}(x,x) + \tfrac{1}{2}(n-1)\,[\xi - \xi(n)]\,\Box G^{(1)}(x,x)\,. \tag{2.154}$$

$G^{(1)}(x,x)$ has been derived by Dowker and Critchley (1976),

$$G^{(1)}(x,x) = \frac{2\alpha^2}{(4\pi\alpha^2)^{n/2}}\frac{\Gamma\left(\nu(n) - \frac{1}{2} + n/2\right)\Gamma\left(-\nu(n) - \frac{1}{2} + n/2\right)}{\Gamma\left(\frac{1}{2} + \nu(n)\right)\Gamma\left(\frac{1}{2} - \nu(n)\right)}\Gamma(1 - n/2)\,, \tag{2.155}$$

where $[\nu(n)]^2 = \frac{1}{4}(n-1)^2 - m^2\alpha^2 - \xi n(n-1)$ and α is the radius of de Sitter space.[9] Since $G^{(1)}$ is independent of x, only the first term in equation (2.154) contributes, and $T = \frac{1}{2}m^2 G^{(1)}(x,x)$.

The last Gamma function in the Green's function (2.155) obviously has a pole for $n = 4$. This divergence can be removed by expanding equation (2.155) about $n = 4$ and subtracting from it the adiabatic expansion to order four of $G^{(1)}_{\rm DS}$, also expanded about $n = 4$. This is equivalent to renormalising $L_{\rm eff}$, in that the low-order terms in the adiabatic expansion of $G^{(1)}_{\rm DS}$ form, via equation (2.117), the divergent part of the Lagrangian,

[9] Consider, for example, 4-dimensional de Sitter space, which can be represented as a hyperboloid $z_0{}^2 - z_1{}^2 - z_2{}^2 - z_3{}^2 - z_4{}^2 = -\alpha^2$ embedded in a 5-dimensional Minkowski space with line element $ds^2 = dz_0{}^2 - dz_1{}^2 - dz_2{}^2 - dz_3{}^2 - dz_4{}^2$. For a more in-depth discussion of de Sitter space, see section 3.3.

L_{div}. The procedure of removing these terms from $G^{(1)}$ is thus equivalent to the subtraction (2.136).

The result of this calculation is (see Birrell and Davies (1984, section 6.4) and Dowker and Critchley (1976))

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = (g_{\mu\nu}/64\pi^2)\Big\{m^2[m^2+(\xi-\tfrac{1}{6})R][\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)-\ln(12m^2R^{-1})] - m^2(\xi-\tfrac{1}{6})R - \tfrac{1}{18}m^2R - \tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2 + \tfrac{1}{2160}R^2\Big\}\,, \tag{2.156}$$

where the de-Sitter-space relation $R = 12\alpha^{-2}$ has been inserted.

For the case of a massless, conformally coupled ($\xi = \frac{1}{6}$) scalar field, equation (2.156) yields

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \frac{g_{\mu\nu}}{64\pi^2}\frac{R^2}{2160} = \frac{g_{\mu\nu}}{960\pi^2}\frac{1}{\alpha^4}\,, \tag{2.157}$$

where α is still the radius of de Sitter space. Hence, a conformally invariant scalar field in de Sitter space fills the universe with a uniformly distributed, positive vacuum energy.

It is interesting to note that the form of equation (2.157) is very similar to the renormalised vacuum energy (2.65) we found for the ordinary Casimir experiment with two conducting plates in an otherwise flat and empty Minkowski spacetime. The main difference is that, since the arrangement of plates in the z-direction breaks the maximal symmetry of Minkowski space, the energy-momentum tensor can no longer be proportional to the metric tensor, the way it is in the maximally-symmetric de Sitter case. Apart from this difference in the degree of symmetry, the two results are remarkably similar. Both even exhibit the same scaling behaviour, one with the inverse fourth power of the separation between the plates, the other with the same power of the radius of the de Sitter universe. We will further elaborate on this parallel in section 3.3.

Chapter 3

Casimir Energy and Cosmology

This chapter is dedicated to giving a few more specific examples of the Casimir effect, already discussed in section 2.3.2. We shall first consider, in section 3.1, a circular chain of harmonic oscillators. We will find that this chain possesses a vacuum energy differing from that of a linear chain by a finite amount proportional to the inverse of the radius R' of the circle. We will then interject a short speculation on a similar energy appearing in the surface of a cylindrical shell of harmonic oscillators in section 3.2. We will find that the Casimir effect of section 3.1 should give rise to a negative Casimir energy of essentially phononic origin within the surface of the cylinder, which supplements the ordinary electromagnetic Casimir energy within the volume of a cylindrical shell.

In section 3.3, we will take a look at Casimir energy in de Sitter spacetime, which we are interested in as the vacuum-dominated spacetime of the inflationary universe. We shall briefly introduce the motivation for and concepts of the inflationary period our universe is thought to have gone through in its earliest stages, and then discuss the effect that the Casimir energy of equation (2.157) might have on the evolution of the de Sitter universe.

In section 3.4, we will try to generalise the findings of this chapter and speculate on the possibility that Casimir-type arguments may shed some light on the origins of dark energy.

3.1. Casimir Effect for a Circular Chain of Harmonic Oscillators

As an illustration of how Casimir energies might emerge in condensed matter systems, let us consider the simple model of a circular chain of N point masses m connected by ideal springs with zero proper length. Left to its own devices, a chain like this would collapse to radius $r = 0$, where all the springs would be at their equilibrium length, zero. In order to shift the equilibrium of the system to radius $r = R$, we introduce an external potential, which could, for example, be an ordinary Mexican hat potential; we are only interested in its behaviour about its minimum at $r = R'$, though, and will therefore approximate it as a harmonic potential.

The total potential energy of the system is then given by

$$V = \frac{k'}{2}(R - R')^2 + \frac{k}{2}\sum_{i=1}^{N}\ell_i^2, \tag{3.1}$$

where ℓ_i is the length of the ith spring, given by

$$\ell_i = R\Delta\varphi_i, \tag{3.2}$$

where $\Delta\varphi_i = \varphi_{i+1} - \varphi_i$ is the angular separation of two neighbouring masses, and we fix $\sum_{i=1}^{N}\Delta\varphi_i = 2\pi$; in equilibrium, $\Delta\varphi_i = 2\pi/N$. We have assumed that $N \gg 1$ and thus $\Delta\varphi_i \ll 1$, allowing us to approximate $\sin(\Delta\varphi_i/2) \approx \Delta\varphi_i/2$. Note, that the external harmonic potential and the intrinsic oscillator potential will, in general, have different force constants k and k'.

With the potential (3.1), we can now write down the Lagrangian for the system,

$$\mathscr{L} = \underbrace{\frac{1}{2}Nm\dot{R}^2 + \frac{m}{2}R^2\sum_{i=1}^{N}\dot{\varphi}_i^2}_{\text{kinetic energy}} \underbrace{- \frac{k'}{2}(R' - R)^2 - \frac{k}{2}R^2\sum_{i=1}^{N}\Delta\varphi_i^2}_{-\text{potential energy}, V}, \tag{3.3}$$

In equilibrium, the condition

$$\frac{\partial V}{\partial R} = k'(R - R') + \sum_{i=1}^{N} 4kR\left(\frac{\Delta\varphi_i}{2}\right)^2 \stackrel{!}{=} 0 \tag{3.4}$$

holds, and all the oscillators are in their equilibrium positions, i.e., $\Delta\varphi_i = 2\pi/N$. We can then express the equilibrium radius of the system in terms of the radius of the external potential,

$$R_{\text{eq}} = \frac{Nk'}{4\pi^2 k + Nk'}R'. \tag{3.5}$$

A simple calculation shows that upon inserting equation (3.5) and the equilibrium separations $\Delta\varphi_i = 2\pi/N$, the potential energy of the system in equilibrium is

$$V_{\text{eq}} = \frac{2\pi^2 kk'R'^2}{4\pi^2 k + Nk'}. \tag{3.6}$$

We will refer to this as the classical vacuum energy of the system.

We shall now expand the potential V about the equilibrium at V_{eq} by introducing small perturbations δr and $\delta\varphi$ of the radius of the external potential and the angular position of the oscillators, respectively. Thus, we replace $R \to R_{\text{eq}} + \delta r$ and $\Delta\varphi_i \to 2\pi/N + \tilde{\Delta}\varphi_i$, where $\tilde{\Delta}\varphi_i = \delta\varphi_{i+1} - \delta\varphi_i$ is now the deviation of the separation of neighbouring oscillators from the equilibrium separation $2\pi/N$, and we require $\sum_{i=1}^{N} \tilde{\Delta}\varphi_i = 0$. We find for the perturbed potential

$$V = V_{\text{eq}} + \left(\frac{2\pi^2}{N}k + \frac{1}{2}k'\right)(\delta r)^2 + \frac{1}{2}\frac{kk'^2N^2R'^2}{\left(4\pi^2k + Nk'\right)^2}\sum_{i=1}^{N}\left(\delta\varphi_{i+1} - \delta\varphi_i\right)^2 + \mathcal{O}\left(\delta^3\right), \tag{3.7}$$

where $\mathcal{O}\left(\delta^3\right)$ generically represents terms of higher than second order in the perturbations. Taking the limit $N \to \infty$ and dropping the higher-order terms, this potential simplifies to

$$V = V_{\text{eq}} + \frac{1}{2}k'(\delta r)^2 + \frac{1}{2}kR'^2\sum_{i=1}^{N}\left(\delta\varphi_{i+1} - \delta\varphi_i\right)^2, \tag{3.8}$$

which yields the linearised Lagrangian

$$\mathcal{L} = \frac{1}{2}Nm(\delta\dot{r})^2 + \sum_{i=1}^{N}\frac{1}{2}mR^2\left(\delta\dot{\varphi}_i\right)^2 - V_{\text{eq}} - \frac{1}{2}k'(\delta r)^2 - \sum_{i=1}^{N}\frac{1}{2}kR^2\left(\delta\varphi_{i+1} - \delta\varphi_i\right)^2. \tag{3.9}$$

Using the Euler-Lagrange equations, we arrive at the equations of motion

$$mN(\delta\ddot{r}) = k(\delta r), \tag{3.10}$$

$$m\left(\delta\ddot{\varphi}_i\right) = -k(2\delta\varphi_i - \delta\varphi_{i+1} - \delta\varphi_{i-1}). \tag{3.11}$$

In preparation for the quantisation of the system, let us rewrite the angular perturbations as a superposition of plane waves,

$$\delta\varphi_l(t) = \frac{1}{R}\frac{1}{\sqrt{N}}\sum_s\left(e^{isl}B_s^{\varphi}(t) + e^{-isl}B_s^{\varphi *}(t)\right), \tag{3.12}$$

where we include the factor $1/R$ in order to more elegantly obtain the correct dimensions in our final result; the operators B_s^{φ} have dimensions of (length)^1 or (mass)$^{-1}$. Since, on the circle, periodic boundary conditions hold, we determine the form of s by requiring $\delta\varphi_0 \stackrel{!}{=} \delta\varphi_N$, that is, $\sum_s\left(B_s^{\varphi}(t) + B_s^{\varphi *}(t)\right) \stackrel{!}{=} \sum_s\left(e^{isN}B_s^{\varphi}(t) + e^{-isN}B_s^{\varphi *}(t)\right)$, which is only possible if the exponentials on the right-hand side vanish. Hence, sN has to be a multiple of 2π, and we deduce

$$s = 2\pi n/N, \tag{3.13}$$

where n takes values $0 \leq n < N$.

Inserting the Fourier decomposition (3.12) into the equation of motion (3.11) and using the fact that the resulting equation has to hold for the B_s^{φ} and the $B_s^{\varphi *}$ separately, we obtain

$$
\begin{aligned}
& m\ddot{B}_s^{\varphi}(t) = -k\left(2 - \mathrm{e}^{-\mathrm{i}s} - \mathrm{e}^{\mathrm{i}s}\right) B_s^{\varphi}(t) \\
\Rightarrow \quad & \ddot{B}_s^{\varphi}(t) = 4\frac{k}{m}\sin^2\left(\frac{s}{2}\right) B_s^{\varphi}(t).
\end{aligned}
\tag{3.14}
$$

Assuming periodic behaviour $B_s^{\varphi}(t) = \mathrm{e}^{-\mathrm{i}\omega_s^{\varphi} t} A_s^{\varphi}$ with frequencies ω_s^{φ} and amplitudes A_s^{φ} which are constant in time, equation (3.14) implies the dispersion relation

$$
\left(\omega_s^{\varphi}\right)^2 = 4\omega_0^2 \sin^2\left(\frac{s}{2}\right), \tag{3.15}
$$

where we have introduced $\omega_0^2 = k/m$.

Let us now express the Hamiltonian of the system,

$$
\mathcal{H} = \frac{1}{2}Nm(\delta\dot{r})^2 + \sum_{i=1}^{N}\frac{1}{2}mR^2(\delta\dot{\varphi}_i)^2 + V_{\mathrm{eq}} + \frac{1}{2}k'(\delta r)^2 + \sum_{i=1}^{N}\frac{1}{2}kR^2\left(\delta\varphi_{i+1} - \delta\varphi_i\right)^2, \tag{3.16}
$$

in terms of the Fourier modes. In addition to equation (3.12), we will need its time derivative

$$
\delta\dot{\varphi}_l(t) = \frac{1}{R}\frac{1}{\sqrt{N}}\sum_s \mathrm{i}\omega_s^{\varphi}\left(-\mathrm{e}^{\mathrm{i}sl}B_s(t) + \mathrm{e}^{-\mathrm{i}sl}B_s^*(t)\right). \tag{3.17}
$$

We give the complete calculation in appendix B. The unsurprising result for the angular part of the Hamiltonian is

$$
\mathcal{H}^{\varphi} = \sum_s m\left(\omega_s^{\varphi}\right)^2\left(B_s^{\varphi}B_s^{\varphi *} + B_s^{\varphi *}B_s^{\varphi}\right). \tag{3.18}
$$

For convenience, we introduce the dimensionless quantities b_s^{φ} by

$$
B_s^{\varphi} = \frac{1}{\sqrt{2m\omega_s^{\varphi}}}b_s^{\varphi} \tag{3.19}
$$

and rewrite equation (3.18) as

$$
\mathcal{H}^{\varphi} = \sum_s \frac{1}{2}\omega_s^{\varphi}\left(b_s^{\varphi}b_s^{\varphi *} + b_s^{\varphi *}b_s^{\varphi}\right). \tag{3.20}
$$

Finally, we quantise the angular part of the motion by promoting b_s^{φ} and $b_s^{\varphi *}$ to operators $\hat{b}_s^{\varphi}$ and $\hat{b}_s^{\varphi\dagger}$, and imposing the commutation relations

$$\left[\hat{b}_s^{\varphi}, \hat{b}_{s'}^{\varphi\dagger}\right] = \delta_{s,s'} . \tag{3.21}$$

This turns the angular Hamiltonian into the standard form

$$\mathscr{H}^{\varphi} = \sum_s \omega_s^{\varphi} \left(\hat{b}_s^{\varphi\dagger} \hat{b}_s^{\varphi} + \frac{1}{2} \right) . \tag{3.22}$$

The quantisation of the radial part of the Hamiltonian (3.16), comprising just a single degree of freedom, can be treated in the standard way to obtain an analogous expression with the frequency given by

$$(\omega^r)^2 = \frac{k'}{Nm} . \tag{3.23}$$

We can then use the number operators $n = \hat{b}^{\dagger}\hat{b}$ to write the total Hamiltonian of our system of oscillators on a circle as

$$\mathscr{H} = V_{\text{eq}} + \omega^r \left(n^r + \frac{1}{2} \right) + \sum_s \omega_s^{\varphi} \left(n^{\varphi} + \frac{1}{2} \right) . \tag{3.24}$$

The total vacuum energy of the system is obtained for the case $n^r = 0 = n^{\varphi}$,

$$E_{\text{vac}} = V_{\text{eq}} + \frac{1}{2}\omega^r + \frac{1}{2}\sum_s \omega_s^{\varphi} . \tag{3.25}$$

Let us introduce the length scale $\alpha = R'/N$, which is, apart from a factor 2π, the equilibrium separation of the point masses of the system. We may think of α as the distance of the atoms in a condensed matter system, or as the Planck length in a theory where spacetime is made up of some sort of microscopic degrees of freedom, accessible beyond the Planck scale (see references in the introduction for extensive details on various approaches to theories with discrete spacetime). α can thus be considered an ultraviolet cutoff, restricting the number of modes available to the system.

Taking the limit of large radius, $R' \to \infty$, or, alternatively, small cutoff, $\alpha \to 0$, we increase the number of oscillators, $R'/\alpha = N \to \infty$, while keeping their linear density constant. We shall refer to this as the continuum limit. The classical vacuum energy V_{eq} of equation (3.6) then simplifies to,

$$V_{\text{eq}} = \frac{2\pi^2 k k' R'^2}{4\pi^2 k + k' \frac{R'}{\alpha}} \quad \stackrel{R' \to \infty}{\longrightarrow} \quad 2\pi^2 k \alpha R' . \tag{3.26}$$

In the same limit, the second term in equation (3.25) vanishes due to the dispersion relation (3.23),

$$\frac{1}{2}\omega^{r} = \frac{1}{2}\sqrt{\frac{k'}{Nm}} = \frac{1}{2}\sqrt{\frac{k'\alpha}{mR'}} \quad \overset{R'\to\infty}{\longrightarrow} \quad 0. \tag{3.27}$$

The last term in equation (3.25) is of particular interest to us. We find, owing to equations (3.13) and (3.15),

$$\begin{aligned}\sum_s \frac{1}{2}\omega_s^{\varphi} &= \omega_0 \sum_s \sin\left(\frac{s}{2}\right) = \omega_0 \sum_n \sin\left(\frac{\pi n}{N}\right) = \omega_0 \cot\left(\frac{\pi}{2N}\right) = \omega_0 \cot\left(\frac{\pi\alpha}{2R'}\right) \\ &= \omega_0 \left(\frac{2R'}{\alpha\pi} - \frac{\alpha\pi}{6R'} + \mathcal{O}\left(\frac{\alpha}{R'}\right)^3\right),\end{aligned} \tag{3.28}$$

where we have expanded $\cot(\pi\alpha/(2R'))$ for large values of R'.

We shall deal with the vacuum energy E_{vac} in a manner analogous to our treatment of the vacuum-energy divergence in the cylindrical spacetime of section 2.3.1. Consider, in particular, the renormalisation of $\langle 0_L|T_{tt}|0_L\rangle$ by subtraction of the Minkowski-space expression $\lim_{L\to\infty}\langle 0_L|T_{tt}|0_L\rangle$ (see equation (2.44)). There, we determined the vacuum contribution in Minkowski space by taking the limit of large extent of the spatial dimension. By analogy, we obtain in the infinite-radius limit of our circular chain of oscillators, the total vacuum energy of an infinite linear chain. Inserting the relations (3.26)–(3.28) back into equation (3.25), we find for large radii R',

$$E_{\text{vac}}(R' \gg \alpha) = 2\pi^2 k\alpha R' + \omega_0 \frac{2R'}{\alpha\pi}. \tag{3.29}$$

Subtracting this "straight"-chain contribution from equation (3.25), we arrive at the renormalised vacuum energy[1]

$$E_{\text{vac}}^{\text{ren}} = \omega_0 \left(-\frac{\alpha\pi}{6R'} + \mathcal{O}\left(\frac{\alpha}{R'}\right)^3\right). \tag{3.30}$$

As it was obtained by subtracting from the vacuum energy of a finite, constrained system that of the unconstrained, Minkowski analogue, we shall refer to this energy as the Casimir energy of the system.

[1]Even though it may appear that way at first sight, the circular chain considered here, and the cylinder universe of section 2.3.1 are not easily comparable. The only scale available to construct an energy from in the case of the flat $(\mathbb{R}^1 \times S^1)$-universe populated by a massless scalar field is the size $L$of the compactified spatial dimension, which corresponds to the radius of the external potential in the case of the circular chain. Hence, the Casimir energy of that system can only depend on that scale, and turns out to be proportional to L^{-2}. In the case of the chain of oscillators, however, we have more scales at our disposal. There is the spring constant k of the oscillator potentials, as well as the mass m of the oscillators; these two quantities form the eigenfrequency $\omega_0^2 = k/m$. Furthermore, there is the equilibrium separation α, which, combined with the radius R', gives the total number $N \propto R'/\alpha$ of oscillators, so that the Casimir energy turns out to be proportional to $N\omega_0$. With all these differences between the system of this section and that of section 2.3.1, it is hardly surprising that the resulting energies look different.

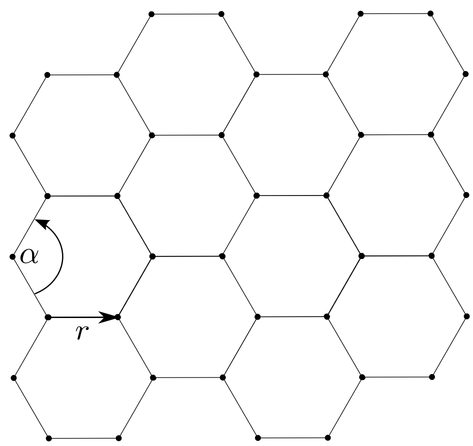


Figure 3.1.: Hexagonal structure of graphene; each vertex represents a carbon atom. With periodic boundary conditions in one direction, this structure turns into that of a carbon nanotube.

3.2. Casimir Effect for a Cylindrical Structure

We saw in section 3.1 that in a circular chain of oscillators, a Casimir energy proportional to the inverse of the circle's radius appears. It seems reasonable, then, to speculate that a similar energy might emerge in a cylindrical tube, which could be considered a stacking of many circles connected to one another by harmonic forces similar to those connecting the point masses of the original circular chain. We shall conjecture that the periodicity of the structure in the angular direction gives rise to a Casimir energy that behaves in the same way as on the circle, that is, we assume it is of the form $-C/R$, where C is a small positive constant and R is the radius of the tube.

Furthermore, we shall simplistically assume that the cylinder in question has been constructed by curling up a sheet of atoms interacting with one another via the Lennard-Jones potential

$$V(r) = 4\epsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right], \tag{3.31}$$

where r is the separation of the carbon atoms. The equilibrium separation r_0 of the atoms satisfies the condition $\mathrm{d}V/\mathrm{d}r|_{r=r_0} = 0$, which places it at

$$r_0 = 2^{1/6}\sigma\,. \tag{3.32}$$

Naturally, the potential can be approximated as harmonic about the equilibrium position r_0.

The Casimir energy contributes the additional term $-C/R$ to the potential energy of the tube, where the tube radius R is related to the bond length r by

$$R = \frac{N}{2\pi} \sin\left(\frac{\alpha}{2}\right) r\,, \tag{3.33}$$

in which α is the angle between two neighbouring bonds, as illustrated in figure 3.1, and N is the number of atoms that fit on the circumference of the tube. In a planar hexagonal structure like graphene, for example, the angle is $\alpha = 120°$. The potential energy of two neighbouring atoms is then given by

$$V(r) = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^{6} \right] - \frac{C'}{r}, \tag{3.34}$$

where we have defined $C' \equiv 2\pi C/(N \sin \frac{\alpha}{2})$. The negative Casimir term in the potential will, of course, change the equilibrium separation of the atoms. Expanding the numerical solution of the equation $\mathrm{d}V/\mathrm{d}r|_{r=r_0} = 0$ in C'/ϵ yields

$$r_0 = 2^{1/6}\sigma - \frac{C'}{72\epsilon} + \mathcal{O}\left(C'^2\right). \tag{3.35}$$

In this rather simplified model, we would therefore expect the Casimir effect due to the periodic structure in the angular direction to reduce the diameter of the tube. This effect would thus produce a reduction of the bond length of the atoms in smaller-diameter tubes relative to larger-diameter tubes.

We have to point out, however, that the phononic Casimir energy within the cylindrical surface is not the only contracting effect at work here. Similar to the conducting plates of section 2.3.2, a negative Casimir energy of electromagnetic origin is expected within the volume of a cylindrical shell. In fact, it is found that for a perfectly conducting cylindrical shell, the Casimir energy per unit length is $-A/R^2$ with A a positive numerical factor (DeRaad and Milton, 1981; Milton *et al.*, 1999). Hence, this effect would also tend to contract the tube. Unfortunately, without a more rigorous calculation on our part, it seems difficult to predict what the relative magnitude of these two energies would be and whether our surface effect may actually be discernible in experiments.

It may be interesting to investigate the effect of this change in the bond length of the atoms of carbon nanotubes. Unfortunately, our derivation of a Casimir energy for the circle relied on the limit $N \to \infty$, which is not applicable in nanotubes, because ordinarily these contain no more than a few tens of atoms per circumference. If we assume, nonetheless, that the energy scales as $1/R$, we expect the equilibrium bond length of carbon atoms in a small carbon nanotube to be less than in a large carbon nanotube.

3.3. Casimir Energy in de Sitter space

We saw in section 2.5.5 that renormalisation of the energy-momentum tensor of a massless, conformally coupled scalar field leaves us with the finite vacuum energy (2.157). We can think of this energy as being due, like the Casimir energies of sections 2.3.1 and 2.3.2, to a modification of the zero-modes of the field.

Comparing equations (2.157) and (1.10), we see that this vacuum energy contribution to de Sitter space is of the same form as a cosmological constant, i.e., the metric tensor multiplied by a constant. In order to better understand the possible relevance of this term, we shall now take a closer look at de Sitter space.

3.3.1. De Sitter Spacetime

First investigated by de Sitter (1917), the de Sitter spacetime is a vacuum solution of the Einstein equations with positive vacuum energy. De Sitter space can be thought of as the hyperboloid ${z_0}^2 - {z_1}^2 - {z_2}^2 - {z_3}^2 - {z_4}^2 = -\alpha^2$ embedded in a 5-dimensional Minkowski space with line element $ds^2 = d{z_0}^2 - d{z_1}^2 - d{z_2}^2 - d{z_3}^2 - d{z_4}^2$. De Sitter spacetime, like Minkowski spacetime, possesses 10 Killing vectors, making it a maximally-symmetric spacetime.

Choosing the coordinates of the ambient 5-dimensional space to be

$$\begin{aligned} z_0 &= \alpha \sinh(t/\alpha) + \tfrac{1}{2}\alpha^{-1} e^{t/\alpha} |\boldsymbol{x}|^2 , \\ z_4 &= \alpha \cosh(t/\alpha) - \tfrac{1}{2}\alpha^{-1} e^{t/\alpha} |\boldsymbol{x}|^2 , \\ z_i &= e^{t/\alpha} x_i , \quad i = 1,2,3, \quad -\infty < t, x_i < \infty , \end{aligned} \tag{3.36}$$

we can rewrite the line element as

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^{3} (dx^i)^2 . \tag{3.37}$$

Comparing equation (3.37) to the Robertson-Walker line element $ds^2 = dt^2 - a(t)^2 (d\boldsymbol{x})^2$, we realise that the de Sitter universe is simply a Robertson-Walker universe with flat spatial sections and scale factor

$$a(t) = e^{t/\alpha} . \tag{3.38}$$

This implies for the Hubble parameter,

$$H = \frac{\dot{a}}{a} = \frac{1}{\alpha} = \text{constant} . \tag{3.39}$$

3.3.2. Why We Need Inflation

Now that we know we can have an exponentially expanding universe, why would we even want one? There are two main motivations for introducing an early epoch of exponential, or inflationary, expansion: the flatness problem and the horizon problem.

Using equations (1.4), (1.8), (1.22), it is easy to derive the evolution of the total density parameter Ω with the scale factor a,

$$\begin{aligned} \frac{\partial \Omega}{\partial \ln a} &= a \frac{d}{da} \left(1 + \frac{k}{a^2 H^2} \right) = -\frac{2k}{a^2 H^2} \left(1 + \frac{a}{H} \frac{dH}{da} \right) = -2(\Omega - 1) \left(1 + \frac{\dot{H}}{H^2} \right) \\ &= (1 + 3w) \Omega (\Omega - 1) . \end{aligned} \tag{3.40}$$

As we saw from equation (1.22), the universe is flat for $\Omega = 1$. Now, since for matter ($w = 0$) and radiation ($w = 1/3$) the factor $(1+3w) > 0$, equation (3.40) implies that the flat universe is an unstable fixed point, i.e., for $\Omega > 1$, the right-hand side of equation (3.40) is positive, which means that Ω increases over time, evolving ever further away from 1. Similarly, for $\Omega < 1$, the right-hand side of equation (3.40) is negative, also driving the universe away from flatness. Thus, any initial deviation from flatness would have been significantly amplified over the course of the evolution of the universe, making a curvature density as close to zero as observed by Komatsu *et al.* (2010) a rather unnatural state of affairs. This is what we refer to as the flatness problem.

Obviously, the problem vanishes if $(1+3w) < 0$. Then, the sign of the right-hand side of equation (3.40) is always opposite to that of $(\Omega - 1)$, and $\Omega = 1$ turns into a stable fixed point—the universe is driven towards flatness. This situation occurs whenever the universe is dominated by a component with sufficiently negative pressure, $p < -\rho/3$, which according to equation (1.4) happens to be the condition for accelerated expansion.

In order to understand the nature of the horizon problem – the other main reason for postulating an early inflationary epoch – we need to delve a bit into the concept of horizons (Linde, 2005).

From the finiteness of the speed of light we conclude that there must be some distance from beyond which a given point in space cannot have received any information until a given time t. This distance is called the particle horizon, and to determine it, we integrate the light-cone condition $\mathrm{d}s^2 = 0$, (Schmidt-May, 2010)

$$D_\mathrm{p} = a(t)\int_0^{r(t)} \frac{\mathrm{d}r'}{\sqrt{1-kr^2}} = a(t)\int_0^t \frac{\mathrm{d}t'}{a(t')}. \tag{3.41}$$

We found in equation (1.27) that the scale factor evolves with time as $a(t) \propto t^n$ with $n = \frac{2}{3(1+w)}$, and hence

$$D_\mathrm{p} = \begin{cases} \frac{1}{1-n}t\,, & \text{for } n < 1\,, \\ \infty\,, & \text{for } n \geq 1\,. \end{cases} \tag{3.42}$$

Thus, for decelerated expansion ($w > -1/3$, $n < 1$), the particle horizon in physical coordinates grows; so does the comoving particle horizon $D_{\mathrm{p,cm}} = D_\mathrm{p}/a \propto t^{1-n}$. An accelerated universe ($w \leq -1/3$, $n \geq 1$) has no particle horizon.

Hence, in a universe without an accelerated period in its past, spacetime points separated by a distance greater than the current particle horizon have never been in causal contact—their past light-cones don't overlap. Why is it, then, that CMB measurements indicate approximate homogeneity on scales far beyond the particle horizon at the time of decoupling, when the CMB photons were released from their local thermodynamically equilibrated regions?

To explain this discrepancy, we need to consider the event horizon, the maximum distance from which we can ever receive information about events taking place now (at time t),

$$D_{\rm e} = a(t)\int_t^\infty \frac{{\rm d}t'}{a(t')}\,. \tag{3.43}$$

For $a(t) \propto t^n$, this yields

$$D_{\rm e} = \begin{cases} \frac{1}{n-1}t\,, & \text{for } n > 1\,, \\ \infty\,, & \text{for } n \leq 1\,. \end{cases} \tag{3.44}$$

Thus, the decelerated universe ($w > -1/3$, $n < 1$) has no event horizon—information about the events at one point in space may eventually reach any other point. Meanwhile, the event horizon in the accelerated universe ($w \leq -1/3$, $n \geq 1$) is finite—knowledge of the events at a given point in space can never pervade the entire universe. The comoving event horizon in an accelerated universe is $D_{\rm e,cm} = D_{\rm e}/a \propto t^{1-n}$; it shrinks with time.

For the special case of $w = -1$ and $a \propto {\rm e}^{-Ht}$, the de Sitter universe, we find that the physical event horizon is constant, $D_{\rm e} = 1/H$, while the comoving one decreases exponentially, $D_{\rm e} \propto {\rm e}^{-Ht}$. This exponential decrease of the event horizon means that during the brief inflationary period the horizon shrinks by an enormous amount and hence regions of space which had previously been causally connected, because their size was well below the horizon size, may easily end up being vastly greater than the horizon by the end of inflation.

Thus, an early exponential expansion solves the problem of how regions that seem like they ought to have always been disconnected, can nevertheless be in near thermodynamic equilibrium with one another: there was an era in the distant past when the size of regions permitted to be in causal contact was much larger than we would expect from a simple extrapolation of a matter- or radiation-dominated, decelerated expansion back to early times. Only when the exponential expansion of the universe shrunk the event horizon, was the connection severed—but by then, thermodynamic equilibrium had already been established.

3.3.3. Inflation from Scalar Fields

We saw in section 1.1.3, already, that exponential expansion occurs whenever the universe is dominated by an energy component with equation of state $w = -1$, i.e., $\rho = -p$. The component driving the early inflationary expansion, however could not have been an exact cosmological constant: dominance of a cosmological constant, once established, would never end, as all other energy components are diluted by the expansion of the cosmos, while a cosmological constant is not. In a universe undergoing exponential expansion from very early on, though, no structure formation would have been possible, which is in stark contradiction to the observations of modern cosmology—and all other natural sciences, for that matter. Fortunately, the equation of state of a cosmological constant can easily be mimicked by a minimally coupled ($\xi = 0$) scalar field ϕ with potential $V(\phi)$. In the context of inflation, the field ϕ is usually referred to as the inflaton field. Take for its Lagrangian

$$\mathscr{L} = \tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{3.45}$$

which yields the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2 \phi + \frac{\partial V}{\partial \phi} = 0. \tag{3.46}$$

The energy-momentum tensor of ϕ is given by $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathscr{L}$. For a homogeneous field, this takes the perfect fluid form of equation (1.3) with pressure and energy density

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{3.47}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{3.48}$$

Hence, the equation of state $w = p/\rho$ of the scalar field assumes the correct value for driving exponential expansion when the potential of the scalar field dominates over its kinetic energy, $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. This is known as the slow-roll condition; the field is said to slowly roll down its potential towards the minimum.

In the slow-roll approximation, the Friedmann equation (1.5) simplifies to

$$H^2 \simeq \frac{8\pi G}{3} V(\phi). \tag{3.49}$$

With the added assumption that the friction term $3H\dot{\phi}$ in the equation of motion dominates over the acceleration $\ddot{\phi}$, the equation of motion (3.46) takes the form

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} \simeq 0. \tag{3.50}$$

In order to test the validity of the slow-roll condition, one usually introduces the slow-roll parameters (Liddle and Lyth, 2000)

$$\varepsilon(\phi) = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \qquad \eta(\phi) = \frac{1}{8\pi G}\frac{V''}{V}, \tag{3.51}$$

where the prime denotes differentiation with respect to ϕ. We then require that the following conditions hold:

$$\varepsilon(\phi) \ll 1, \quad |\eta(\phi)| \ll 1. \tag{3.52}$$

Inflation ends when these conditions are no longer satisfied. It is then followed by a brief period of reheating, during which the inflaton field's oscillating about the minimum of its potential creates, via a small coupling to the Standard Model fields, the field content required for the subsequent evolution along the lines laid out in section 1.1.

Note that in many cases, particularly power law potentials, the necessary flatness of the potential is achieved simply by taking very large field values. The derivatives $V' = \frac{\partial V}{\partial \phi}$ and $V'' = \frac{\partial^2 V}{\partial \phi^2}$ are then much smaller than the potential itself simply because they contain lower powers of the field. It is not uncommon, therefore, for the inflaton field to assume values beyond the Planck scale. This is, however, not as worrying as it may seem and does not, in itself, herald the breakdown of our ability to describe the physics of inflation, since the relevant physical quantity is not the field itself, but rather its energy density. Thus, only when the energy density, in slow-roll inflation given by the potential V, exceeds the Planck density $M_{\rm Pl}^4$, does current physics cease to describe the world.

A quick remark on another important use of scalar fields in a cosmological context seems in order at this point. As we have seen, scalar fields in an appropriately flat potential can mimic the behaviour of a cosmological constant. This, of course, makes them a prime candidate for causing accelerated expansion not only in the early, but also in the late universe—the accelerated universe we live in today. Indeed, scalar fields in the role of dark energy have been discussed for some time now under the name of quintessence (first by Ratra and Peebles (1988); later contributions came from Caldwell *et al.* (1998) and many others).

Quintessence models achieve accelerated expansion in much the same way we just described for the inflaton field; the equations (3.45)-(3.48), with the inflaton field ϕ replaced by the quintessence field Q, continue to hold. An important difference, however, is that, while we need close to pure de-Sitter behaviour, i.e. $w \approx -1$, during inflation, the equation of state in quintessence models with standard kinetic terms may lie between $+1$ and -1, and late-time acceleration strictly only requires $w \leq -1/3$. Depending on the potential $V(Q)$ and the cosmic era, the equation of state may be constant, slowly- or rapidly-varying or even oscillatory. This less stringent criterion allows for a more lenient constraint than the slow-roll condition $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ of inflation. In fact, in order satisfy $w \leq -1/3$, we only need $\dot{Q}^2 < V(Q)$. Of course, in order to comply with cosmological data, realistic quintessence models should reproduce $w_0 \approx -1$ today (Komatsu *et al.*, 2010).

Some time after the original concept had been suggested, a special class of quintessence models was discovered (Steinhardt *et al.*, 1999; Zlatev *et al.*, 1999) which exhibit what is called tracking behaviour: It was found that for certain potentials, the solutions to the equations of motion converge on a common cosmic evolutionary track for a wide range of initial conditions, including the natural scenario of equipartition of energy between all degrees of freedom at the end of reheating.

The energy density of a quintessence field following the tracker solution is typically subdominant for most of the history of the universe, tracking the evolution of the background density, but diluting somewhat less quickly. It then starts to dominate the energy density in the recent past. Note, that it is quite possible for the quintessence density to start out significantly above or below the tracker density and still converge to it eventually; in fact, the allowed range for initial Q spans over 100 orders of magnitude.

3.3.4. The Effect of Casimir Energy

Following the short introduction to the standard concepts of inflation in section 3.3.3, let us now investigate whether the Casimir-like vacuum energy of equation (2.157) has a significant influence on the cosmic evolution during this epoch of vacuum dominance. Unfortunately, the inflaton is a classical field, while the calculation that led to equation (2.157) assumed a purely quantum field. We will therefore have to consider a toy model which contains one purely classical scalar field slowly rolling down its potential V and driving the inflationary expansion, and an additional, purely quantum, massless, conformally coupled $\left(\xi = \frac{1}{6}\right)$ scalar field that gives rise to the Casimir energy (2.157).

In order to determine the effect of the Casimir energy on the expansion, we need to compare it to the energy density of the inflaton field. For convenience, we will give all expressions in terms of the reduced Planck mass $M_{\mathrm{Pl}}^2 = 1/(8\pi G)$.

First of all, let us rewrite the Casimir energy using the relation (3.39)

$$\langle T_{\mu\nu}\rangle_{\mathrm{ren}} = \frac{g_{\mu\nu}}{960\pi^2}\frac{1}{\alpha^4} = \frac{g_{\mu\nu}}{960\pi^2}H^4 . \tag{3.53}$$

Using the slow-roll Friedmann equation (3.49), we may express the vacuum energy density $\langle\rho\rangle = \langle T_{00}\rangle_{\mathrm{ren}}$ in terms of the potential V,

$$\langle\rho\rangle_{\mathrm{Casimir}} = \frac{1}{8640\pi^2}\frac{V^2}{M_{\mathrm{Pl}}^4} . \tag{3.54}$$

Thus, in order for the Casimir energy $\langle\rho\rangle_{\mathrm{Casimir}}$ to affect the inflationary epoch, i.e., in order for $V \sim \langle\rho\rangle_{\mathrm{Casimir}}$, we would need

$$V \sim 10^5 M_{\mathrm{Pl}}^4 . \tag{3.55}$$

Clearly, this energy density far beyond the Planck density is untenable in current physical theory, if not thoroughly unphysical, which forces us to conclude that the Casimir energy cannot have played a significant part in driving inflation in the early universe.

We may reach the same sobering conclusion in a slightly different manner by considering a specific example for the inflaton potential. Take, for instance, the quartic potential $V(\phi) = \lambda\phi^4$. Comparison with the normalisation of the CMB power spectrum requires the extremely small coupling $\lambda \sim 10^{-15}$ (Kinney, 2009). Thus, for inflaton field values of no more than a few M_{Pl}, the potential is no greater than $V(\phi) \sim 10^{-12} M_{\text{Pl}}^4$, assuming $\phi \lesssim 5M_{\text{Pl}}$. This is a reasonable field value, because $\phi \approx 5M_{\text{Pl}}$ is where the field needs to start out in order to achieve inflation of 60 e-folds, i.e., a total increase of the scale factor over the course of the inflationary epoch by a factor e^{60}; 60 e-folds is commonly considered the minimum amount of inflation necessary to cure the flatness and horizon problems. Thus, the Casimir contribution, proportional to the square of the potential would be $\langle \rho \rangle_{\text{Casimir}} \sim 10^{-29} M_{\text{Pl}}^4$, and therefore utterly negligible by comparison.

We conclude that, interesting as it may be, the Casimir energy of a massless, conformally invariant field during inflation is completely dominated by the inflaton potential, and hence irrelevant. Nevertheless, it may still have some influence in later eras of the universe, when inflation has ended and the inflaton has settled near the minimum of its potential.

3.4. General Features of Casimir Energies, and Relevance to Cosmology

The finite vacuum energy (2.157), like the Casimir energies of sections 2.3.1 and 2.3.2, can be thought of as being due to a modification of the zero-modes of the field. In the case of section 2.3.1, this modification was effected by the finite size of the spatial sections of the cylindrical Minkowski universe, which imposed periodic boundary conditions upon the field, while in section 2.3.2 its cause was the introduction of Dirichlet boundary conditions on the two conducting plates placed in Minkowski space.

A noteworthy parallel between these three instances of Casimir-like energies is their scale dependence: in equation (2.55), we found that the Casimir energy for the two-dimensional cylinder universe is proportional to L^{-2}, where L was the extent of the compactified spatial dimension; equation (2.65) indicated an energy density proportional to a^{-4} for plates placed at a separation of a in four-dimensional Minkowski space; similarly, equation (2.157) showed that the vacuum energy density of a de Sitter universe populated by a conformally invariant scalar field scales as α^{-4}, where α is the radius of the four-dimensional universe embedded within a five-dimensional Minkowski space.

In all of these cases, the Casimir energy density is proportional to a^d, where a is the extent of the spatial region available to the field, or the separation of the boundaries constraining the field, while d is the dimension of the spacetime the field propagates in.

If we were to postulate an infrared cutoff at the Hubble scale, that is, assume that fields propagating in our universe have to satisfy some sort of boundary condition around the Hubble radius, we might expect the resulting Casimir energy due to the modification of the field's zero-modes to scale as $(1/H)^{-4} = H^4$. Of course, given that the Hubble parameter today is $H_0 \approx 10^{-42}\,\text{GeV}$, an energy of order H^4 is an utterly negligible contribution to the observed cosmological constant of order $H^2 M_{\text{Pl}}^2$. We shall, however, give a recent example of a model in which similar considerations give rise to a Casimir-type energy of the correct order of magnitude.

The straightforward way an infrared cutoff might enter our theory is by means of the spacetime we inhabit not being the traditional $\mathbb{R}^4$, but instead having some non-trivial topology.[2] The most intuitive non-trivial topology we might think of is that of a torus in one or more of the three spatial dimensions, which is similar to our treatment of the cylinder spacetime $\mathbb{R}^1 \times S^1$ in section 2.3.1 with additional spatial dimensions.

Consider, for instance, the toroidal universe $\mathbb{R}^1 \times T^3$, where T^3 represents the 3-torus. This spacetime is obtained from ordinary Minkowski spacetime by the identifications

$$x = x + L_1, \quad y = y + L_2, \quad z = z + L_3. \tag{3.56}$$

We shall assume that $L_1, L_2 \gg L_3$, so we are effectively left with a 1-torus T^1 of linear size $L \equiv L_3$. Interestingly, many topologically non-trivial universes with diameter less than 24 Gpc were already excluded by early WMAP data (Cornish *et al.*, 2004) via the non-observation of "circles in the sky" (Cornish *et al.*, 1998).

It has recently been proposed (Urban and Zhitnitsky, 2009, 2010)[3] that the so-called Veneziano ghost of QCD, originally introduced for entirely different reasons (Veneziano, 1979), might be used to explain the cosmological constant. The authors argue that as an exactly massless degree of freedom, the Veneziano ghost may probe the large-scale structure of spacetime and is therefore susceptible to its topology. Propagating in a finite compact manifold of size L would then give rise to a vacuum energy deviation from Minkowski space of order

$$\rho_\Lambda \simeq H\Lambda_{\text{QCD}}^3 \simeq c \cdot N_{\text{f}} \cdot 3.4 \times 10^{-46}\,\text{GeV}^4,$$

where N_{f} is the number of light quark flavours and c is a coefficient of order 1. Interestingly, this result is only by a factor of $\sim 4N_{\text{f}}$ greater than the observed value of the cosmological constant. Subtracting from the vacuum energy of the torus that of Minkowski space, then, would yield a vacuum energy of roughly the correct magnitude.

As a side note, in Urban and Zhitnitsky (2009), the authors use the deviation of their result from the observed dark energy density to predict $L \approx 17 H_0^{-1} \approx 74\,\text{Gpc}$ for the size of the torus. They note, that at this size, the Planck satellite should be able to detect the CMB anisotropy due to the orientation of the manifold.

The model of Urban and Zhitnitsky (2009, 2010) has a few shortcomings, however. For one thing, it is based on a 2-dimensional model—4-dimensional calculations have not been performed, although the authors expect them to yield similar results, since the

[2] For a review of topology, its influence on cosmology, and how we might test it via the cosmic microwave background, consult Levin (2002).

[3] Note that, although published in the opposite order, Urban and Zhitnitsky (2010) actually precedes Urban and Zhitnitsky (2009) historically.

relevant ghost exists in four dimensions as well. Furthermore, the calculations have only been done in a static, non-expanding background—the realistic calculation is complicated by the necessary knowledge of the dynamics of the ghost coupled to gravity on a finite, expanding manifold. Nevertheless, this seems to us a good illustration of how Casimir energies in a universe with non-trivial topology may ultimately provide an explanation for the observed vacuum energy.

An alternative way one might try to find a Casimir energy of the right order of magnitude, $H^2 M_{\mathrm{Pl}}^2$, is the introduction of an ultraviolet cutoff of order M_{Pl} in addition to the infrared cutoff obtained from boundary conditions.

We found in section 3.1 that the Casimir energy of the circular chain of oscillators investigated there is proportional to $\omega_0 \alpha / R'$, where ω_0 was the eigenfrequency of the individual oscillators; R' was proportional to the radius and thus the circumference of the chain, representing the maximum wavelength of any oscillation modes and therefore an infrared cutoff; α stood for the separation of the point masses and therefore provided a short-distance, ultraviolet cutoff.

We might then speculate that similar energy contributions exist in a discrete $(1+1)$-dimensional spacetime with compactified spatial dimension of Hubble size, where the infrared cutoff would be given by the periodic boundary condition at the scale of the Hubble radius $1/H$, and the ultraviolet cutoff by the separation of the microscopic degrees of freedom of the spacetime, presumably the Planck length $1/M_{\mathrm{Pl}}$. Unfortunately, the naïve replacements $\alpha \to 1/M_{\mathrm{Pl}}$, $\omega_0 \to M_{\mathrm{Pl}}$, $R' \to 1/H$ lead to an energy of $M_{\mathrm{Pl}} H / M_{\mathrm{Pl}} = H$, from which the ultraviolet cutoff is entirely absent. This energy—particularly after converting to an energy density by dividing by the volume $\sim 1/H$—is again altogether too small to be relevant for cosmology.

Chapter 4

Conclusions and Outlook

It has become apparent, in the course of this work, that in an interesting variety of situations, placing a field in a spacetime other than flat, unbounded Minkowski spacetime induces a non-vanishing change in the zero-point energy of the field. We first made this observation in section 2.3, where both the periodicity of a compactified spatial dimension and the Dirichlet boundary conditions on conducting plates in Minkowski space introduced a vacuum energy we referred to as Casimir energy.

We found in section 2.5.5 that a related energy, scaling as the fourth power of the Hubble parameter, emerges in de Sitter spacetime. Following a short review of scalar field inflation, section 3.3 discussed the possible relevance of this additional contribution to the vacuum energy of de Sitter space. We showed that the Casimir energy of a conformally coupled, massless scalar field is far too small to significantly influence the cosmic evolution during inflation.

In section 3.1, we investigated the zero-point energy of a chain of oscillators arranged into a circle by some external potential. We discovered that the zero-point energy differs from that of a chain with infinite radius by a Casimir contribution proportional to the inverse of the circle's radius. In attempting, in section 3.2, to apply this finding to a cylinder constructed by naïvely stacking many such oscillator chains, we were led to the conclusion that, while the zero-modes of the oscillators should indeed give rise to a Casimir effect within the surface of the cylinder, it is not obvious how this would compare to the ordinary Casimir effect within its volume. We feel confident, however, that the former like the latter effect would tend to contract the structure with respect to a hypothetical object free from Casimir energies.

A rather straightforward way to expand on the work presented here, would be a more rigorous consideration of Casimir energies within 2- or higher-dimensional condensed matter structures. A useful generalisation of our results in section 3.1 and our speculation in section 3.2 might be achieved by dropping the assumption of the number of oscillators being very large. The outcome of these more complicated calculations may provide some insight into whether the phononic Casimir effect we postulated to exist within the surface of carbon nanotubes may actually be observable.

Another very interesting system that might be expected to exhibit a Casimir effect is a layer of graphene. In a realistic model of the graphene sheet, one would expect the separation of the individual carbon atoms to change if the layer were bent. This change in the bond length of the atoms should give rise to a change in the zero-point energy with respect to the unbent sheet. It would be interesting to investigate the dependence of this energy shift on the amount of deformation applied to the graphene layer.

Another intriguing possibility is that compactified extra dimensions might contribute to the vacuum energy of the cosmos. Owing to their finite size, a Casimir energy should emerge in the same way as in the $(\mathbb{R}^1 \times S^1)$-universe of section 2.3. Of course, it may be difficult to obtain for the Casimir energy from compactified extra dimensions the scaling behaviour and magnitude of the observed dark energy density $\sim H^2 M_{\mathrm{Pl}}^2$, but it seems far from inconceivable.

Intuitively, its compactification radius is the only scale the extra dimension can contribute to the form of the Casimir energy. As extra dimensions of size $\gtrsim 2 \times 10^{-4}$ m are excluded by measurements of the gravitational force (Kribs, 2006), the energy scale associated with a possible extra dimension has to be $M_{\mathrm{ED}} \gtrsim 10^{-12}\,\mathrm{GeV}$. These measurements obviously preclude the Hubble scale $H \sim 10^{-42}\,\mathrm{GeV}$ from being the size of the extra dimension. Therefore, the extra-dimensional Casimir energy would certainly involve some more complicated combination of the Hubble scale, as the extent of the observable universe, and the size of the extra dimension. Additionally, it seems quite possible that in a model of discrete spacetime, the separation of the discrete spacetime points, presumably the Planck length $1/M_{\mathrm{Pl}}$, would enter into the Casimir energy. The vacuum energy contribution originating from the compactification of the extra spatial dimension would thus involve some combination of the three scales H, M_{ED}, and M_{Pl}. This combination may end up being of the correct order of magnitude for the Casimir energy to be considered as a candidate for dark energy.

In analogy to the graphene sheet, one would be tempted to contemplate the influence vacuum energy might have on the shape of a spacetime. In other words: would the Casimir energy arising from a possible microscopic substructure of spacetime exert a force on the spacetime itself?

It is, of course, hard to speculate on such profound matters, but let us, nevertheless, outline a possible approach to this issue. It seems perfectly feasible, if certainly complicated, to derive the Casimir energy within the surface of a sphere made up of quantum harmonic oscillators. The phononic modes travelling within this surface would have to be decomposed in spherical harmonics, rather than plane waves, but that should not be a fundamental obstacle to the procedure. In addition to the ultraviolet modes being cut off by the finite separation of the underlying spacetime points, the infrared modes would be constrained by the periodicity imposed on them by the finite extent of the sphere. Subtracting from the vacuum energy of this spacetime the vacuum energy of the infinite-curvature-radius limit of the same microscopically structured spacetime, we would presumably be left with a finite Casimir energy.

If this energy, like the usual electromagnetic Casimir energy of a sphere of positive curvature, turned out to be positive, one might surmise that Casimir energies tend to stretch a spherical space, making a flat spacetime with large curvature radius the stable endpoint of the dynamical evolution of a microscopically structured spacetime. If the phononic Casimir energy turned out to be negative, however, we would have to conclude that the curvature radius would be shrunk by the same effect—flat space would be unstable.

Even though we cannot, at this point, predict what the outcome of these investigations would be, it seems to us a very worthwhile path of inquiry.

Appendix A

Conventions, Notation, Definitions

Throughout this work, we set $c = \hbar = 1$.

For convenience, we frequently write partial derivatives as

$$\frac{\partial V_\mu}{\partial x^\lambda} = \partial_\lambda V_\mu = V_{\mu,\lambda},$$

and covariant derivatives as

$$D_\lambda V_\mu = V_{\mu;\lambda}.$$

The covariant derivative is defined as

$$V^\mu{}_{;\lambda} \equiv \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} V^\kappa, \tag{A.1}$$

where the affine connection is

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu} \right). \tag{A.2}$$

Furthermore, this work follows the sign convention of Birrell and Davies (1984), which is $(-,-,-)$ in the classification of Misner, Thorne, and Wheeler (1973), where the three signs are those of the metric, the curvature tensor, and the right-hand side of the Einstein equations:

$$g_{\mu\nu} = [S1] \times \mathrm{diag}(-1,+1,+1,+1), \tag{A.3}$$

$$R^\lambda{}_{\mu\nu\kappa} = [S2] \times \left(\Gamma^\lambda_{\mu\nu,\kappa} - \Gamma^\lambda_{\mu\kappa,\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} \right), \tag{A.4}$$

$$G_{\mu\nu} = [S3] \times 8\pi G T_{\mu\nu}. \tag{A.5}$$

Specifically, our metric is $g_{\mu\nu} = (+1,-1,-1,-1)$.

The Ricci tensor and the Ricci or curvature scalar are defined as

$$R_{\mu\kappa} = [S2] \times [S3] \times R^{\lambda}{}_{\mu\lambda\kappa} \tag{A.6}$$

$$R = g^{\mu\kappa} R_{\mu\kappa} \tag{A.7}$$

Rewriting the FRW metric (1.2) as $g_{tt} = 1$, $g_{it} = 0$, $g_{ij} = a^2(t)\tilde{g}_{ij}(x)$, we find that the Ricci tensor is given, in our conventions, by (compare Weinberg (1972, chapter 15.1), who uses sign conventions $(+,-,-)$)

$$R_{tt} = -\frac{3\ddot{a}}{a}, \quad R_{ti} = 0, \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2k)\tilde{g}_{ij}\,. \tag{A.8}$$

This yields for the Ricci scalar

$$R = -\frac{6}{a^2}\left(\ddot{a}a + 2\dot{a}^2 + 2k\right)\,. \tag{A.9}$$

Appendix B

Calculation of the Hamiltonian of a Circular Chain of Oscillators

We begin with the angular part of the Hamiltonian (3.16),

$$\mathscr{H}^{\varphi} = \sum_{l=1}^{N} \left(\frac{1}{2} m R^2 (\delta \dot{\varphi}_l)^2 + \frac{1}{2} k R^2 (\delta \varphi_{l+1} - \delta \varphi_l)^2 \right) . \tag{B.1}$$

Using equation (3.17) and dropping the time-dependence, we find for the kinetic energy

$$\begin{aligned}
\sum_{l=1}^{N} (\delta \dot{\varphi}_l)^2 &= \frac{1}{N} \sum_{l} \sum_{s} \mathrm{i}\omega_s^{\varphi} \left(-\mathrm{e}^{\mathrm{i}sl} B_s + \mathrm{e}^{-\mathrm{i}sl} B_s^* \right) \sum_{s'} \mathrm{i}\omega_{s'}^{\varphi} \left(-\mathrm{e}^{\mathrm{i}s'l} B_{s'} + \mathrm{e}^{-\mathrm{i}s'l} B_{s'}^* \right) \\
&= \frac{1}{N} \sum_{l} \sum_{s,s'} \omega_s^{\varphi} \omega_{s'}^{\varphi} \left[-B_s B_{s'} \mathrm{e}^{\mathrm{i}(s+s')l} + B_s B_{s'}^* \mathrm{e}^{\mathrm{i}(s-s')l} + B_s^* B_{s'} \mathrm{e}^{\mathrm{i}(-s+s')l} - B_s^* B_{s'}^* \mathrm{e}^{\mathrm{i}(-s-s')l} \right] \\
&= \sum_{s,s'} \omega_s^{\varphi} \omega_{s'}^{\varphi} \left[-B_s B_{s'} \delta_{s,-s'} + B_s B_{s'}^* \delta_{s,s'} + B_s^* B_{s'} \delta_{-s,-s'} - B_s^* B_{s'}^* \delta_{-s,s'} \right] \\
&= \sum_{s} \left[-\omega_s^{\varphi} \omega_{-s}^{\varphi} B_s B_{-s} + (\omega_s^{\varphi})^2 B_s B_s^* + (\omega_s^{\varphi})^2 B_s^* B_s - \omega_s^{\varphi} \omega_{-s}^{\varphi} B_s^* B_{-s}^* \right] ,
\end{aligned} \tag{B.2}$$

where $\delta_{s,s'}$ is the Kronecker symbol, and we have used $\frac{1}{N}\sum_l \mathrm{e}^{\mathrm{i}(s-s')l} = \delta_{s,s'}$. From equation (3.15), we learn that $\omega_s^{\varphi} = \omega_{-s}^{\varphi}$, and therefore the kinetic energy of the angular motion is

$$\frac{1}{2} m R^2 \sum_{l=1}^{N} (\delta \dot{\varphi}_l)^2 = \frac{1}{2} m R^2 \sum_{s} (\omega_s^{\varphi})^2 \left[-B_s B_{-s} + B_s B_s^* + B_s^* B_s - B_s^* B_{-s}^* \right] . \tag{B.3}$$

We separate the potential into three parts,

$$
\begin{aligned}
V^{\varphi} &= \sum_{l=1}^{N} \frac{1}{2} k R^2 \left(\delta\varphi_{l+1} - \delta\varphi_l\right)^2 \\
&= \sum_{l=1}^{N} \frac{1}{2} k R^2 \left((\delta\varphi_{l+1})^2 + (\delta\varphi_l)^2 - 2\delta\varphi_{l+1}\delta\varphi_l\right) \\
&= \frac{1}{2} k R^2 \left(\underbrace{\sum_{l=1}^{N} 2(\delta\varphi_l)^2}_{=V_1^{\varphi}} - \underbrace{\sum_{l=1}^{N} \delta\varphi_{l+1}\delta\varphi_l}_{=V_2^{\varphi}} - \underbrace{\sum_{l=1}^{N} \delta\varphi_l \delta\varphi_{l-1}}_{=V_3^{\varphi}} \right) ,
\end{aligned}
\tag{B.4}
$$

where, in the last step, we have symmetrised with respect to l by using

$$
\begin{aligned}
2\sum_{l=1}^{N} \delta\varphi_{l+1}\delta\varphi_l &= \sum_{l=1}^{N} \delta\varphi_{l+1}\delta\varphi_l + \sum_{l=2}^{N} \delta\varphi_{l+1}\delta\varphi_l + \underbrace{\delta\varphi_2\delta\varphi_1}_{=\delta\varphi_{N+2}\delta\varphi_{N+1}} \\
&= \sum_{l=1}^{N} \delta\varphi_{l+1}\delta\varphi_l + \sum_{l=2}^{N+1} \delta\varphi_{l+1}\delta\varphi_l \\
&= \sum_{l=1}^{N} \delta\varphi_{l+1}\delta\varphi_l + \sum_{l=1}^{N} \delta\varphi_l\delta\varphi_{l-1} ,
\end{aligned}
\tag{B.5}
$$

where the identity $\delta\varphi_2\delta\varphi_1 = \delta\varphi_{N+2}\delta\varphi_{N+1}$ is due to the N-periodicity of the chain, and in the last step we simply replaced the summation index in the second term $l \to l-1$.

Let us now consider the three parts of the potential separately,

$$
\begin{aligned}
V_1^{\varphi} &= \frac{2}{N} \sum_{l} \sum_{s,s'} \left[B_s B_{s'} \mathrm{e}^{\mathrm{i}(s+s')l} + B_s B_{s'}^* \mathrm{e}^{\mathrm{i}(s-s')l} + B_s^* B_{s'} \mathrm{e}^{\mathrm{i}(-s+s')l} + B_s^* B_{s'}^* \mathrm{e}^{\mathrm{i}(-s-s')l} \right] \\
&= 2 \sum_{s} \left[B_s B_{-s} + B_s B_s^* + B_s^* B_s + B_s^* B_{-s}^* \right] ,
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
V_2^{\varphi} &= \frac{1}{N} \sum_{l} \sum_{s,s'} \left[B_s B_{s'} \mathrm{e}^{\mathrm{i}(s+s')l+\mathrm{i}s} + B_s B_{s'}^* \mathrm{e}^{\mathrm{i}(s-s')l+\mathrm{i}s} + B_s^* B_{s'} \mathrm{e}^{\mathrm{i}(-s+s')l-\mathrm{i}s} + B_s^* B_{s'}^* \mathrm{e}^{\mathrm{i}(-s-s')l-\mathrm{i}s} \right] \\
&= \sum_{s} \left[B_s B_{-s} \mathrm{e}^{\mathrm{i}s} + B_s B_s^* \mathrm{e}^{\mathrm{i}s} + B_s^* B_s \mathrm{e}^{-\mathrm{i}s} + B_s^* B_{-s}^* \mathrm{e}^{-\mathrm{i}s} \right] ,
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
V_3^{\varphi} &= \frac{1}{N} \sum_{l} \sum_{s,s'} \left[B_s B_{s'} \mathrm{e}^{\mathrm{i}(s+s')l-\mathrm{i}s} + B_s B_{s'}^* \mathrm{e}^{\mathrm{i}(s-s')l-\mathrm{i}s} + B_s^* B_{s'} \mathrm{e}^{\mathrm{i}(-s+s')l+\mathrm{i}s} + B_s^* B_{s'}^* \mathrm{e}^{\mathrm{i}(-s-s')l+\mathrm{i}s} \right] \\
&= \sum_{s} \left[B_s B_{-s} \mathrm{e}^{-\mathrm{i}s} + B_s B_s^* \mathrm{e}^{-\mathrm{i}s} + B_s^* B_s \mathrm{e}^{\mathrm{i}s} + B_s^* B_{-s}^* \mathrm{e}^{\mathrm{i}s} \right] .
\end{aligned}
\tag{B.8}
$$

Hence, using $e^{is} + e^{-is} = 2\cos(s)$ and $(1 - \cos(s)) = 2\sin^2\left(\frac{s}{2}\right)$, we obtain

$$\begin{aligned} V^\varphi &= \frac{1}{2} k R^2 \left(V_1^\varphi + V_2^\varphi + V_3^\varphi\right) \\ &= kR^2 \sum_s (1 - \cos(s)) \left[B_s B_{-s} + B_s B_s^* + B_s^* B_s + B_s^* B_{-s}^*\right] \\ &= 2kR^2 \sum_s \sin^2\left(\frac{s}{2}\right) \left[B_s B_{-s} + B_s B_s^* + B_s^* B_s + B_s^* B_{-s}^*\right]. \end{aligned} \tag{B.9}$$

Substituting equations (B.3) and (B.9) back into equation (B.1), we find

$$\begin{aligned} \mathcal{H}^\varphi = \sum_s \Bigg[& \underbrace{\left(-\frac{m}{2} R^2 \omega_s^{\varphi 2} + 2kR^2 \sin^2\left(\frac{s}{2}\right)\right)}_{=0} B_s B_{-s} + \underbrace{\left(\frac{m}{2} R^2 \omega_s^{\varphi 2} + 2kR^2 \sin^2\left(\frac{s}{2}\right)\right)}_{= m\omega_s^{\varphi 2}} B_s B_s^* \\ & + \underbrace{\left(\frac{m}{2} R^2 \omega_s^{\varphi 2} + 2kR^2 \sin^2\left(\frac{s}{2}\right)\right)}_{= m\omega_s^{\varphi 2}} B_s^* B_s + \underbrace{\left(-\frac{m}{2} R^2 \omega_s^{\varphi 2} + 2kR^2 \sin^2\left(\frac{s}{2}\right)\right)}_{=0} B_s^* B_{-s}^* \Bigg], \end{aligned} \tag{B.10}$$

where the simplification of the coefficients to 0 or $m\omega_s^{\varphi 2}$ results from the dispersion relation (3.15). Hence, we can finally write

$$\mathcal{H}^\varphi = \sum_s mR^2 \left(\omega_s^\varphi\right)^2 \left(B_s^\varphi B_s^{\varphi *} + B_s^{\varphi *} B_s^\varphi\right), \tag{B.11}$$

which is the result (3.18).

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 01.12.2010 ....

(Nico Kronberg)